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On the Structural Information Contained in the Output of $GI/G/\infty$

By

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1. Introduction

It is a familiar fact that the output of the stochastic service system $GI/G/\infty$ may not by itself contain sufficient information to identify the interarrival-time and service-time distributions. For example, the output of $M/G/\infty$ is a Poisson stream ([6]; see also [3]) and so contains no information about the form of the service-time distribution; one can learn nothing from it beyond the intensity of the Poisson output, and this does not depend on the service-time distribution. It has been suggested [3] that complete structural information about the system may be obtained if we have access to the infinite permutation P which converts the order of arrival (with an arbitrary customer labelled 0) into the order of departure (with this same customer labelled 0). The permutation P may itself contain structural information; here, however, we are going to assume that *we are supplied with one complete output-record*

$$(\ldots, t_{-2}, t_{-1}, t_0, t_1, t_2, \ldots) \tag{1}$$

together with the associated permutation P; we shall prove that then (i) *the inter-arrival-time distribution* dA *is completely identifiable*, and (ii) *the service-time distribution* dB *is completely identifiable up to a location parameter*. (Alteration of the location of dB shifts the output rigidly along the time-scale and leaves P unchanged; obviously we could never detect this.)

As was explained in [3], the output of $GI/G/\infty$ can be interpreted as a *randomly delayed renewal process*. Such point-processes often form the input to a queueing system [2, 4]; it is this fact which is responsible for our interest in the present problem.

We are here concerned only with the uniqueness of dA and dB, and not with their estimation from empirical data. That task would call for quite different methods, and will be discussed elsewhere [5].

2. Characteristic functions of non-negative random variables***

If $\Phi(t) = E e^{itu}$, where u is a non-negative random variable and t is real, we shall say that Φ belongs to the class $\mathscr{K}_+$. This sub-class of the whole family $\mathscr{K}$ of characteristic functions has some very special properties; for example, SMITH [7] has remarked that the set of zeros of Φ cannot contain a non-degenerate interval,

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*** The arguments used here have much in common with those of ZYGMUND [8].

and that two members Φ_1 and Φ_2 of $\mathcal{K}_+$ must coincide if they agree on a non-degenerate interval. We here record two similar but much stronger results; we shall make use of them in § 3 *.

Theorem A. *Let $\Phi \in \mathcal{K}_+$; then the set $\{t : \Phi(t) = 0, -\infty < t < \infty\}$ has Lebesgue measure zero.*

Theorem B. *Let Φ_1 and Φ_2 belong to $\mathcal{K}_+$; then $\Phi_1 \equiv \Phi_2$ if $\Phi_1(t) = \Phi_2(t)$ throughout a t-set of positive Lebesgue measure.*

Proofs. Both theorems follow if we prove the assertion of Theorem A for the Fourier-Stieltjes transform Φ of a non-null totally-finite signed measure μ on $[0, \infty)$. Let

$$f(z) = \int_{0-}^{\infty} \exp\left(-\frac{1-z}{1+z}u\right)\mu(du) \qquad (|z| \leqq 1, z \neq -1); \tag{2}$$

then f is analytic for $|z| < 1$ and bounded and continuous on the punctured disk on which it is here defined. At a point $z = e^{i\theta}(-\pi < \theta < \pi)$ of the perimeter of the disk, $(1-z)/(1+z) = -i \tan \theta/2$, and thus

$$\lim_{r\to 1} f(re^{i\theta}) = f(e^{i\theta}) = \Phi(\tan \theta/2) \qquad (-\pi < \theta < \pi). \tag{3}$$

From a theorem proved by the brothers Riesz in 1916 (for which see, for example, [*1*], p. 46) we know that the bounded analytic function f must vanish identically if its Fatou radial limit $\lim_{r\to 1} f(re^{i\theta})$ vanishes on a θ-set of positive Lebesgue measure, and so μ must vanish (because $\int e^{-su}\mu(du)$ will vanish for all real $s > 0$) if $\Phi(t) = 0$ on a t-set of positive Lebesgue measure.

3. The identification problem

We now return to the identification problem formulated in the introduction. We know the epochs of departure of the successively departing customers, and we know the permutation P and so know in what order those customers arrived at the system. Thus, if C, C', and C'' are three customers who arrived consecutively in that order, we will be able to observe the epochs at which they each departed. Now suppose that in fact

C arrived at T and departed at $T + v$,

C' arrived at $T + u'$ and departed at $T + u' + v'$,

and

C'' arrived at $T + u' + u''$ and departed at $T + u' + u'' + v''$.

From observations on triplets like this we can determine the joint distribution of the differences

$$x = u' + (v' - v), \quad y = u'' + (v'' - v'), \tag{4}$$

* It is only fair to add that we could equally well have used Smith's theorem in § 3 at the cost of an extra step or two.

of the elements in the second column (by an appeal to the strong law of large numbers) ⋆. We shall prove that the joint distribution of x and y uniquely determines dA and dB up to an error in the location of the latter.

Let us write Φ and Ψ for the characteristic functions of dA and dB, so that $\Phi(t) = Ee^{itu}$ and $\Psi(t) = Ee^{itv}$ (t real). The characteristic function of the distribution of x will be

$$L(t) = \Phi(t)\,|\Psi(t)|^2, \tag{5}$$

and the joint characteristic function of the distribution of (x, y) will be

$$M(t, \tau) = \Phi(t)\,\Phi(\tau)\,\Psi(-t)\,\Psi(t-\tau)\,\Psi(\tau), \tag{6}$$

so that both functions L and M can be considered known.

If we know M, then we must also know the function defined by

$$N(t) = M(t, t) = \Phi(t)\,L(t). \tag{7}$$

Now Φ and Ψ and therefore also L vanish at most in a set of measure zero, and so $\Phi(t)$ is given almost everywhere by $N(t)/L(t)$, and so is determined for all real t by continuity; that is, dA is uniquely determined by the data supposed given.

Exactly the same argument applied to (5) shows that $|\Psi(t)|^2$ is uniquely determined (it is equal almost everywhere to $\{L(t)\}^2/N(t)$), and so $\Psi(t)$ is known for every real t save for a phase-factor. In order to show that dB is known up to a shift in location we have to prove that the undetermined phase-factor has the form e^{ibt}. To establish this we have found it necessary to make use of the bivariate function M. It should be noted that the facts $\Psi \in \mathscr{K}_+$, $|\Psi|^2$ known, are not in themselves sufficient to determine the distribution dB up to a shift in location. For a counter-example, take the distinct characteristic functions Ψ_1, $\Psi_2 \in \mathscr{K}_+$ given by

$$\Psi_1(t) = \xi(t)\,\eta(t), \quad \Psi_2(t) = \xi(t)\,\eta(-t)\,e^{ict},$$

where ξ is any member of $\mathscr{K}_+$, and η is any asymmetric characteristic function with range $[0, c]$; for example,

$$\eta(t) = 1 - p + p\,e^{ict}\,(0 < p < 1, p \neq \tfrac{1}{2}).$$

Suppose then that two solutions $\Psi = \lambda$ and $\Psi = \mu$ are compatible with the data, so that $|\lambda| = |\mu|$, $= |L|/|N|^{1/2}$ a. e. We shall have

$$\lambda(-t)\,\lambda(t-\tau)\,\lambda(\tau) = \frac{M(t,\tau)}{\Phi(t)\,\Phi(\tau)} = \mu(-t)\,\mu(t-\tau)\,\mu(\tau)$$

provided that each of t and τ lies outside a certain null-set (the zero-set for Φ), whence by continuity

$$\lambda(-t)\,\lambda(t-\tau)\,\lambda(\tau) = \mu(-t)\,\mu(t-\tau)\,\mu(\tau) \tag{8}$$

for all real t and τ. From this we obtain

$$\begin{aligned}\lambda(t-\tau)\,\mu(t)\,\mu(-\tau)\,&|\lambda(-t)\,\lambda(\tau)|^2 \\ &= \mu(t-\tau)\,\lambda(t)\,\lambda(-\tau)\,|\mu(-t)\,\mu(\tau)|^2,\end{aligned}$$

⋆ By considering triplets with arrival-ordinals congruent to 0, 1, and 2 (modulo 3) we can find the probability that the pair of random variables (x, y) lies in any rectangle R having rational vertex-coordinates. We can do this for *every* R because there are only countably many such rectangles. The distribution of (x, y) is then uniquely determined.

and so (the two squared moduli being equal) we find that

$$\lambda(t-\tau)\,\mu(t)\,\mu(-\tau) = \mu(t-\tau)\,\lambda(t)\,\lambda(-\tau)\,, \tag{9}$$

first everywhere save when $|\lambda(-t)| = |\mu(-t)| = 0$ or when $|\lambda(\tau)| = |\mu(\tau)| = 0$, and then (trivially) everywhere else.

We now change the sign of τ, and put $\beta(t) = \lambda(t)/\mu(t)$ save in the null-set Z on which $|\lambda(t)| = |\mu(t)| = 0$. From (9) we then find that

$$\beta(t+\tau) = \beta(t)\,\beta(\tau) \tag{10}$$

except when some one of t, τ, or $t+\tau$ lies in Z.

Now $\lambda(0) = 1$ and λ is continuous, so that $|\lambda| = |\mu| > 0$ throughout some neighbourhood $(-\varepsilon, \varepsilon)$ of the point $t = 0$; no point of Z can lie in this neighbourhood. It follows that the function β is continuous and satisfies the Cauchy functional equation (10) in the halved neighbourhood $(-\varepsilon/2, \varepsilon/2)$, and that $\beta(0) = 1$, and so we must have

$$\beta(t) = e^{ibt}\,, \quad \lambda(t) = e^{ibt}\mu(t) \neq 0 \tag{11}$$

throughout the halved neighbourhood.

Put $\mu_1(t) = e^{ibt}\mu(t)$ for all real t, so that μ_1 coincides with λ on $(-\varepsilon/2, \varepsilon/2)$. Going back to (9) (which holds when μ is replaced by μ_1) and changing the sign of τ we find that

$$\lambda(t+\tau)\,\mu_1(t)\,\mu_1(\tau) = \mu_1(t+\tau)\,\lambda(t)\,\lambda(\tau) \tag{12}$$

for all real t and τ. Let J be the set on which $\lambda(t) = \mu_1(t)$. Then

(i) $(-\varepsilon/2, \varepsilon/2) \subseteq J\backslash Z$,

and

(ii) $t+\tau \in J$ *whenever both* t *and* τ *lie in* $J\backslash Z$. Now let

$$\begin{aligned} t_1 &= t \quad + \tau_1\,,\\ t_2 &= t_1 \quad + \tau_2\,,\\ &\dots\\ t_n &= t_{n-1} + \tau_n\,; \end{aligned}$$

then iteration of (ii) shows that

(iii) $t_n \in J$ *provided that* t *and all* τ*'s lie in* $J\backslash Z$ *and further provided that all of* $t_1, \dots, t_{n-1}$ *lie outside* Z.

Choose any t in $(-\varepsilon/2, \varepsilon/2)$, and let t' be arbitrary. When n is sufficiently large we can put $t' = t_n$ and satisfy the conditions of (iii) by ensuring that all the τ's lie in $(-\varepsilon/2, \varepsilon/2)$ and that all the "bridging" points $t_1, \dots, t_{n-1}$ avoid Z. Thus t' is in J, which must therefore be the whole line, and so

$$\lambda(t) = \mu_1(t) = e^{ibt}\mu(t)$$

for all real t, as required.

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