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Some Characterization Theorems for Wiener Process in a Hilbert Space

B. L. S. PRAKASA RAO

1. Introduction

Stochastic integrals for real-valued stochastic processes have been defined and an extensive discussion of the various types of stochastic integrals is given in Lukacs [1]. Recently various types of characterizations of real-valued stochastic processes have been obtained by Laha, Lukacs, Prakasa Rao, Skitovich etc. either through independence or identical distribution of stochastic integrals. In particular, characterizations of Wiener process in the real line are known. For a discussion of these results, the reader is referred to Lukacs [1]. All these results deal with stochastic processes which are real-valued.

Recently Vakhaniya and Kandelski [3] have given a definition of a stochastic integral for operator-valued functions with respect to stochastic processes which are Hilbert-space valued. Using this type of stochastic integral, we shall obtain characterization theorems for Wiener processes which are Hilbert-space Valued.

Section 2 contains some preliminary lemmas. In Section 3, Kolmogorov type representation for infinitely divisible distributions μ with finite second moment (i.e. $E_\mu[\|X\|^2]$ is finite) has been obtained. Definitions and relevant results for homogeneous processes with independent increments and Wiener processes which are Hilbert-space valued are given in Section 4. Section 5 contains derivation of characteristic functional for stochastic integrals as defined in Vakhaniya and Kandelski [3]. Three characterization theorems of Wiener process are given in Section 6.

2. Preliminaries

Let H be a real seperable Hilbert space and let (x, y) denote the inner product between $x \in H$ and $y \in H$. Let $\|x\|$ denote the norm of x. Let $(\Omega, \mathscr{I}, P)$ be a probability space. X is said to be random element in H if X is a measurable mapping from $(\Omega, \mathscr{I})$ to (H, ζ) where ζ is the σ-field of Borel subsets of H. For more details regarding probability measures on H, the reader is referred to Parthasarathy [2].

Suppose, X, Y are random elements taking values in the Hilbert space H and A and B are bounded linear operators on H.

Lemma 2.1. *If X and Y are independent, then AX and BY are independent.*

Proof. For any Borel set F, $\{x\colon Ax \in F\}$ is also a Borel set in H since A is a bounded linear operator and hence a continuous operator. Hence for any two Borel sets F_1, F_2

$$P[AX \in F_1, BY \in F_2] = P[X \in G_1, Y \in G_2]$$

where
$$G_1=\{x\colon Ax\in F_1\} \quad\text{and}\quad G_2=\{y\colon By\in F_2\}.$$

Since X and Y are independent, it follows that

$$\begin{aligned}P[X\in G_1, Y\in G_2]&=P[X\in G_1]\,P[Y\in G_2]\\&=P[AX\in F_1]\,P[BY\in F_2].\end{aligned}$$

Lemma 2.2. *If X has characteristic functional $\hat{\mu}(y)$, then AX has the characteristic functional $\hat{\mu}(A'y)$.*

Proof. Let $\hat{\mu}_A(y)$ denote the characteristic functional of AX. Clearly

$$\begin{aligned}\hat{\mu}_A(y)&=E[e^{i(AX,y)}]\\&=E[e^{i(X,A'y)}]=\hat{\mu}(A'y).\end{aligned}$$

Lemma 2.3. *If X has an infinitely divisible characteristic functional, then AX has an infinitely divisible characteristic functional.*

Proof. Let $\hat{\mu}(y)$ denote the characteristic functional of X. For any n there exists a characteristic functional $\hat{\mu}_n$ such that $\hat{\mu}(y)=[\hat{\mu}_n(y)]^n$ for all $y\in H$. Now

$$\hat{\mu}_A(y)=\hat{\mu}(A'y)=[\hat{\mu}_n(A'y)]^n.$$

Since $\hat{\mu}_n(A'y)$ is the characteristic functional of AZ when Z has the characteristic functional $\hat{\mu}_n(y)$, it follows that $\hat{\mu}_A(y)$ is an infinitely divisible characteristic functional (i.d.c.f.).

Lemma 2.4. *If $\hat{\mu}(y)$ is an i.d.c.f., then $[\hat{\mu}(y)]^t$ is an i.d.c.f. for real number $t>0$.*

Proof. Since $\hat{\mu}$ is i.d., $\hat{\mu}$ is different from zero for all $y\in H$ (Theorem 4.2, p. 171, Parthasarathy [2]) and $\log\hat{\mu}$ can be represented in the form

$$\log\hat{\mu}(y)=i(x_0,y)-\tfrac{1}{2}(Sy,y)+\int_H K(x,y)\,M(dx)$$

where $x_0\in H$, S is an S-operator and M is a σ-finite measure with finite mass outside every neighbourhood of the origin and

$$\int_{\{\|x\|\leq 1\}}\|x\|^2 M(dx)<\infty$$

where

$$K(x,y)=e^{i(x,y)}-1-\frac{i(x,y)}{1+\|x\|^2}$$

and every functional of the above form is the log of an i.d.c.f. Consider

$$\begin{aligned}\log[\hat{\mu}(y)]^t&=t\log\hat{\mu}(y)\\&=t\left\{i(x_0,y)-\tfrac{1}{2}(Sy,y)+\int_H K(x,y)\,M(dx)\right\}\\&=i(x',y)-\tfrac{1}{2}(S'y,y)+\int_H K(x,y)\,M'(dx)\end{aligned}$$

where $x'=tx_0$, $S'=tS$ and $M'=tM$. Clearly S' is an S-operator and M' satisfies conditions similar to those of M.

Hence $[\hat{\mu}(y)]^t$ is an i.d.c.f. for any $t>0$.

Lemma 2.5. *If μ_n converges weakly to μ and μ_n are i.d., then μ is i.d.*

Proof. (Theorem 4.1, p. 170, Parthasarathy [2].)

We shall now give some relations between various types of convergence for random elements in H.

Definition 2.1. $X_n \to X$ in quadratic means as $n \to \infty$ if

$$E\|X_n - X\|^2 \to 0.$$

In such an event we write $X_n \xrightarrow{\text{q.m.}} X$.

Definition 2.2. $X_n \to X$ in probability if for every $\varepsilon > 0$,

$$P(\|X_n - X\| > \varepsilon) \to 0 \quad \text{as } n \to \infty.$$

In such an event we write $X_n \xrightarrow{p} X$.

Definition 2.3. $X_n \to X$ in law if the measures μ_n induced by X_n converge weakly to the measure μ induced by X. In such an event we write $X_n \xrightarrow{\mathscr{L}} X$.

Lemma 2.6. $X_n \xrightarrow{\text{q.m.}} X \Rightarrow X_n \xrightarrow{p} X \Rightarrow X_n \xrightarrow{\mathscr{L}} X$.

Proof. By Chebyshev's inequality, for any $\varepsilon > 0$

$$P(\|X_n - X\| > \eta) \leqq \frac{1}{\eta^2} E\|X_n - X\|^2$$

which clearly proves that $X_n \xrightarrow{\text{q.m.}} X \Rightarrow X_n \xrightarrow{p} X$.

Now suppose $X_n \xrightarrow{p} X$. Choose any $\eta > 0$. Then for n sufficiently large $P(\|X_n - X\| \leqq \eta) > 1 - \varepsilon$. Let B be any open set in H. Then $B = \{\cup S_\alpha; \alpha \in B\}$ where S_α are open spheres contained in B. Let $B^{-\eta}$ denote the union of spheres $S_\alpha^{-\eta}$ where $S_\alpha^{-\eta} = \Phi$ if the radius of S_α is smaller than or equal to η and $S_\alpha^{-\eta}$ denotes the sphere with the same centre as S_α but radius decreased by η if the radius of S_α is greater than η. Clearly $B^{-\eta} \uparrow B$ as $\eta \downarrow 0$. Now

$$\begin{aligned} &P[\{\|X_n - X\| \leqq \eta\} \cup \{X \in B^{-\eta}\}] \\ &\qquad = P[\|X_n - X\| \leqq \eta] + P[X \in B^{-\eta}] - P[\{\|X_n - X\| \leqq \eta\} \cap \{X \in B^{-\eta}\}]. \end{aligned}$$

Hence

$$P[\{\|X_n - X\| \leqq \eta\} \cap \{X \in B^{-\eta}\}] \geqq P[X \in B^{-\eta}] - \varepsilon.$$

But $\|X_n - X\| \leqq \eta$ and $X \in B^{-\eta}$ imply that $X_n \in B$. Therefore

$$\begin{aligned} P(X_n \in B) &\geqq P(\{\|X_n - X\| \leqq \eta\} \cap \{X \in B^{-\eta}\}) \\ &\geqq P[X \in B^{-\eta}] - \varepsilon \end{aligned}$$

which shows that

$$\liminf_{n \to \infty} P(X_n \in B) \geqq P[X \in B^{-\eta}] - \varepsilon.$$

Taking limit as $\eta \downarrow 0$, we obtain that

$$\liminf_{n \to \infty} P(X_n \in B) \geqq P(X \in B) - \varepsilon.$$

Since ε is arbitrary, it follows that for every open set B

$$\liminf_{n\to\infty} P(X_n \in B) \geqq P(X \in B).$$

Hence $X_n \xrightarrow{\mathscr{L}} X$ by Theorem 6.1, p. 40 of Parthasarathy [2].

3. Some Theorems on Representation of Infinitely Divisible Distributions

Theorem 3.1. *A function $\varphi(y)$ is the characteristic functional of an infinitely divisible distribution μ on X if and only if it is of the form*

$$\varphi(y) = \exp\left[i(x_0, y) - \tfrac{1}{2}(Sy, y) + \int_H K(x, y)\, M(dx)\right]$$

where $x_0 \in H$, S is an S-operator, M is a σ-finite measure with finite mass outside every neighbourhood of the origin and

$$\int_{\{\|x\| \leqq 1\}} \|x\|^2 M(dx) < \infty. \qquad \textit{Here} \qquad K(x, y) = e^{i(x, y)} - 1 - i\frac{(x, y)}{1 + \|x\|^2}.$$

The representation is unique. (*Note that* $M\{0\} = 0$.)

Proof. See Theorem 4.10, p. 181 of Parathasarathy [2].

Lemma 3.1. *If μ is an infinitely divisible distribution with*

$$\int_H \|x\|^2 \mu(dx) < \infty, \qquad \textit{then} \qquad \int_H \|x\|^2 M(dx) < \infty$$

where M is the σ-finite measure in the canonical representation.

Proof. We shall write $\mu = [x_0, S, M]$ to denote the canonical representation of any infinitely divisible distribution μ. Suppose X has distribution μ. Then $-X$ has distribution $\bar{\mu}$ with characteristic functional $\hat{\mu}(-y)$ where $\hat{\mu}(y)$ denotes the characteristic functional of μ. Let $\tau = \mu * \bar{\mu}$. Then τ has the representation $[0, 2S, M + \bar{M}]$. In particular τ is a symmetric infinitely divisible distribution. Furthermore

$$\int_H \|x\|^2 \tau(dx) = 2\int_H \|x\|^2 \mu(dx).$$

Let ν denote the Gaussian distribution with the characteristic functional $e^{-(Sy, y)}$. Then $\tau = \nu * e(M^*)$ where $M^* = M + \bar{M}$ and $e(M^*)$ has the characteristic functional $\exp\{\int_H K^*(x, y)\, M^*(dx)\}$ where $K^*(x, y) = \cos(x, y) - 1$. Furthermore

$$\int_H \|x\|^2 M^*(dx) = 2\int_H \|x\|^2 M(dx)$$

whenever any of the numbers is finite. In view of these remarks it is enough to prove the Lemma when μ is of the form $[0, 0, M]$ where M is symmetric. Let us first suppose that M is finite measure. Let $M(H) = t$ and $F = t^{-1} M$. Then

$$\mu = e^{-t} \sum_{r=0}^{\infty} t^r \frac{F^r}{r!}$$

where F^r denotes the convolution of F for r times and hence

$$\int_H \|x\|^2 \mu(dx) = e^{-t} \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_H \|x\|^2 F^r(dx)$$
$$= e^{-t} \sum_{r=0}^{\infty} \frac{t^r r}{r!} \int_H \|x\|^2 F(dx)$$
$$= e^{-t} \sum_{r=1}^{\infty} \frac{t^{r-1}}{(r-1)!} \int_H \|x\|^2 M(dx)$$
$$= \int_H \|x\|^2 M(dx)$$

which shows that, $\int_H \|x\|^2 \mu(dx)$ is finite implies that

$$\int_H \|x\|^2 M(dx) < \infty.$$

If M is not finite, then we can find an increasing sequence of finite symmetric measures M_n approximating M. Let $\mu_n = [0, 0, M_n]$. Then we obtain that

$$\int_H \|x\|^2 \mu_n(dx) = \int_H \|x\|^2 M_n(dx).$$

We note that μ_n is an increasing sequence of finite measure converging to measure μ. Hence

$$\int_H \|x\|^2 \mu_n(dx) \to \int_H \|x\|^2 \mu(dx)$$

as $n \to \infty$ which implies that

$$\int_H \|x\|^2 M_n(dx) \to \int_H \|X\|^2 \mu(dx)$$

as $n \to \infty$. But

$$\int_H \|x\|^2 M_n(dx) \to \int_H \|x\|^2 M(dx)$$

which proves that

$$\int_H \|x\|^2 \mu(dx) = \int_H \|x\|^2 M(dx)$$

even when M is not a finite measure. This proves in particular that

$$\int_H \|x\|^2 M(dx) < \infty.$$

Theorem 3.2. *μ is an infinitely divisible distribution with*

$$\int_H \|x\|^2 \mu(dx) < \infty$$

if and only if the characteristic functional of μ viz. $\hat{\mu}(y)$ can be written in the form

$$\hat{\mu}(y) = \exp\left[i(x_1, y) + \int_H L(x, y)\, R(dx)\right]$$

where $x_1 \in H$ and $0 \leqq R(H) < \infty$ and $R\{0\} = 1$, $L(0, y) = -\frac{1}{2}(Sy, y)$ where S is an S-operator. Here

$$L(x, y) = [e^{i(x, y)} - 1 - i(x, y)]\, \|x\|^{-2} \quad \text{for } x \neq 0,\ x, y \in H.$$

This representation is unique.

Proof. By Theorem 3.1, there exists x_0, S and M giving the canonical representation for $\hat{\mu}(y)$. Further this triple is unique. Since $\int_H \|x\|^2 \mu(dx) < \infty$, by the previous lemma $\int_H \|x\|^2 M(x) < \infty$. Now

$$\log \hat{\mu}(y) = i(x_0, y) - \tfrac{1}{2}(Sy, y) + \int_{H-\{0\}} K(x, y)\, M(dx).$$

Define

$$N(A) = \int_A \frac{\|x\|^2}{1+\|x\|^2} M(dx)$$

for Borel sets A not containing $0 \in H$. Then

$$\log \hat{\mu}(y) = i(x_0, y) + \int_{H-\{0\}} K(x, y) \frac{1+\|x\|^2}{\|x\|^2} N(dx) - \frac{1}{2}(Sy, y).$$

Define $R\{0\} = 1$ and

$$R(A) = \int_A (1+\|x\|^2)\, N(dx)$$

for Borel sets A not containing 0. Therefore

$$\log \hat{\mu}(y) = i(x_0, y) + \int_H \frac{K(x, y)}{\|x\|^2} R(dx)$$

where the integrand is appropriately defined at $x=0$. Hence

$$\log \hat{\mu}(y) = i(x_0, y) + \int_H \left\{ e^{i(x, y)} - 1 - \frac{i(x, y)}{1+\|x\|^2} \right\} \|x\|^{-2} R(dx)$$

$$= i(x_0, y) + i \int_H \frac{(x, y)}{1+\|x\|^2} R(dx) + \int_H L(x, y)\, R(dx).$$

It is clear that

$$R(H) = 1 + \int_{H-\{0\}} (1+\|x\|^2)\, N(dx)$$
$$= 1 + \int_{H-\{0\}} \|x\|^2 M(dx)$$
$$= 1 + \int_H \|x\|^2 M(dx)$$

is finite by Lemma 3.1. Therefore

$$\left| \int_H \frac{(x, y)}{1+\|x\|^2} R(dx) \right| \leqq \|y\| \int_H \frac{\|x\|}{1+\|x\|^2} R(dx)$$
$$\leqq \|y\|\, R(H)$$

which shows that

$$\int_H \frac{(x, y)}{1+\|x\|^2} R(dx)$$

is a bounded linear functional on H. Hence there exists an element $x_0' \in H$ such that

$$(x_0', y) = \int_H \frac{(x, y)}{1+\|x\|^2} R(dx)$$

for all $y \in H$. This implies that

$$\log \hat{\mu}(y) = i(x_0, y) + i(x'_0, y) + \int_H L(x, y)\, R(dx)$$
$$= i(x_1, y) + \int_H L(x, y)\, R(dx)$$

where $x_1 = x_0 + x'_0 \in H$. The uniqueness of the representation follows from the uniqueness of the representation in Theorem 3.1.

4. Homogeneous Process with Independent Increments

Let Λ be the interval [0, 1] and $\mathscr{B}$ denote the class of Borel subsets of [0, 1]. For each $\Delta \in \mathscr{B}$, let $\Phi(\Delta)$ be a random element with the following properties.

(4.1) If Δ and Δ' are disjoint Borel subsets of [0, 1] then $\Phi(\Delta)$ and $\Phi(\Delta')$ are independent and $\Phi(\Delta \cup \Delta') = \Phi(\Delta) + \Phi(\Delta')$.

(4.2) $\Phi(\Delta)$ has stationary increments i.e. $\Phi(\Delta)$ and $\Phi(\Delta')$ are identically distributed if the Lebesgue measure of Δ and Δ' are equal.

(4.3) Let μ_t denote the probability measure of $\Phi([0, t])$. Then μ_t converges weakly to the distribution degenerate at the origin as $t \to 0$.

Clearly $\hat{\mu}_t(y) \neq 0$ for all $t \geqq 0$ and for all $y \in H$. Furthermore $\{\mu_t, t > 0\}$ forms a one-parameter convolution semi-group of distributions. In fact μ_t is infinitely divisible for each t and we have the following theorem.

Theorem 4.1. *Let $\{\mu_t, t > 0\}$ be a one-parameter convolution semi-group of distributions such that μ_t converges weakly to the distribution degenerate at the origin as $t \to 0$. Then $\hat{\mu}_t(y)$ has the canonical representation*

$$\hat{\mu}_t(y) = \exp t\left[\int_H K(x, y)\, M(dx) - \tfrac{1}{2}(S_0 y, y) + i(x_0, y)\right]$$

where $x_0 \in H$, M is a σ-finite measure with finite mass outside every neighbourhood of the origin and $\int_{\|x\| \leqq 1} \|x\|^2 M(dx) < \infty$. Here x_0, S and M are uniquely determined and S_0 is an S-operator.

Proof. See Theorem 7.1, p. 201 of Parthasarathy [2].

Theorem 4.2. *Suppose $\{\mu_t, t > 0\}$ is a one-parameter convolution semigroup of distributions with $\int_H \|x\|^2 \mu_1(dx) < \infty$ and such that μ_t converges weakly to the distribution degenerate at 0 at $t \to 0$. Then $\hat{\mu}_t(y)$ has the canonical representation*

$$\hat{\mu}_t(y) = \exp t\left[i(x_1, y) + \int_H L(x, y)\, R(dx)\right]$$

where $x_1 \in H$, $R(H) < \infty$ and $L(0, y) = -\frac{1}{2}(Sy, y)$ where S is an S-operator and $R\{0\} = 1$. Here x_1, S and R are unique.

Proof. This follows from Theorem 3.2 as Theorem 4.1 follows from Theorem 3.1 (cf. Theorem 7.1, p. 201 of Parthasarathy [2]).

Definition 4.1. A process Φ on Λ with the properties (4.1), (4.2) and (4.3) is said to be a homogeneous process with independent increments. The process is said to have mean zero if $x_1=0$ in the representation of $\Phi([0,1])$. The S-operator S in the representation is called the associated S-operator.

Definition 4.2. A homogeneous process Φ on Λ with independent increments is said to be a Wiener process with mean 0 if the characteristic functional $\hat{\mu}_t(y)$ of $\Phi([0,t])$ has the representation

$$\hat{\mu}_t(y)=\exp\left[-\tfrac{1}{2}t(Sy,y)\right],$$

where S is an S-operator.

5. Stochastic Integrals for Operator-Valued Functions

Let Φ be a homogeneous process on Λ with independent increments with mean 0 and with $E_\mu[\|X\|^2]<\infty$ where μ is the distribution of $\Phi([0,1])$. Let S denote the S-operator associated with Φ. For any bounded linear operator A, define

$$n(A)=[\mathrm{Tr}(ASA')]^{\frac{1}{2}}+[\mathrm{Tr}(A'SA)]^{\frac{1}{2}}.$$

Then the set $\{A: n(A)=0\}$ is a linear semi-group in the linear group of all bounded linear operators A. The function n is a norm in the corresponding factor group. We shall not distinguish between a coset and the individual operators in the coset. In this sense n is a norm in the linear set of all bounded linear operators. Let $\mathscr{A}_s$ denote the completion of this set in the norm n. Consider the space $\mathscr{L}_2=\mathscr{L}_2(\Lambda,\mathscr{B},\mathscr{M},\mathscr{A}_s)$ of functions $A(\cdot)$ with values in $\mathscr{A}_s$ which are strongly measurable and such that

$$|A|^2=\int_\Lambda n^2(A(\lambda))\,d\lambda<\infty$$

where $\mathscr{M}$ is the Lebesgue measure on Λ. The norm in $\mathscr{L}_2$ is as defined above. The set of functions whose values are bounded linear operators and which are piecewise constant functions is dense in $\mathscr{L}_2$. Using this set up, Vakhaniya and Kandelski [3] have defined stochastic integrals of the form

$$J\equiv\int_\Lambda A(\lambda)\,\Phi(d\lambda)$$

for functions $A(\cdot)$ in $\mathscr{L}_2$. They have proved that

(i) $E[J]=0$ i.e. $E[(J,x)]=0$ for all $x\in H$ (ii) $E\|J\|^2\leq|A|^2$ and (iii) J has a finite S-operator S_J which has the representation

$$S_J=\int_\Lambda A(\lambda)\,SA'(\lambda)\,d\lambda$$

where S_J is understood to be a Bochner integral under convergence in the space $\mathscr{A}_s$. We shall now obtain the characteristic functional of J.

Theorem 5.1. *J has an i.d.c.f. and the logarithm of the characteristic functional of J is*

$$\int_\Lambda v(A'(\lambda)\,y)\,d\lambda$$

where $v(y)=\log\hat{\mu}(y)$.

Proof. We can find a sequence of simple functions A_n such that $|A_n-A|^2\to 0$ where $|A|$ is the norm of A as defined before. Let

$$J_n=\int_\Lambda A_n(\lambda)\,\Phi(d\lambda)$$

and

$$J = \int_{\Lambda} A(\lambda)\, \Phi(d\lambda).$$

It also follows that $J_n \xrightarrow{\text{q.m.}} J$ and hence by Lemma 2.6, $J_n \xrightarrow{\mathscr{L}} J$. Hence the characteristic functional of J_n converges to the characteristic functional of J. Let $A_n(\lambda) = A_n^{(k)}$ for $\lambda \in \Delta_n^k$, $1 \leqq k \leqq k_n$ where $\bigcup_{k=1}^{k_n} \Delta_n^{(k)} = [0, 1]$ and Δ_n^k are disjoint in k for any fixed n. Then by definition $J_n = \sum_{k=1}^{n} A_n^{(k)} \Phi(\Delta_n^k)$. Since $\Delta_n^{(k)}$ are disjoint in k for any fixed, n, the random elements $\Phi(\Delta_n^k)$ are independent and hence by Lemma 2.1, $A_n^{(k)} \Phi(\Delta_n^k)$ are independent. Hence the characteristic functional of J_n is the product of the characteristic functionals of $A_n^{(k)} \Phi(\Delta_n^{(k)})$. Let $\hat{\mu}$ denote the characteristic functional of $\Phi([0, 1])$ and let $v(y) = \log \hat{\mu}(y)$. By Lemma 2.3, characteristic functional of $A_n^{(k)} \Phi(\Delta_n^{(k)})$ is i.d. since $\hat{\mu}$ is an i.d.c.f. and hence characteristic functional of J_n is infinitely divisible. Hence J has an i.d.c.f. and is nonzero. The characteristic functional of J_n is

$$\prod_{k=1}^{k_n} \hat{\mu}_{\Delta_n^{(k)}}(A_n^{(k)\prime} y)$$

by Lemma 2.2 where $\hat{\mu}_{\Delta_n^{(k)}}$ denotes the characteristic functional of $A_n^{(k)\prime} X$ when X has the distribution of $\Phi(\Delta_n^{(k)})$. But

$$\hat{\mu}_{\Delta_n^{(k)}}(y) = [\hat{\mu}_1(y)]^{|\Delta_n^{(k)}|}$$

where $|\Delta_n^{(k)}|$ denotes the Lebesgue measure of $\Delta_n^{(k)}$. Hence the characteristic functional of J_n is

$$\prod_{k=1}^{k_n} [\hat{\mu}(A_n^{(k)\prime} y)]^{|\Delta_n^{(k)}|}.$$

Therefore log of characteristic functional of J_n (well-defined since it is not zero) is given by $\sum_{k=1}^{k_n} v(A_n^{(k)\prime} y)\, |\Delta_n^{(k)}|$. We know that this converges to the log of characteristic functional of J. Hence the log of c.f. of J is

$$\int_{\Lambda} v(A'(\lambda)\, y)\, d\lambda$$

since $A_n' \to A'$ in the norm in $\mathscr{L}_2$ and $\sum_{k=1}^{k_n} v(A_n^{(k)\prime} y)\, |\Delta_n^{(k)}|$ is an approximating sum for the above integral.

6. Characterization Theorems

Theorem 6.1. *Suppose Φ is a homogeneous process on $\Lambda = [0, 1]$ with independent increments with mean* 0 *and $\int_H \|x\|^2 \mu(dx) < \infty$ where μ is the distribution of $\Phi([0, 1])$. Let $A(\cdot)$ and $B(\cdot)$ be functions in $\mathscr{L}_2$ satisfying the following properties.*

(6.1) $$a \equiv \sup_{\lambda} \|A(\lambda)\| < \infty; \qquad b \equiv \sup_{\lambda} \|B(\lambda)\| < \infty.$$

(6.2) $$H_\lambda^A = H_\lambda^B = H \quad \textit{for all } \lambda \in \Lambda$$

where H_λ^A denotes the subspace spanned by the operator $A(\lambda)$ etc.

(6.3)
$$\int_\Lambda [\|A(\lambda)\,x\|^2 - \|B(\lambda)\,x\|^2]\,d\lambda$$

is either strictly greater than zero or strictly less than zero for all $x \in H - \{0\}$. Then

$$\int_\Lambda A(\lambda)\,\Phi(d\lambda) \quad \text{and} \quad \int_\Lambda B(\lambda)\,\Phi(d\lambda)$$

are identically distributed if and only if Φ is a Wiener process and $A(\cdot)$ and $B(\cdot)$ satisfy the relation

$$\int_\Lambda A(\lambda)\,S A'(\lambda)\,d\lambda = \int_\Lambda B(\lambda)\,S B'(\lambda)\,d\lambda$$

where S is the S-operator associated with Φ.

Proof. Let $v(y) = \log \hat{\mu}(y)$. Suppose further that

$$\int_\Lambda A(\lambda)\,\Phi(d\lambda) \quad \text{and} \quad \int_\Lambda B(\lambda)\,\Phi(d\lambda)$$

are identically distributed. Then by Theorem 5.1, it follows that

$$\int_\Lambda v\big(A'(\lambda)\,y\big)\,d\lambda = \int_\Lambda v\big(B'(\lambda)\,y\big)\,d\lambda$$

for all $y \in H$ and each of them is the logarithm of an infinitely divisible characteristic functional. Let

$$v(y) = \int_H L(x, y)\,R(dx)$$

where $R\{0\} = 1$ and $L(0, y) = -\frac{1}{2}(S y, y)$ where S is the associated S-operator and $R(H)$ is finite and $L(x, y)$ is as defined before for $x \neq 0$.

We shall write A_λ for $A(\lambda)$ when it is convenient to do so. Now consider

$$\begin{aligned}
\int_\Lambda v\big(A'(\lambda)\,y\big)\,d\lambda &= \int_\Lambda \Big[\int_H L\big(x, A'(\lambda)\,y\big)\,R(dx)\Big]\,d\lambda \\
&= \int_\Lambda \left[\int_{H-\{0\}} \frac{e^{i(x, A'(\lambda)\,y)} - 1 - i\big(x, A'(\lambda)\,y\big)}{\|x\|^2}\,R(dx)\right] d\lambda \\
&\quad -\tfrac{1}{2} \int_\Lambda \big(S A'(\lambda)\,y, A'(\lambda)\,y\big)\,d\lambda \\
&= \int_\Lambda \left[\int_{H-\{0\}} \frac{e^{i(A(\lambda)\,x, y)} - 1 - i\big(A(\lambda)\,x, y\big)}{\|x\|^2}\,R(dx)\right] d\lambda \\
&\quad -\tfrac{1}{2} \int_\Lambda \big(S A'(\lambda)\,y, A'(\lambda)\,y\big)\,d\lambda \\
&= \int_\Lambda \left[\int_{H-\{0\}} \frac{e^{i(z, y)} - 1 - i(z, y)}{\|A_\lambda^{-1} z\|^2}\,R(dA_\lambda^{-1} z)\right] d\lambda \\
&\quad -\tfrac{1}{2} \int_\Lambda \big(A(\lambda)\,S A'(\lambda)\,y, y\big)\,d\lambda \\
&= \int_\Lambda \left[\int_{H-\{0\}} \frac{e^{i(z, y)} - 1 - i(z, y)}{\|z\|^2}\,\frac{\|z\|^2}{\|A_\lambda^{-1} z\|^2}\,R(dA_\lambda^{-1} z)\right] d\lambda \\
&\quad -\tfrac{1}{2} \int_\Lambda \big(A(\lambda)\,S A'(\lambda)\,y, y\big)\,d\lambda.
\end{aligned}$$

Let

$$R_\lambda^A(dz) = \frac{\|z\|^2}{\|A_\lambda^{-1} z\|^2} R(dA_\lambda^{-1} z).$$

Then

$$\int_\Lambda v(A'(\lambda)\, y)\, d\lambda = \int_\Lambda \Big[\int_{H-\{0\}} L(z, y)\, R_\lambda^A(dz)\Big]\, d\lambda - \tfrac{1}{2} \int_\Lambda (A(\lambda)\, S A'(\lambda)\, y, y)\, d\lambda. \tag{6.4}$$

Since $|L(z, y)| \leqq \|y\|^2$ uniformly in z, and

$$\begin{aligned}\int_\Lambda \left[\int_{H-\{0\}} \frac{\|z\|^2}{\|A_\lambda^{-1} z\|^2} R(dA_\lambda^{-1} z)\right] d\lambda &= \int_\Lambda \left[\int_{H-\{0\}} \frac{\|A_\lambda x\|^2}{\|x\|^2} R(dx)\right] d\lambda \\ &\leqq \int_\Lambda \left[\int_{H-\{0\}} \frac{a^2 \|x\|^2}{\|x\|^2} R(dx)\right] d\lambda \\ &= a^2 R(H) < \infty\end{aligned}$$

the order of integration in R.H.S. of (6.4) can be interchanged. Define

$$R_A(dz) = \int_\Lambda R_\lambda^A(dz)\, d\lambda,$$

i.e. for any Borel set F in $H - \{0\}$,

$$R_A(F) = \int_\Lambda R_\lambda^A(F)\, d\lambda.$$

Then in view of the previous remarks it follows that

$$\int_\lambda v(A'(\lambda)\, y)\, d\lambda = \int_{H-\{0\}} L(z, y)\, R_A(dz) - \tfrac{1}{2} \int_\Lambda (A(\lambda)\, S A'(\lambda)\, y, y)\, d\lambda.$$

Similarly we obtain that

$$\int_\Lambda v(B'(\lambda)\, y)\, d\lambda = \int_{H-\{0\}} L(z, y)\, R_B(dz) - \tfrac{1}{2} \int_\Lambda (B(\lambda)\, S B'(\lambda)\, y, y)\, d\lambda.$$

Let S_A be defined by the relation

$$S_A = \int_\Lambda A(\lambda)\, S A'(\lambda)\, d\lambda$$

where the integral defined is understood to be a Bochner Integral under convergence in the space $\mathscr{A}_s$ as defined in Vakhaniya and Kandelski [3]. S_B is similarly defined. Then

$$\int_\Lambda v(A'(\lambda)\, y)\, d\lambda = \int_{H-\{0\}} L(z, y)\, R_A(dz) - \tfrac{1}{2}(S_A\, y, y)$$

where S_A is an S-operator.

Now

$$\begin{aligned}R_A(H - \{0\}) &= \int_\Lambda R_\lambda^A(H - \{0\})\, d\lambda \\ &= \int_\Lambda \left[\int_{H-\{0\}} \frac{\|z\|^2}{\|A_\lambda^{-1} z\|^2} R(dA_\lambda^{-1} z)\right] d\lambda \\ &= \int_\Lambda \left[\int_{H-\{0\}} \frac{\|A_\lambda x\|^2}{\|x\|^2} R(dx)\right] d\lambda \\ &= \int_{H-\{0\}} \left[\int_\Lambda \frac{\|A_\lambda x\|^2}{\|x\|^2} d\lambda\right] R(dx).\end{aligned}$$

The interchange in the order of integration is valid since $\sup_\lambda \|A(\lambda)\| \equiv a < \infty$. Hence

$$R_A(H-\{0\}) = \int_{H-\{0\}} \frac{1}{\|x\|^2} \left[\int_\Lambda \|A_\lambda x\|^2 \, d\lambda\right] R(dx).$$

We know that the representation

$$\int_{H-\{0\}} L(z, y)\, R_A(dz) - \tfrac{1}{2}(S_A y, y)$$

of the logarithm of the characteristic functional of $\int_\Lambda A(\lambda)\, \Phi(d\lambda)$ is unique. Hence it follows that $R_A \equiv R_B$ on $H-\{0\}$ and $S_A \equiv S_B$. Next for any Borel set F in $H-\{0\}$,

$$\begin{aligned}
R_A(F) &= \int_\Lambda R_\lambda^A(F)\, d\lambda \\
&= \int_\Lambda \left[\int_F \frac{\|z\|^2}{\|A_\lambda^{-1} z\|^2} R(dA_\lambda^{-1} z)\right] d\lambda \\
&= \int_\Lambda \left[\int_{A_\lambda^{-1} F} \frac{\|A_\lambda x\|^2}{\|x\|^2} R(dx)\right] d\lambda \\
&= \int_\Lambda \left[\int_{H-\{0\}} \frac{\|A_\lambda x\|^2}{\|x\|^2} I[x\colon A_\lambda x \in F]\, R(dx)\right] d\lambda \\
&= \int_{H-\{0\}} \left[\int_\Lambda \frac{\|A_\lambda x\|^2}{\|x\|^2} I[\lambda\colon A_\lambda x \in F]\, d\lambda\right] R(dx).
\end{aligned}$$

Here $I(G)$ denotes the indicator function of the set G. Let

$$a(x, F) = \int_\Lambda \|A_\lambda x\|^2 I[\lambda\colon A_\lambda x \in F]\, d\lambda.$$

Then

$$R_A(F) = \int_{H-\{0\}} \frac{a(x, F)}{\|x\|^2} R(dx).$$

Hence we have for any Borel set F in $H-\{0\}$,

$$\int_{H-\{0\}} \frac{a(x, F)}{\|x\|^2} R(dx) = \int_{H-\{0\}} \frac{b(x, F)}{\|x\|^2} R(dx)$$

where $b(x, F)$ is defined for the operator B. Therefore

$$\int_{H-\{0\}} \frac{b(x, F) - a(x, F)}{\|x\|^2} R(dx) = 0$$

for all Borel sets F.

In particular it follows that

$$\int_{H-\{0\}} \frac{b(x, H-\{0\}) - a(x, H-\{0\})}{\|x\|^2} R(dx) = 0. \tag{6.5}$$

By the hypothesis for any $x \neq 0$ in H,

$$\begin{aligned} b(x, H-\{0\}) - a(x, H-\{0\}) \\ &= \int_\Lambda (\|B_\lambda x\|^2 I[\lambda: B_\lambda x \in H-\{0\}] - \|A_\lambda x\|^2 I[\lambda: A_\lambda x \in H-\{0\}])\, d\lambda \\ &= \int_\Lambda (\|B_\lambda x\|^2 - \|A_\lambda x\|^2)\, d\lambda \end{aligned}$$

is either strictly greater than zero or strictly less than zero. Hence it follows from (6.5) that $R(H-\{0\})=0$. This in turn proves that

$$v(y) = -\tfrac{1}{2}(S y, y)$$

which shows that Φ is a Wiener process with mean 0. We also note $\int A(\lambda)\, \Phi(d\lambda)$ is normal random element with the S-operator $\int_\Lambda A(\lambda)\, SA'(\lambda)\, d\lambda$. Conversely suppose Φ is a Wiener process with mean 0. Then it is easy to see that $\int A(\lambda)\, \Phi(d\lambda)$ and $\int B(\lambda)\, \Phi(d\lambda)$ are normal random elements and they are identically distributed provided

$$\int_\Lambda A(\lambda)\, SA'(\lambda)\, d\lambda = \int_\Lambda B(\lambda)\, SB'(\lambda)\, d\lambda.$$

Theorem 6.2. *Suppose Φ is a homogeneous process with independent increments with mean 0 and $\int_H \|x\|^2 \mu(dx) < \infty$ where μ is the distribution of $\Phi([0,1])$. Let $A(\cdot) \in \mathscr{L}_2$ such that*

(i) $a \equiv \sup_\lambda \|A(\lambda)\| < \infty$.

(ii) *$H_\lambda^A = H$ for all λ where H_λ^A denote the space spanned by $A(\lambda)$. Let B be a bounded linear operator with $H_B = H$ where H_B is the space spanned by the operator B. Further suppose that*

$$\int_\Lambda \|A(\lambda)\, x\|^2\, d\lambda - \|Bx\|^2$$

is either strictly greater than zero or strictly less than zero for all $x \in H-\{0\}$. Then

$$\int_\Lambda A(\lambda)\, \Phi(d\lambda) \quad \textit{and} \quad B\Phi([0,1])$$

are identically distributed if and only if Φ is a Wiener Process and A and B satisfy the relation

$$\int_\Lambda A(\lambda)\, SA'(\lambda)\, d\lambda = BSB'$$

where S is the S-operator associated with Φ.

Proof. This follows from the previous theorem by choosing

$$B(\lambda) = B \qquad \text{for all } \lambda \in [0,1].$$

Theorem 6.3. *Let X be a random element with mean 0 and finite S-operator S i.e. $\int_H \|x\|^2 \mu(dx) < \infty$ where μ is the distribution of X. Let A and B be bounded linear operators with $H_A = H_B = H$ where H_A and H_B denote the subspace spanned by A and B respectively. Further suppose that*

$$\|Ax\|^2 - \|Bx\|^2$$

is either strictly greater than zero or strictly less than zero for all $x \in H - \{0\}$. *Then* AX *and* BX *are identically distributed if and only if* X *has a normal distribution i.e.*

$$\hat{\mu}(y) = e^{-\frac{1}{2}(Sy, y)}$$

for all $y \in H$ *and* A *and* B *satisfy the relation*

$$ASA' = BSB'.$$

Proof. This follows from Theorem 6.1 by taking

$$A(\lambda) \equiv A \quad \text{and} \quad B(\lambda) \equiv B.$$

Remarks. The theorems we have obtained do not seem to hold good when $A(\lambda)x \equiv a(\lambda)x$ and $B(\lambda)x \equiv b(\lambda)x$ where $a(\lambda)$ and $b(\lambda)$ are real or complex-valued functions on Λ. Further these theorems are not the natural generalizations of characterization theorems for Wiener processes in the real line.

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Dr. B.L.S. Prakasa Rao
Department of Mathematics
Indian Institute of Technology
Kanpur, U.P./India

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