A Form of Regular Variation and Its Application to the Domain of Attraction of the Double Exponential Distribution*

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Introduction

In 1930 Karamata introduced the concept of regular variation of a positive function at infinity. He found a striking characterisation of the class of regularly varying functions. Several related forms of regular behaviour at infinity can be defined. At first a new type of regular behaviour is studied in Section 1.

The results of this section are applied to a problem in extreme value theory: we consider a sequence of independent random variables

$$X_1, X_2, X_3, \dots$$

with the same distribution function $F(x)$ and define

$$Y_n = \max(X_1, X_2, \dots, X_n).$$

Then

$$P\{Y_n \leq x\} = F^n(x).$$

A distribution function F is said to belong to the domain of attraction of a non-degenerate distribution function G (notation $F \in D(G)$) when there exist constants $a_n > 0$ and b_n such that

$$F^n(a_n x + b_n) \rightarrow G(x)$$

weakly. Gnedenko [2] proved in 1943 that only three types of distribution functions have non-empty domains of attraction. In addition he gave characterizations of their domains of attraction, but he remarked that the characterization of the domain of attraction of the third type

$$A(x) = \exp(-e^{-x})$$

cannot be regarded as final and simple enough for applications. In 1949 Meijer [6] gave another characterization in terms of the inverse function of F . In this paper we give a comparatively simple characterization of $D(A)$ involving only the distribution function itself. Furthermore we show that this criterion can be used also to characterize the domains of attraction of the two other limit types.

1. A Kind of Regular Variation

First we give the main results of Karamata's papers about regular variation ([5] and [6]).

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Definition 1. A positive function U , defined on $(0, \infty)$ is *regularly varying* at infinity when

$$\lim_{t \rightarrow \infty} \frac{U(tx)}{U(t)} = x^\rho \quad (1)$$

for all $x > 0$. The real number ρ is called the *exponent of regularity*. When $\rho = 0$ U is called *slowly varying* at infinity.

Theorem 1. For a positive function U defined on $(0, \infty)$ and summable on finite intervals the following assertions are equivalent.

a) U is regularly varying with exponent $\rho > -1$,

$$b) \quad \lim_{x \rightarrow \infty} \frac{x \cdot U(x)}{\int_0^x U(t) dt} = \rho + 1 > 0. \quad (2)$$

c) There exist real functions $c(x)$ and $a(x)$ with

$$\lim_{x \rightarrow \infty} c(x) = c \quad (0 < c < \infty) \quad (3)$$

$$\lim_{x \rightarrow \infty} a(x) = \rho > -1$$

such that

$$U(x) = c(x) \cdot \exp \left\{ \int_1^x \frac{a(t)}{t} dt \right\}. \quad (4)$$

Remark 1. For $\rho < -1$ the theorem holds with (2) replaced by

$$\lim_{x \rightarrow \infty} \frac{x U(x)}{\int_x^\infty U(t) dt} = -\rho - 1 > 0. \quad (5)$$

Remark 2. It can be proved that the summability of U on finite subintervals of some terminal interval (a, ∞) is implied by the regular variation and the measurability of U (see e.g. [1]). Hence for measurable U a slightly different form of Theorem 1 holds.

Corollary 1. If U is regularly varying with exponent $\rho > -1$

$$U_1(x) = \int_0^x U(t) dt$$

is regularly varying with exponent $\rho + 1$. If U is regularly varying with exponent $\rho < -1$

$$U_2(x) = \int_x^\infty U(t) dt$$

is regularly varying with exponent $\rho + 1$.

Definition 1 can be extended to $\rho = \pm \infty$. We define for $x > 0$

$$x^\infty = \begin{cases} 0 & \text{for } x < 1 \\ 1 & \text{for } x = 1 \\ \infty & \text{for } x > 1 \end{cases}$$

and

$$x^{-\infty} = \begin{cases} \infty & \text{for } x < 1 \\ 1 & \text{for } x = 1 \\ 0 & \text{for } x > 1. \end{cases}$$

Definition 2. A positive function U defined on $(0, \infty)$ is *rapidly varying* at infinity when

$$\lim_{t \rightarrow \infty} \frac{U(tx)}{U(t)} = x^\rho$$

for all $x > 0$ with $\rho = \pm \infty$.

For rapidly varying functions we have a much weaker version of Theorem 1.

Theorem 2. A non-decreasing positive function U defined on $(0, \infty)$ is rapidly varying with $\rho = \infty$ iff

$$\lim_{x \rightarrow \infty} \frac{x U(x)}{\int_0^x U(t) dt} = \infty. \quad (6)$$

Proof. a) Suppose that U is rapidly varying with $\rho = +\infty$. By Lebesgue's theorem on dominated convergence

$$\lim_{x \rightarrow \infty} \frac{\int_0^x U(t) dt}{x U(x)} = \lim_{x \rightarrow \infty} \int_0^1 \frac{U(xt)}{U(x)} dt = \int_0^1 \lim_{x \rightarrow \infty} \frac{U(xt)}{U(x)} dt = 0.$$

b) On the other hand if there exists a value of t ($0 < t < 1$) and a sequence $x_n \rightarrow \infty$ such that

$$\lim_{n \rightarrow \infty} \frac{U(x_n t)}{U(x_n)} = c > 0, \quad (7)$$

then in view of the monotonicity of the lefthand part of (7) as a function of t

$$\liminf_{n \rightarrow \infty} \int_0^1 \frac{U(x_n s)}{U(x_n)} ds > 0.$$

This contradicts (6).

Remark 3. For $\rho = -\infty$ Theorem 2 holds with (6) replaced by

$$\lim_{x \rightarrow \infty} \frac{x^{-1} U(x)}{\int_x^\infty \frac{dt}{t^2}} = \infty.$$

A related form of Karamata's theorem (Theorem 1) is given in the next theorem.

Theorem 3. For a real-valued function V defined on $(0, \infty)$ and summable on finite intervals the following assertions are equivalent.

a) For every $a > 0$

$$\lim_{x \rightarrow \infty} \{V(ax) - V(x)\} = \rho \log a \quad (8)$$

where ρ is a real constant.

$$\text{b) } \lim_{x \rightarrow \infty} \left\{ V(x) - \frac{1}{x} \int_0^x V(t) dt \right\} = \rho. \quad (9)$$

c) There exist real functions $c(x)$ and $a(x)$ with

$$\begin{aligned} \lim_{x \rightarrow \infty} c(x) &= c \quad (-\infty < c < \infty) \\ \lim_{x \rightarrow \infty} a(x) &= \rho \end{aligned} \quad (10)$$

such that

$$V(x) = c(x) + \int_0^x \frac{a(t)}{t} dt. \quad (11)$$

Proof. Relation (8) holds iff

$$U(x) = \exp \{V(x)\}$$

is regularly varying at infinity with exponent ρ . Hence the equivalence of a) and c) is contained in Theorem 1. The implication c) $\Rightarrow$ b) is a matter of standard calculation. For the proof of the implication b) $\Rightarrow$ c) we define

$$g(x) = V(x) - \frac{1}{x} \int_0^x V(t) dt. \quad (12)$$

Then

$$\begin{aligned} \int_1^x \frac{g(t)}{t} dt &= \int_1^x \frac{V(t)}{t} dt + \frac{1}{x} \int_0^x V(t) dt - \int_0^1 V(t) dt - \int_1^x \frac{V(t)}{t} dt \\ &= - \int_0^1 V(t) dt + V(x) - g(x), \end{aligned}$$

hence

$$V(x) = \int_0^1 V(t) dt + g(x) + \int_1^x \frac{g(t)}{t} dt. \quad (13)$$

Remark. If (8) holds with $\rho > 0$, V is slowly varying at infinity.

The next theorem can be seen as an attempt to characterize a subclass of the class of slowly varying functions with functions which behave even more regularly.

Theorem 4. For a real-valued strictly increasing function V which is defined on $(0, \infty)$ the following assertions are equivalent.

a) For every positive a and $b \neq 1$

$$\lim_{x \rightarrow \infty} \frac{V(ax) - V(x)}{V(bx) - V(x)} = \frac{\log a}{\log b}. \quad (14)$$

b) *The function*

$$V(x) - \frac{1}{x} \int_0^x V(t) dt \quad (15)$$

is slowly varying at infinity.

c) *There exists a slowly varying function g such that*

$$V(x) = c + g(x) + \int_1^x \frac{g(t)}{t} dt. \quad (16)$$

d) *For every $a > 0$*

$$\lim_{x \rightarrow \infty} \frac{V(ax) - V(x)}{V(x) - \frac{1}{x} \int_0^x V(t) dt} = \log a. \quad (17)$$

Proof. a) $\Rightarrow$ b). Writing ($b > 0, 0 < a < 1$)

$$\frac{V(bx) - V(ax)}{V(x) - V(ax)} = \frac{V(x) - V(ax)}{V(x) - V(ax)} - \frac{V(x) - V(bx)}{V(x) - V(ax)}$$

and using (14) we see that the function

$$h(x) = V(x) - V(ax)$$

is slowly varying for every $0 < a < 1$.

By Theorem 1 this implies

$$\lim_{x \rightarrow \infty} \frac{\frac{1}{x} \int_0^x V(t) dt - \frac{1}{ax} \int_0^{ax} V(t) dt}{V(x) - V(ax)} = 1,$$

hence

$$\lim_{x \rightarrow \infty} \left\{ \frac{V(x) - \frac{1}{x} \int_0^x V(t) dt}{V(x) - V(ax)} - \frac{V(ax) - \frac{1}{ax} \int_0^{ax} V(t) dt}{V(x) - V(ax)} \right\} = 0. \quad (18)$$

By Fatou's lemma we have

$$\liminf_{x \rightarrow \infty} \frac{V(x) - \frac{1}{x} \int_0^x V(t) dt}{V(x) - V(ax)} \geq \int_0^1 \liminf_{x \rightarrow \infty} \frac{V(x) - V(tx)}{V(x) - V(ax)} dt = \int_0^1 \frac{\log t}{\log a} dt > 0. \quad (19)$$

Combining (18) and (19) we obtain

$$\lim_{x \rightarrow \infty} \frac{\left\{ V(x) - \frac{1}{x} \int_0^x V(t) dt \right\} - \left\{ V(ax) - \frac{1}{ax} \int_0^{ax} V(t) dt \right\}}{V(x) - \frac{1}{x} \int_0^x V(t) dt} = 0.$$

b) $\Rightarrow$ c). Defining

$$g(x) = V(x) - \frac{1}{x} \int_0^x V(t) dt$$

we get (see (13))

$$V(x) = c + g(x) + \int_1^x \frac{g(t)}{t} dt.$$

c) $\Rightarrow$ d). For $a > 1$ we have

$$\frac{V(ax) - V(x)}{V(x) - \frac{1}{x} \int_0^x V(t) dt} = \frac{g(ax)}{g(x)} - 1 + \int_1^a \frac{g(tx)}{g(x)} \frac{dt}{t}. \quad (20)$$

Since the relation

$$\lim_{x \rightarrow \infty} \frac{g(tx)}{g(x)} = 1.$$

holds uniformly on $[1, a]$ (see [1]), we obtain (17) by letting $x \rightarrow \infty$ in (20).

d) $\Rightarrow$ a). Trivial.

Remark. The requirement that V is strictly increasing is only used to ensure the finiteness of the expression to the right of the $\lim$ sign in (14) and may be replaced by the requirement: for each $a > 1$ there exists a $M(a)$ such that for $x > M(a)$

$$V(ax) - V(x) > 0.$$

Remark. It is not difficult to show that a positive function V satisfying (14) is slowly varying at infinity. It is not true that (14) implies (8) for $a \in [0, \infty]$.

Corollary 2. *Theorem 4 remains valid if we replace everywhere $\lim_{x \rightarrow \infty}$ by $\lim_{x \downarrow 0}$ (of course now “slowly varying” has to be read as “slowly varying at $x=0$ ”).*

Proof. If $h(x)$ is slowly varying at $x=0$ i.e. if for each $x > 0$

$$\lim_{t \downarrow 0} \frac{h(tx)}{h(t)} = 1,$$

then $x^{-2} h\left(\frac{1}{x}\right)$ is regularly varying with $\rho = -2$ and by Remark 1

$$\lim_{x \downarrow 0} \frac{x h(x)}{\int_0^x h(t) dt} = \lim_{y \rightarrow \infty} \frac{h(y)}{y \int_y^\infty h\left(\frac{1}{t}\right) \frac{dt}{t^2}} = 1.$$

The remainder is easy.

2. Preliminaries

First we list some well-known results on the domain of attraction of the double exponential law used in the sequel (cf. [3]).

Lemma 1. *Let $\{F_n\}$ be a sequence of distribution functions. Suppose that there exist sequences of real numbers $\{a_n\}$ and $\{b_n\}$ with*

$$a_n > 0 \quad \text{for } n=1, 2, 3, \dots,$$

such that

$$\lim_{n \rightarrow \infty} F_n(a_n x + b_n) = G(x) \quad (21)$$

weakly where G is non-degenerate distribution function.

Then

$$\lim_{n \rightarrow \infty} F_n(\alpha_n x + \beta_n) = G^*(x) \quad (22)$$

holds with G^* non-degenerate and real numbers $\alpha_n > 0$ and β_n iff

$$\lim_{n \rightarrow \infty} \frac{\alpha_n}{a_n} = A > 0, \quad \lim_{n \rightarrow \infty} \frac{\beta_n - b_n}{a_n} = B \quad (23)$$

and

$$G^*(x) = G(Ax + B). \quad (24)$$

We say that a distribution function F belongs to the domain of attraction of a non-degenerate distribution function G (notation $F \in D(G)$) if for suitably chosen constants $a_n > 0$ and b_n

$$\lim_{n \rightarrow \infty} F^n(a_n x + b_n) = G(x) \quad (25)$$

for all continuity points x of G .

Theorem 5. *A distribution function F can belong only to the domain of attraction of one of the following types of distribution functions:*

$$\phi_\alpha(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ \exp(-x^{-\alpha}) & \text{for } x > 0. \end{cases} \quad (26)$$

$$\psi_\alpha(x) = \begin{cases} \exp(-(-x)^\alpha) & \text{for } x \leq 0 \\ 1 & \text{for } x > 0. \end{cases} \quad (27)$$

$$A(x) = \exp(-e^{-x}). \quad (28)$$

In (26) and (27) α is a positive constant.

In the sequel we use the notation

$$x_0 = x_0(F) = \sup \{x | F(x) < 1\} \leq \infty. \quad (29)$$

Theorem 6. a) *The distribution function F belongs to the domain of attraction of $\phi_\alpha(x)$ iff $1 - F(x)$ is regularly varying with exponent $-\alpha$.*

b) *The distribution function F belongs to the domain of attraction of $\psi_\alpha(x)$ iff $x_0 < \infty$ and $1 - F\left(x_0 - \frac{1}{x}\right)$ is regularly varying with exponent $-\alpha$.*

Theorem 7. *The distribution function F belongs to the domain of attraction of $A(x)$ iff*

$$\lim_{n \rightarrow \infty} F^n(a_n x + b_n) = A(x)$$

or equivalently

$$\lim_{n \rightarrow \infty} n \{1 - F(a_n x + b_n)\} = e^{-x} \quad (30)$$

with

$$\begin{aligned} b_n &= \inf \left\{ x \mid 1 - F(x) \geq \frac{1}{n} \right\} \\ a_n &= \inf \left\{ x \mid 1 - F(x) \geq \frac{1}{n e} \right\} - b_n. \end{aligned} \quad (31)$$

Remark. It can be seen (cf. [4]) that the choice (31) for the stabilizing coefficients also holds for distribution functions attracted to the other limit types.

Theorem 8. *The distribution function F belongs to the domain of attraction of $A(x)$ iff it is possible to choose a function A with*

$$z A(z) > 0 \quad \text{when } z \neq 0$$

and

$$\lim_{z \uparrow x_0} A(z) = 0 \quad (32)$$

such that for every x

$$\lim_{z \uparrow x_0} \frac{1 - F(z + z A(z) x)}{1 - F(z)} = e^{-x}. \quad (33)$$

Remark. In Gnedenko's paper the continuity of A is required but in his proof this property is not used.

The next theorem is due to von Mises ([8]). The theorem is extended in Gnedenko's paper.

Theorem 9. *Suppose that the distribution function F is twice differentiable. Define*

$$f(x) = \frac{F'(x)}{1 - F(x)}. \quad (34)$$

If

$$\lim_{x \uparrow x_0} \frac{d}{dx} \left(\frac{1}{f(x)} \right) = 0, \quad (35)$$

then $F \in D(A)$.

3. The Domain of Attraction of A

Lemma 2. *If $F \in D(A)$, there exists a continuous and strictly increasing distribution function G such that*

$$1 - F(x) \sim 1 - G(x) \quad \text{for } x \uparrow x_0. \quad (36)$$

Proof. Suppose first $x_0 = \infty$. By Theorem 8 there is a positive function A with

$$\lim_{z \rightarrow \infty} A(z) = 0.$$

such that for all x

$$\lim_{z \rightarrow \infty} \frac{1 - F(z + z A(z) x)}{1 - F(z)} = e^{-x}. \quad (37)$$

Both sides of (37) are monotone functions of x hence (37) holds uniformly for $0 \leq x \leq 1$. Taking $x(z) = z^{-1}$ we obtain

$$\lim_{z \rightarrow \infty} \frac{1 - F(z + A(z))}{1 - F(z)} = 1. \quad (38)$$

From this we see

$$\lim_{z \rightarrow \infty} \frac{1 - F(z)}{1 - F(z-0)} = 1. \quad (39)$$

Let $\{z_n\}_{n=1}^{\infty}$ be an enumeration of the points of discontinuity of F . We define $F_1(x)$ in the following way: $F_1(z_1) = F(z_1 - 0)$, $F_1(z)$ linear on $[z_1, z_1 + A(z_1)]$ with $F_1(z_1 + A(z_1)) = F(z_1 + A(z_1))$ and $F_1(z) = F(z)$ when $z \notin [z_1, z_1 + A(z_1)]$.

Now $F_2(z)$ equals $F_1(z)$ if $z_1 < z_2 \leq z_1 + A(z_1)$; otherwise $F_2(z)$ is constructed by putting $F_2(z_2) = F_1(z_2 - 0)$ and making F_1 linear on $[z_2, z_2 + A(z_2)]$ or $[z_2, z_1]$ if $z_2 < z_1 \leq z_2 + A(z_2)$. In this way we construct a sequence of distribution functions F_n . As the intervals I_n where the function is changed are disjoint, $H(z) = \lim_{n \rightarrow \infty} F_n(z)$ exists and is a continuous distribution function. If $H(z) \neq F(z)$ for a z then there exists an n such that $H(z) = F_n(z)$ and $H(z) \neq F_k(z)$ for $k < n$, so

$$F(z_n - 0) \leq H(z) \leq F(z_n + A(z_n))$$

and

$$\frac{1 - F(z_n - 0)}{1 - F(z_n + A(z_n))} \geq \frac{1 - H(z)}{1 - F(z)} \geq \frac{1 - F(z_n + A(z_n))}{1 - F(z_n)}.$$

Hence by (38) and (39)

$$\lim_{z \rightarrow \infty} \frac{1 - H(z)}{1 - F(z)} = 1. \quad (40)$$

In an analogous way we proceed to make the function strictly increasing. Let $\{u_n\}$ be an enumeration of the initial points of intervals where H is constant and $\{v_n\}$ the corresponding endpoints. The construction of a sequence H_n is analogous to the construction of the sequence F_n using now the intervals $[u_n, v_n + A(v_n)]$ instead of $[z_n, z_n + A(z_n)]$. The function $G(z) = \lim_{n \rightarrow \infty} H_n(z)$ is a continuous and strictly increasing distribution function and as above we see

$$\lim_{z \rightarrow \infty} \frac{1 - G(z)}{1 - H(z)} = 1. \quad (41)$$

If $x_0 < \infty$ we have

$$\lim_{z \uparrow x_0} \frac{1 - F(z + z A(z) x)}{1 - F(z)} = e^{-x}$$

with

$$z A(z) > 0$$

and

$$\lim_{z \uparrow x_0} A(z) = 0.$$

By taking $x(z) = x_0 - z$ we obtain

$$\lim_{z \uparrow x_0} \frac{1 - F(z + z A(z)(x_0 - z))}{1 - F(z)} = 1. \quad (42)$$

As

$$z A(z)(x_0 - z) > 0$$

and

$$\lim_{z \uparrow x_0} z A(z)(x_0 - z) = 0,$$

relation (42) implies

$$\lim_{z \uparrow x_0} \frac{1 - F(z)}{1 - F(z - 0)} = 1;$$

the remainder of the proof is analogous to the proof for the case $x_0 = \infty$.

Lemma 3. Let $\{F_t\}$ be a family of distribution functions $(-\infty < t < t_0 \leq \infty)$. Suppose that there exist real-valued functions $a(t) > 0$ and $b(t)$ such that

$$\lim_{t \uparrow t_0} F_t(a(t)x + b(t)) = G(x)$$

weakly, where G is a non-degenerate distribution function. Then

$$\lim_{t \uparrow t_0} F_t(\alpha(t)x + \beta(t)) = G^*(x)$$

holds with G^* non-degenerate and real-valued functions $\alpha(t) > 0$ and $\beta(t)$ iff

$$\lim_{t \uparrow t_0} \frac{\alpha(t)}{a(t)} = A > 0, \quad \lim_{t \uparrow t_0} \frac{\beta(t) - b(t)}{a(t)} = B$$

and

$$G^*(x) = G(Ax + B).$$

Proof. Analogous to the proof of Lemma 2 (see Feller [2] p. 246).

Corollary 3. If for a distribution function F

$$\lim_{z \uparrow x_0} \frac{1 - F(z + z A(z)x)}{1 - F(z)} = e^{-x} \quad (43)$$

holds with

$$z A(z) > 0$$

and

$$\lim_{z \uparrow x_0} A(z) = 0,$$

then

$$\lim_{z \uparrow x_0} \frac{1 - F(f(z) + z B(z)x)}{1 - F(z)} = e^{-x-b} \quad (44)$$

holds iff

$$\lim_{z \uparrow x_0} \frac{B(z)}{A(z)} = 1 \quad (45)$$

$$\lim_{z \uparrow x_0} \frac{f(z) - z}{A(z)} = b.$$

Proof. Take in Lemma 3

$$F_t(x) = 1 - \frac{1 - F(x)}{1 - F(t)} \quad \text{and} \quad G(x) = 1 - e^{-x}.$$

The fact that $F_t(-\infty) > 0$ is immaterial.

Lemma 4¹. If $F \in D(A)$ then for all $a > 1$

$$\lim_{y \downarrow 0} \frac{V(a^{-1}y) - V(y)}{V(e^{-1}y) - V(y)} = \log a \quad (46)$$

with for $0 < y < 1$

$$V(y) = \inf \{x | 1 - F(x) \geq y\}.$$

Proof. We use Theorem 7 and make first the remark that if a_n in (30) is replaced by

$$\alpha_n = V\left(\frac{1}{an}\right) - V\left(\frac{1}{n}\right)$$

for an arbitrary $a > 1$ then

$$\lim_{n \rightarrow \infty} n \{1 - F(\alpha_n x + b_n)\} = a^{-x}; \quad (47)$$

this can be seen by a simple adaptation of Gnedenko's proof.

Combination of (30) and (47) gives by Lemma 1

$$\lim_{n \rightarrow \infty} \frac{V\left(\frac{1}{na}\right) - V\left(\frac{1}{n}\right)}{V\left(\frac{1}{ne}\right) - V\left(\frac{1}{n}\right)} = \log a. \quad (48)$$

Using the monotonicity of V we see from (48)

$$\lim_{n \rightarrow \infty} \frac{V\left(\frac{1}{n+1}\right) - V\left(\frac{1}{n}\right)}{V\left(\frac{1}{ne}\right) - V\left(\frac{1}{n}\right)} = 0. \quad (49)$$

From (47) we obtain also

$$\lim_{n \rightarrow \infty} n \{1 - F(\alpha_{n+1} x + b_{n+1})\} = a^{-x}.$$

Using Lemma 1 again we get

$$\lim_{n \rightarrow \infty} \frac{V\left(\frac{1}{(n+1)a}\right) - V\left(\frac{1}{n+1}\right)}{V\left(\frac{1}{na}\right) - V\left(\frac{1}{n}\right)} = 1. \quad (50)$$

¹ Cf. Meizler [7].

Now defining

$$U(x) = V\left(\frac{1}{x}\right)$$

we see (using the entier function)

$$\frac{U(a[y]) - U([y] + 1)}{U(e[y]) - U([y])} \leq \frac{U(ay) - U(y)}{U(e[y]) - U([y])} \leq \frac{U(a([y] + 1)) - U([y])}{U(e[y]) - U([y])}.$$

The righthand member is the same as

$$\frac{U(a([y] + 1)) - U([y] + 1)}{U(e[y]) - U([y])} + \frac{U([y] + 1) - U([y])}{U(e[y]) - U([y])}$$

and this by (48), (49) and (50) tends to $\log a$ as y tends to infinity. The lefthand side tends to the same limit so

$$\lim_{y \rightarrow \infty} \frac{U(ay) - U(y)}{U(e[y]) - U([y])} = \log a \quad (51)$$

and this in combination with (48) gives the assertion of the lemma.

Lemma 5. *If $F \in D(A)$ then*

$$\lim_{z \uparrow x_0} \frac{1 - F(z + z A(z)x)}{1 - F(z)} = e^{-x} \quad (52)$$

for all x with

$$A(z) = \frac{\int_0^{x_0} \{1 - F(t)\} dt}{z \{1 - F(z)\}}. \quad (53)$$

Proof. By Lemma 2 and Corollary 3 we need only consider continuous and strictly increasing distribution functions.

By Lemma 4

$$\lim_{y \downarrow 0} \frac{V(a^{-1}y) - V(y)}{V(e^{-1}y) - V(y)} = \log a \quad (54)$$

for all $a > 1$ with

$$V(y) = (1 - F)^{-1}(y).$$

By Theorem 4 and Corollary 2 (used for $-V$) relation (54) is equivalent with

$$\lim_{y \downarrow 0} \frac{V(a^{-1}y) - V(y)}{\frac{1}{y} \int_0^y V(t) dt - V(y)} = \log a. \quad (55)$$

Taking $a = e$ and using Lemma 1 we see

$$\lim_{n \rightarrow \infty} n \{1 - F(a_n x + b_n)\} = e^{-x}$$

with

$$b_n = V\left(\frac{1}{n}\right)$$

$$a_n = n \int_0^{1/n} V(t) dt - V\left(\frac{1}{n}\right).$$

Now it is not difficult to see (using (49), (51), (55) and Lemma 1) that

$$\lim_{y \downarrow 0} y^{-1} \{1 - F(a(y)x + b(y))\} = e^{-x} \quad (56)$$

with

$$b(y) = V(y)$$

$$a(y) = \frac{1}{y} \int_0^y V(t) dt - V(y). \quad (57)$$

Putting $y = 1 - F(z)$ in (56) and (57) we obtain (52) and (53).

Lemma 6. *If for a distribution function F*

$$\lim_{z \uparrow x_0} \frac{1 - F(z + z A(z)x)}{1 - F(z)} = e^{-x} \quad (58)$$

for all x with $z A(z) > 0$ for $z \neq 0$ and

$$\lim_{z \uparrow x_0} A(z) = 0,$$

then

$$\frac{1 - F_1(z + z A(z)x)}{1 - F_1(z)} = e^{-x} \quad (59)$$

where

$$F_1(x) = 1 - \int_x^{x_0} \{1 - F(t)\} dt. \quad (60)$$

Proof. By the Lemma's 3 and 5

$$A(z) \sim \frac{\int_z^{x_0} \{1 - F(t)\} dt}{z \{1 - F(z)\}} \quad \text{for } z \uparrow x_0. \quad (61)$$

In (58) we substitute $z = y + y A(y)s$ with s an arbitrary real number; then:

$$\lim_{y \uparrow x_0} \frac{1 - F(y + y A(y)s + (y + y A(y)s) A(y + y A(y)s)x)}{1 - F(y + y A(y)s)} = e^{-x}.$$

With (58) this becomes

$$\lim_{y \uparrow x_0} \frac{1 - F(y + y A(y)s + (y + y A(y)s) A(y + y A(y)s)x)}{1 - F(y)} = e^{-(x+s)}. \quad (62)$$

Applying Corollary 3 we obtain from (58) and (62)

$$\lim_{y \uparrow x_0} \frac{A(y + y A(y) s)}{A(y)} = 1 \quad (63)$$

for all s .

Combining (58), (61) and (63) we obtain (59).

Lemma 7. *If for a positive non-decreasing function f defined on $(-\infty, x_0)$ with*

$$\lim_{z \uparrow x_0} f(z) = \infty$$

(where $x_0 \leq \infty$), the relation

$$\lim_{z \uparrow x_0} \frac{f(z + z A(z) x)}{f(z)} = e^x \quad (64)$$

holds for all x and a properly chosen function A with

$$z A(z) > 0 \quad \text{for } z \neq 0$$

and

$$\lim_{z \uparrow x_0} A(z) = 0,$$

then also

$$\lim_{z \uparrow x_0} \frac{f_1(z + z A(z) x)}{f_1(z)} = e^x \quad (65)$$

for all x where

$$f_1(x) = \int_{x_1}^x f(t) dt$$

and

$$x_1 = \begin{cases} x_0 - 1 & \text{when } x_0 < \infty \\ 1 & \text{when } x_0 = \infty. \end{cases}$$

Proof. Analogous to the proof of Lemma 6. First we use Theorem 4 to find

$$A(z) \sim \frac{\int_{x_1}^z f(t) dt}{z f(z)} \quad \text{for } z \uparrow x_0,$$

then by Lemma 3

$$\lim_{y \uparrow x_0} \frac{A(y + y A(y) s)}{A(y)} = 1$$

for all s . As in the proof of Lemma 6 the assertion of this lemma follows.

Now we are able to prove the main theorem.

Theorem 10. *A distribution function F is in the domain of attraction of the double exponential distribution iff*

$$\lim_{x \uparrow x_0} \frac{\{1 - F(x)\} \left\{ \int_x^{x_0} \int_y^{x_0} \{1 - F(t)\} dt dy \right\}}{\left\{ \int_x^{x_0} \{1 - F(t)\} dt \right\}^2} = 1. \quad (66)$$

Proof. a) Suppose $F \in D(\lambda)$. By Lemma 5 this is equivalent to

$$\lim_{z \uparrow x_0} \frac{1 - F(z + z A(z) x)}{1 - F(z)} = e^{-x} \quad (67)$$

with

$$A(z) = \frac{\int_z^{x_0} \{1 - F(t)\} dt}{z \{1 - F(z)\}}. \quad (68)$$

By Lemma 6 we know that

$$F_1(x) = 1 - \int_x^{x_0} \{1 - F(t)\} dt \quad (69)$$

also satisfies (67) and (68). But then by Lemma 5 $F_1(x)$ also satisfies (67) with

$$A_1(z) = \frac{\int_z^{x_0} \int_y^{x_0} \{1 - F(t)\} dt dy}{z \int_z^{x_0} \{1 - F(t)\} dt} \quad (70)$$

and by Lemma 3

$$A(z) \sim A_1(z) \quad \text{for } z \uparrow x_0.$$

This is equivalent with (66).

b) Suppose that (66) holds. By Theorem 8 and Theorem 9 we know that the distribution function F_2 , defined for sufficiently large values of x by

$$F_2(x) = 1 - \int_x^{x_0} \int_y^{x_0} \{1 - F(t)\} dt dy \quad (71)$$

satisfies

$$\lim_{z \uparrow x_0} \frac{1 - F_2(z + z A_2(z) x)}{1 - F_2(z)} = e^{-x} \quad (72)$$

with

$$z A_2(z) > 0 \quad \text{for } z \neq 0$$

$$\lim_{z \uparrow x_0} A_2(z) = 0. \quad (73)$$

As the function

$$f(x) = \frac{1}{1 - F_2(x)}$$

satisfies (64), by Lemma 7 the function

$$f_1(x) = \int_{x_1}^x f(t) dt$$

also satisfies (64). But by (66) this function is asymptotically equivalent to

$$f_2(x) = \int_0^x \{1 - F(t)\} \left\{ \int_t^{x_0} (1 - F(s)) ds \right\}^{-2} dt \sim \left\{ \int_x^{x_0} (1 - F(t)) dt \right\}^{-1} \quad \text{for } x \uparrow x_0.$$

Thus $F_1(x)$ defined by (69) satisfies (72). Hence by (66)

$$\lim_{z \uparrow x_0} \frac{1 - F(z + z A(z) x)}{1 - F(z)} = \lim_{z \uparrow x_0} \left\{ \frac{1 - F_1(z + z A(z) x)}{1 - F_1(z)} \right\}^2 \cdot \lim_{z \uparrow x_0} \frac{1 - F_2(z)}{1 - F_2(z + z A(z) x)} = e^{-x}.$$

This completes the proof.

4. A Unifying Approach

For distribution functions with $x_0 < \infty$ we are now able to combine the results on the domain of attraction of the two limit distributions.

Theorem 11. *Let F be a distribution function with $x_0 < \infty$. The sequence*

$$F^n(a_n x + b_n)$$

tends to a non-degenerate distribution function for a proper choice of the constants $a_n > 0$ and b_n iff

$$\lim_{x \uparrow x_0} \frac{\{1 - F(x)\} \left\{ \int_x^{x_0} \int_y^{x_0} \{1 - F(t)\} dt dy \right\}}{\left\{ \int_x^{x_0} \{1 - F(t)\} dt \right\}^2} = c \quad \text{with } \frac{1}{2} < c \leq 1. \quad (74)$$

F is in the domain of attraction of ψ_α with $\alpha = (1 - c)^{-1} - 2$ if $c < 1$. F is in the domain of attraction of Λ if $c = 1$.

Proof. For $c = 1$ the content of the theorem is part of Theorem 10. Suppose (74) holds with $\frac{1}{2} < c < 1$. If we write $a(x)$ for the function to the right of the lim sign in (74), then for almost all x

$$a(x) = 1 + \frac{d}{dx} \frac{\int_x^{x_0} \int_y^{x_0} \{1 - F(t)\} dt dy}{\int_x^{x_0} \{1 - F(t)\} dt}.$$

Hence

$$\lim_{x \uparrow x_0} \frac{\int_x^{x_0} \int_y^{x_0} \{1 - F(t)\} dt dy}{(x_0 - x) \int_x^{x_0} \{1 - F(t)\} dt} = \lim_{x \uparrow x_0} (x_0 - x)^{-1} \int_x^{x_0} \{1 - a(t)\} dt = 1 - c \quad (75)$$

and by (74)

$$\lim_{x \uparrow x_0} \frac{\int_x^{x_0} \{1 - F(t)\} dt}{(x_0 - x) \{1 - F(x)\}} = c^{-1} (1 - c). \quad (76)$$

From Remark 1 (after a trivial transformation) we see that $1 - F\left(x_0 - \frac{1}{x}\right)$ is regularly varying with exponent $2 - (1 - c)^{-1}$.

Hence by Theorem 6 we have $F \in D(\psi_\alpha)$ with $\alpha = (1 - c)^{-1} - 2$. The converse is a simple application of Theorem 6, Remark 1 and Corollary 1.

For distribution functions with $x_0 = \infty$ there is an additional complication. If for example $F \in D(\phi_{0.5})$ then $a(x)$ is not defined because

$$\int_0^\infty \{1 - F(t)\} dt = \infty.$$

Our final theorem shows that this difficulty is easily overcome.

Theorem 12. *Let F be a distribution function with $x_0 = \infty$. The sequence*

$$F^n(a_n x + b_n)$$

tends to a non-degenerate distribution function for a proper choice of the constants $a_n > 0$ and b_n iff

$$\lim_{x \rightarrow \infty} \frac{\{1 - F(x)\} \left\{ \int_x^\infty \int_y^\infty \{1 - F(t)\} \frac{dt}{t^3} dy \right\}}{x^3 \left\{ \int_x^\infty \{1 - F(t)\} \frac{dt}{t^3} \right\}^2} = c \quad \text{with} \quad 1 \leq c < 2. \quad (77)$$

F is in the domain of attraction of ϕ_α with $\alpha = (c - 1)^{-1} - 1$ if $c > 1$. F is in the domain of attraction of Λ if $c = 1$.

Proof. For $c = 1$ the content of the theorem is a consequence of Theorem 10. Next suppose (77) holds with $1 < c < 2$. As in the proof of Theorem 11 we obtain

$$\lim_{x \rightarrow \infty} \frac{x^2 \int_x^\infty \{1 - F(t)\} \frac{dt}{t^3}}{1 - F(x)} = c^{-1}(c - 1). \quad (78)$$

From Remark 1 we see that $F \in D(\phi_\alpha)$ with $\alpha = (c - 1)^{-1} - 1$. The converse is again a simple application of Remark 1 and Corollary 1.

Corollary 4. *Let $F \in D(\Lambda)$. The function $1 - F(x)$ is rapidly varying at infinity with $\rho = -\infty$ if $x_0 = \infty$. The function $1 - F\left(x_0 - \frac{1}{x}\right)$ is rapidly varying at infinity with $\rho = -\infty$ if $x_0 < \infty$.*

Proof. The relations (76) and (78) which are also true for $c = 1$, are equivalent to the statements in this corollary (see Remark 3 and the transformation in the proof of Corollary 2).

Remark. For distribution functions with $x_0 = \infty$ Corollary 4 is proved in Gnedenko's paper.

A number of related results including other characterisations of $D(\Lambda)$ will be published elsewhere.

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