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Some Convergence Results for Weighted Sums of Independent Random Variables

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1. Introduction

Let X_k for $k=1,2,\ldots$ or $k=0,\pm 1,\ldots$ be independent random variables and let $a_{N,k}$ for $N=1,2,\ldots$ and k as above be real numbers. Define

$$S_N=\sum_k a_{N,k} X_k \tag{1.1}$$

and, when the means of the X_k's exist and are finite, define

$$S'_N=\sum_k a_{N,k}(X_k-EX_k). \tag{1.2}$$

The rates of convergence of $P\{|S_N|\geqq\varepsilon\}$ and $P\{|S'_N|\geqq\varepsilon\}$ to zero under various moment or moment-related assumptions have been studied in various papers including [1], [2], and [4]. The purpose of this paper is to extend some of the results given in the three references. The extended results are given in Section 2. Section 3 contains a short discussion of sharpness. Our proofs are in Section 4.

2. Extensions of Previous Results

Let X_k, $a_{N,k}$, S_N, and S'_N be defined as in Section 1. It should be noted that the existence of first moments has not been assumed. Let F, F' and ρ_N be such that

$$F(y)=\sup_k P\{|X_k|\geqq y\}, \tag{2.1}$$

$$F'(y)=\sup_k P\{|X_k-EX_k|\geqq y\}, \tag{2.2}$$

$$\sum_k |a_{N,k}|^t\leqq\rho_N, \tag{2.3}$$

and let F_k and F'_k be the distribution functions of X_k and X_k-EX_k respectively.

Throughout this paper C will denote various positive constants whose exact numerical values do not matter. Using this notation inequalities such as $1+C\leqq C$ are valid.

Where appropriate, summations will be taken only over those values of k for which $a_{N,k}\neq 0$. Integrals will be Lebesgue Stieltjes integrals.

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Theorem 1. a) *If* $0<t<1$ *and* $y^t F(y) \leqq M < \infty$ *for all* $y>0$, *then* $P\{|S_N|>\varepsilon\} = O(\rho_N)$ *for every* $\varepsilon>0$.

b) (*Case* $t=1$.) *If* $y F(y) \leqq M < \infty$ *for all* $y>0$ *and either*

(2.4) $$\sum_k |a_{N,k}| \log |a_{N,k}| \to 0$$

or

(2.5) $$\overline{\lim_{T\to\infty}} \sup_k \Big| \int_{[-T,T]} y\, dF_k(y) \Big| < \infty,$$

then $P\{|S_N|>\varepsilon\} = O(\rho_N)$ *for every* $\varepsilon>0$.

c) *If* $1<t<2$ *and* $y^t F'(y) \leqq M < \infty$ *for all* $y>0$, *then* $P\{|S'_N|>\varepsilon\} = O(\rho_N)$ *for every* $\varepsilon>0$.

d) (*Case* $t=2$.) *If* $y^2 F'(y) \leqq M < \infty$ *for all* $y>0$ *and there exists a constant* $\lambda>0$ *such that*

(2.6) $$\sum_k a_{N,k}^2 \log |a_{N,k}|^{-1} = O(\rho_N^\lambda)$$

then $P\{|S'_N|>\varepsilon\} = O(\rho_N)$ *for every* $\varepsilon>0$.

Theorem 2. *Assume* $\rho_N \to 0$ *as* $N\to\infty$.

a) *If* $0<t<1$ *and* $y^t F(y) \to 0$ *as* $y\to\infty$, *then* $P\{|S_N|>\varepsilon\} = o(\rho_N)$ *for every* $\varepsilon>0$.

b) (*Case* $t=1$.) *If* $y F(y) \to 0$ *as* $y\to\infty$ *and either* (2.4) *or* (2.5) *holds, then* $P\{|S_N|>\varepsilon\} = o(\rho_N)$ *for every* $\varepsilon>0$.

c) *If* $1<t<2$ *and* $y^t F'(y) \to 0$ *as* $y\to\infty$, *then* $P\{|S'_N|>\varepsilon\} = o(\rho_N)$ *for every* $\varepsilon>0$.

d) (*Case* $t=2$.) *If* $y^2 F'(y) \to 0$ *as* $y\to\infty$ *and there exists a* $\lambda>0$ *such that* (2.6) *holds, then* $P\{|S'_N|>\varepsilon\} = o(\rho_N)$ *for every* $\varepsilon>0$.

Though the assumption is not listed explicitly in his theorem statements, Rohatgi assumes throughout [4] (in his (6)) that $\sum_k |a_{N,k}|^{t'} \leqq C < \infty$ for some $0<t'<t$ when $t \leqq 1$. In our Theorems 1a and 2a (the case $t<1$) we have eliminated this assumption entirely. Our example shows that the assumption cannot be eliminated when $t=1$ but our Theorems 1b and 2b show that in that case it can be replaced by the "weaker" assumption (2.4) or by an additional assumption (2.5) on the distributions. If $\rho_N \to 0$ (an assumption made by Rohatgi when he assumes $\rho>0$) then it is easy to show that our assumption (2.4) is implied by Rohatgi's (6); if $a_{N,k} = [N k (\log k)^3]^{-1}$ for $N, k = 1, 2, \ldots$ then $\sum_k a_{N,k}^t = \infty$ if $t<1$ and is finite if $t=1$, but $\sum_k a_{N,k} \log a_{N,k}$ is finite for all N and converges to zero as $N\to\infty$.

Theorems 1(c, d) and 2(c, d) are improvements of Theorems 1b and 2b respectively in [2].

The proofs of Theorems 1 and 2 for $t<1$ do not use independence. Independence has been used in the proofs of this type of theorem where Markov's inequality is used with other than first moments (as when dealing with (4.7) and (4.11)), and when a sort of double truncation occurs (as results in (4.9)). The proofs given here of Theorems 1 and 2 for $t<1$ (obviously) avoid both of these uses of independence. Is independence necessary when $t=1$?

Other questions of possible interest are:

(2.7) Can (2.4), (2.5), or (2.6) be weakened?

(2.7) What sort of minimal assumption(s) do we need on $\{\gamma_N\}$ when $t \geqq 2$?

The second question is probably more interesting and more important than the first.

The technical report [3] on which this paper is based also contains some "series results" extending and generalizing Rohatgi's Theorems 3 and 4, Theorems 3 and 4 of [1], and Theorems 3 and 4 of [2]. Theorem 5 of [2] can also be extended to cover the case $t<1$.

3. Sharpness of Results

We have not investigated the sharpness of these theorems to the same extent that the sharpness of Theorem 4 of [2] was investigated. Perhaps the main reason for including any results of this type is that Theorems 1 and 2 are "discontinuous" when $t=1$ and $t=2$. It seems desirable to show that (2.4), (2.5), and (2.6) cannot just be omitted.

Theorem 3 shows that the condition "$y^t F(y) \leqq M < \infty$ for all $y>0$" cannot be weakened in Theorem 1 (or in Theorem 1c of [2] which covers the case $t>2$), and that the condition "$y^t F(y) \to 0$ as $y \to \infty$" cannot be weakened in Theorem 2 (or in Theorem 2c of [2] which covers the case $t>2$).

In both Theorem 1 and Theorem 2 the cases $t=1$ and $t=2$ required special treatment. Our example shows that when $t=1$ some additional hypothesis (over those when $t<1$) must be added, and Theorem 4 shows that some additional hypothesis (over those when $1<t<2$) is needed when $t=2$.

Theorem 3. *If* $t>0$, $\rho>0$, $\lim_{y\to\infty} F(y)=0$, *and*

$$\limsup_{y\to\infty} y^t F(y)=\infty \qquad \left(\limsup_{y\to\infty} y^t F(y)>0\right),$$

then there exist a symmetric random variable X and a sequence $\{a_N\}$ of real numbers such that $P\{|X|\geqq x\}\leqq F(x)$ *for all* $x \geqq 0$, $|a_N|^t \leqq C N^{-\rho}$, *and*

$$\limsup_{N\to\infty} N^\rho P\{|a_N X|\geqq 1\}=\infty \qquad \left(\limsup_{N\to\infty} N^\rho P\{|a_N X|\geqq 1\}\neq 0\right).$$

Example. Let $t=1$, let

$$F(y)=\begin{cases} 1 & y\leqq 1 \\ \min\{1, 1/y \log y\} & y>1, \end{cases}$$

and let $X_1, X_2, \ldots$ be independent and identically distributed random variables such that $P\{X_k \geqq y\}=F(y)$. Note that $EX_k=\infty$. If $A_N=\frac{1}{N}\sum_{k=1}^{N} X_k$ then $A_N\to\infty$ with probability one so on some set B of probability at least one half $A_N\to\infty$ uniformly. Let d_N be a uniform lower bound for A_N on B chosen so that $\lim_{N\to\infty} d_N=\infty$. Define

$$a_{N,k}=\begin{cases} \dfrac{1}{N\sqrt{d_N}} & k=1,\ldots,N \quad \text{and} \quad N=1,2,\ldots \\ 0 & \text{otherwise}. \end{cases}$$

Then $\rho_N = 1/\sqrt{d_N} \to 0$, $y^t F(y) \to 0$, but

$$\begin{aligned} P\{|S_N|>1\} &\geqq P[B \cap \{|S_N|>1\}] \\ &\geqq P[B \cap \{\sqrt{d_N}>1\}] \end{aligned}$$

and the last quantity is at least $\frac{1}{2}$ for large enough N.

Theorem 4. *Let $X_1, X_2, \ldots$ be independent, identically distributed, symmetric random variables such that $y^2 P\{|X_1| \geqq y\} \to 0$ as $y \to \infty$ and $EX_1^2 = \infty$. Then there exist a sequence of positive constants $\{a_N\}$ such that $a_N \to \infty$ as $N \to \infty$ and such that*

$$P\left\{\left|\sum_{k=1}^{N} X_k / a_N \sqrt{N}\right| \geqq 1\right\} \to 1 \quad \text{as } N \to \infty.$$

4. Proofs

Proofs of Theorems 1 *and* 2. In Theorem 2 it is assumed that $\rho_N \to 0$. This is not assumed in Theorem 1, however we assume it in our proof of Theorem 1. The case $\rho_N \not\to 0$ can be taken care of as it is in the proof of Theorem 1 in [2].

When $0<t<1$ we note that

$$P\{|S_N|>\varepsilon\} \leqq \sum_k P\{|a_{N,k} X_k|>1\} \tag{4.1}$$

$$+P\{|\sum_k a_{N,k} Y_{N,k}|>\varepsilon\} \tag{4.2}$$

where

$$Y_{N,k} = \begin{cases} X_k & \text{if } |a_{N,k} X_k| \leqq 1 \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

The expression (4.1) is easily shown to be of the right order of magnitude; the method has appeared several times in the literature (see, for example, the handling of expression (2.15) in [2]) and the details will be omitted.

If $0<t<1$ then

$$P\{|\sum_k a_{N,k} Y_{N,k}|>\varepsilon\} \leqq \frac{1}{\varepsilon} \sum_k |a_{N,k}|\, E|Y_{N,k}|.$$

For Theorem 1 we bound $E|Y_{N,k}|$ as follows:

$$\begin{aligned} E|Y_{N,k}| &= \int_{[0, |a_{N,k}|^{-1}]} y\,|dP\{|X_k| \geqq y\}| \\ &\leqq \int_0^{|a_{N,k}|^{-1}} P\{|X_k| \geqq y\}\, dy \\ &\leqq \int_0^{|a_{N,k}|^{-1}} F(y)\, dy \\ &\leqq 1 + \int_1^{|a_{N,k}|^{-1}} M y^{-t}\, dy \\ &\leqq C[1+|a_{N,k}|^{t-1}]. \end{aligned}$$

For large enough N this is bounded by $C|a_{N,k}|^{t-1}$. Thus in this case (4.2) is bounded by $\varepsilon^{-1}\sum_k |a_{N,k}|\,[C|a_{N,k}|^{t-1}]$ which is $O(\rho_N)$. Under the hypotheses of Theorem 2 we note that if $\sup_{y\geqq a} y^t F(y)=M_a$ then $M_a\to 0$ as $a\to\infty$ and thus

$$T^{t-1}\int_0^T F(y)\,dy \leqq T^{t-1}\left[a+\int_a^T M_a\,y^{-t}\,dy\right]$$
$$\leqq a\,T^{t-1}+\frac{M_a}{1-t}.$$

We can make this small by first choosing a large enough that $M_a/(1-t)$ is small and then choosing T large enough that $a\,T^{t-1}$ is small. Thus

$$T^{t-1}\int_0^T F(y)\,dy\to 0 \quad \text{as } T\to\infty. \tag{4.4}$$

We now note that

$$\frac{1}{\varepsilon}\sum_k |a_{N,k}|\,E\,|Y_{N,k}| \leqq \frac{1}{\varepsilon}\,[\sum_k |a_{N,k}|^t]\,\sup_j |a_{N,j}|^{1-t}\int_0^{|a_{N,j}|^{-1}} F(y)\,dy.$$

By assumption $\rho_N\to 0$ so $\sup_j |a_{N,j}|\to 0$, and from (4.4) it then follows that the expression above is $o(\rho_N)$.

So far we have proved Theorems 1 and 2 when $0<t<1$. A centering effect first occurs when $t=1$ and the proof is different from the proof for $0<t<1$. We note that

$$P\{|S_N|>\varepsilon\} \leqq \sum_k P\{|a_{N,k}X_k|>1\} \tag{4.5}$$
$$+P\{|\sum_k a_{N,k}\,E Y_{N,k}|>\varepsilon/2\} \tag{4.6}$$
$$+P\{|\sum_k a_{N,k}(Y_{N,k}-E Y_{N,k})|>\varepsilon/2\}. \tag{4.7}$$

As in the case $0<t<1$ we omit the details relating to expression (4.5).

The probability (4.6) is either 0 or 1. We will show that $\sum_k a_{N,k}\,E Y_{N,k}\to 0$ so that for sufficiently large N the probability is 0. Now

$$|\sum_k a_{N,k}\,E Y_{N,k}| \leqq \sum_k |a_{N,k}|\,\Big|\int_{[-|a_{N,k}|^{-1},\,|a_{N,k}|^{-1}]} y\,dF_k(y)\Big|.$$

Since $\rho_N\to 0$ so does $\sup_k |a_{N,k}|$. It follows from (2.5) that for large enough N the expression $\Big|\int_{[-|a_{N,k}|^{-1},\,|a_{N,k}|^{-1}]} y\,dF_k(y)\Big|$ is uniformly bounded in N and k. Thus in both Theorems 1 and 2

$$|\sum_k a_{N,k}\,E Y_{N,k}| \leqq C\rho_N\to 0$$

so (4.6) is 0 for large enough N if (2.5) is assumed. If (2.4) is assumed instead, then

$$\left|\sum_k a_{N,k} EY_{N,k}\right| \leqq \sum_k |a_{N,k}| \left[1 + \int_1^{|a_{N,k}|^{-1}} M y^{-1} dy\right]$$

as in the proof of these theorems when $0<t<1$. This is bounded by

$$C\sum_k |a_{N,k}| [1-\log |a_{N,k}|] = C\rho_N - C\sum_k |a_{N,k}| \log |a_{N,k}|.$$

Both terms go to 0 as $N\to\infty$ so (4.6) is again 0 for large enough N.

To bound (4.7) we use the second moment in Markov's inequality (the first moment was used when $0<t<1$) and obtain

$$\begin{aligned}
P\{|\textstyle\sum a_{N,k}(Y_{N,k}-EY_{N,k})| > \varepsilon/2\}
&\leqq C\sum_k a_{N,k}^2 E(Y_{N,k}-EY_{N,k})^2 \\
&\leqq C\sum_k a_{N,k}^2 EY_{N,k}^2 \\
&= C\sum_k a_{N,k}^2 \int_{[0,|a_{N,k}|^{-1}]} y^2 \, |dP\{|X_k| \geqq y\}| \\
&\leqq C\sum_k a_{N,k}^2 \int_0^{|a_{N,k}|^{-1}} 2y P\{|X_k|\geqq y\}\, dy \\
&\leqq C\sum_k a_{N,k}^2 \left[1 + \int_1^{|a_{N,k}|^{-1}} 2y(My^{-2})\, dy\right] \\
&\leqq C\sum_k a_{N,k}^2 [1+\log |a_{N,k}|^{-1}] \\
&\leqq C\rho_N \left[\sup_k |a_{N,k}| + \sup_k |a_{N,k}| \log |a_{N,k}|^{-1}\right].
\end{aligned}$$

Since $\sup_k |a_{N,k}| \to 0$ both the expressions in brackets are $o(1)$ and thus (2.18) is $o(\rho_N)$.

When $1<t\leqq 2$ we note that, as in [2],

(4.8) $\quad P\{|S'_N| > 3\varepsilon\} \leqq \sum_k P\{|a_{N,k}(X_k - EX_k)| > \varepsilon\}$

(4.9) $\quad + \sum_{j\neq k} P\{|a_{N,j}(X_j - EX_j)| > \delta_N\}\, P\{|a_{N,k}(X_k - EX_k)| > \delta_N\}$

(4.10) $\quad + P\{|\sum_k a_{N,k} EZ_{N,k}| > \varepsilon\}$

(4.11) $\quad + P\{|\sum_k a_{N,k}(Z_{N,k} - EZ_{N,k})| > \varepsilon\}$

where

$$Z_{N,k} = \begin{cases} X_k - EX_k & \text{if } |a_{N,k}(X_k - EX_k)| \leqq \delta_N \\ 0 & \text{otherwise.} \end{cases}$$

Expression (4.8) is handled as in [2] and as when $0<t<1$.

Taking $\delta_N = \rho_N^{1/2t}$ expression (4.9) is bounded by

$$\left\{\sum_k \left(\frac{|a_{N,k}|}{\delta_N}\right)^t \left[\left(\frac{\delta_N}{|a_{N,k}|}\right)^t P\left(|X_k - EX_k| > \frac{\delta_N}{|a_{N,k}|}\right)\right]\right\}^2$$

$$\leqq \rho_N \left[\sup_k \left(\frac{\delta_N}{|a_{N,k}|}\right)^t P\left\{|X_k - EX_k| > \frac{\delta_N}{|a_{N,k}|}\right\}\right]^2. \tag{4.12}$$

Now $(\delta_N/|a_{N,k}|)^t \geqq \delta_N^t/\rho_N = \rho_N^{-\frac{1}{2}} \to \infty$ as $N \to \infty$. Thus the part of (4.12) in square brackets is $O(1)$ under the assumptions of Theorem 1 and is $o(1)$ under the assumptions of Theorem 2.

To deal with (4.10) we note that for both Theorems 1 and 2

$$\begin{aligned}
|\sum_k a_{N,k} EZ_{N,k}| &\leqq \sum_k |a_{N,k}| \left| \int_{|a_{N,k}(X_k - EX_k)| \leqq \delta_N} (X_k - EX_k)\, dP \right| \\
&= \sum_k |a_{N,k}| \left| \int_{\{|y| > \delta_N/|a_{N,k}|\}} y\, dF_k'(y) \right| \\
&\leqq \sum_k |a_{N,k}| \int_{\delta_N/|a_{N,k}|}^{\infty} y\, |dF'(y)|.
\end{aligned}$$

For large enough N this is bounded by

$$\begin{aligned}
\sum_k |a_{N,k}| \int_{\delta_N/|a_{N,k}|}^{\infty} y\, |d(M y^{-t})| &\leqq C \sum_k |a_{N,k}| (\delta_N/|a_{N,k}|)^{-t+1} \\
&= C\, \delta_N^{-t+1} \rho_N = C \rho_N^{(t+1)/2t}.
\end{aligned}$$

Since $\rho_N \to 0$ and $(t+1)/2t > 0$ the last expression above converges to zero. Thus (4.10) is 0 for large enough N.

We bound (4.11) for $1 < t < 2$ as follows:

$$\begin{aligned}
&P\{|\sum_k a_{N,k}(Z_{N,k} - EZ_{N,k})| > \varepsilon\} \\
&\quad \leqq \frac{1}{\varepsilon^2} \sum_k a_{N,k}^2 E(Z_{N,k} - EZ_{N,k})^2 \\
&\quad \leqq C \sum_k a_{N,k}^2 EZ_{N,k}^2 \\
&\quad = C \sum_k a_{N,k}^2 \int_{[0,\, \delta_N/|a_{N,k}|]} y^2\, |dP\{|X_k - EX_k| \geqq y\}| \\
&\quad \leqq C \sum_k a_{N,k}^2 \left[1 + \int_1^{\delta_N/|a_{N,k}|} y^{-t+1}\, dy\right] \\
&\quad \leqq C \sum_k a_{N,k}^2 \left[1 + \left(\frac{\delta_N}{|a_{N,k}|}\right)^{2-t}\right] \\
&\quad \leqq C \rho_N \left[\sup_k |a_{N,k}|^{2-t} + \delta_N^{2-t}\right] \\
&\quad = o(\rho_N).
\end{aligned}$$

If $t=2$ we use the method used to deal with expression (2.18) of [2]. The integer v is even and will be chosen later.

$$P\{|\sum_k a_{N,k}(Z_{N,k}-EZ_{N,k})|>\varepsilon\} \leqq \frac{1}{\varepsilon^v} E[\sum_k a_{N,k}(Z_{N,k}-EZ_{N,k})]^v$$

$$\leqq C \sum_1 \sum_2 \prod_{k=1}^{a} |a_{N,\beta_k}|^{m_k} E|Z_{N,k}-EZ_{N,k}|^{m_k}$$

where the first sum is taken over all positive integers $a, m_1, \ldots, m_a$ such that $2 \leqq m_k$ for $k=1, \ldots, a$ and $m_1+\cdots+m_a=v$; and where the second sum is taken over $\beta_1, \ldots, \beta_a$. We bound the above expression by

$$C \sum_1 \sum_2 \delta_N^{v-2a} \prod_{k=1}^{a} |a_{N,\beta_k}|^2 E|Z_{N,k}-EZ_{N,k}|^2 \leqq C \sum_1 \delta_N^{v-2a} [\sum_k a_{N,k}^2 EZ_{N,k}^2]^a.$$

Now

$$\sum_k a_{N,k}^2 EZ_{N,k}^2 = \sum_k a_{N,k}^2 \int_{[0,\delta_N/|a_{N,k}|]} y^2 |dP\{|X_k-EX_k| \geqq y\}|$$

$$\leqq C \sum_k a_{N,k}^2 \left[1+\int_1^{\delta_N/|a_{N,k}|} y^{-1}\, dy\right]$$

$$= C \sum_k a_{N,k}^2 \left[1+\log \frac{\delta_N}{|a_{N,k}|}\right].$$

For N large enough this is bounded by

$$C \sum_k a_{N,k}^2 \log |a_{N,k}|^{-1}.$$

Thus

$$P\{|\sum_k a_{N,k}(Z_{N,k}-EZ_{N,k})|>\varepsilon\} \leqq C \sum_1 \delta_N^{v-2a} [\sum_k a_{N,k}^2 \log |a_{N,k}|^{-1}]^a$$

$$\leqq C \sum_1 \delta_N^{v-2a} \rho_N^{\lambda a}$$

$$\leqq C \rho_N^{\frac{v-2a(1-\lambda t)}{2t}}.$$

Choose $v > \max\{2t, 2/\lambda\}$. If $\lambda t \geqq 1$ then the above is bounded by

$$C \rho_N^{v/2t}$$

and if $\lambda t \leqq 1$ it is bounded by

$$C \rho_N^{v[1-(1-\lambda t)]/2t} = C \rho_N^{v\lambda/2}.$$

In both cases the result is $o(\rho_N)$.

Proofs of Theorems 3 *and* 4 may be found in [3].

References

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