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On a Stochastic Integral Equation of the Fredholm Type

W. J. Padgett and Chris P. Tsokos

1. Introduction

Stochastic or random integral equations arise often in the engineering, biological, and physical sciences, and recent attempts have been made to develop and unify the theory of such equations [1, 3–5, 8, 10, 14], to mention a few. In this paper we will be concerned with a certain class of random Fredholm integral equations. Specifically, we will investigate a stochastic integral equation of the Fredholm type of the form

$$x(t;\omega)=h(t;\omega)+\int_0^\infty k_0(t,\tau;\omega)\, e(\tau, x(\tau;\omega))\, d\tau, \qquad t\geqq 0 \tag{1.1}$$

where

(i) $\omega\in\Omega$, where Ω is the supporting set of the probability measure space (Ω, A, P);

(ii) $x(t;\omega)$ is the unknown random variable for each $t\in R_+$, the nonnegative real numbers;

(iii) $h(t;\omega)$ is called the *free random variable* or *stochastic free term* defined for each $t\in R_+$;

(iv) $k_0(t,\tau;\omega)$ is called the *stochastic kernel* and is defined for t and τ in R_+; and

(v) $e(t,x)$ is a scalar function defined for $t\in R_+$ and $x\in R$, the real numbers.

The Eq. (1.1) is a generalization of a stochastic integral equation considered by Anderson [1] in that the kernel is stochastic, the equation is nonlinear, and the interval of integration is $R_+=[0,\infty]$.

We shall actually study a more general stochastic integral equation of the mixed Volterra-Fredholm type of the form

$$\begin{aligned} x(t;\omega)=h(t;\omega)&+\int_0^t k(t,\tau;\omega)\, f(\tau, x(\tau;\omega))\, d\tau \\ &+\int_0^\infty k_0(t,\tau;\omega)\, e(\tau, x(\tau;\omega))\, d\tau, \qquad t\geqq 0 \end{aligned} \tag{1.2}$$

where, in addition to (i)–(v) above,

(vi) $k(t,\tau;\omega)$ is a stochastic kernel defined for t and τ in R_+ such that $0\leqq\tau\leqq t<\infty$; and

(vii) $f(t,x)$ is a scalar function of $t\in R_+$ and $x\in R$.

Thus, Eq. (1.1) is a special case of Eq. (1.2). Also, the random nonlinear Volterra integral equation which was studied by Tsokos [14] is a special case of Eq. (1.2).

We will obtain some very general results concerning the existence and uniqueness of random solutions of Eqs. (1.1) and (1.2). The tools that will be employed are the well-known fixed-point theorems of Banach and Krasnosel'skiĭ [9] and the theory of "*admissibility*" of linear topological spaces which was introduced into the study of integral equations by Corduneanu [6].

Nonstochastic versions of Eq. (1.2) have been considered by Miller, Nohel, and Wong [11], Petrovanu [12], and Corduneanu [7], among others.

2. Preliminaries

Throughout the paper we shall make the following assumptions concerning the functions in Eq. (1.2). The functions $x(t;\omega)$ and $h(t;\omega)$ will have values in the space $L_2 = L_2(\Omega, A, P)$ for each $t \in R_+$. Also, $e(t, x(t;\omega))$ and $f(t, x(t;\omega))$ under certain conditions will be functions of $t \in R_+$ with values in L_2. The stochastic kernels $k(t,\tau;\omega)$ and $k_0(t,\tau;\omega)$ will be bounded except perhaps on a set of probability measure zero for each fixed t and τ satisfying $0 \leqq \tau \leqq t$ and $0 \leqq \tau < \infty$, $0 \leqq t < \infty$, respectively. That is, the values of $k(t,\tau;\omega)$ and $k_0(t,\tau;\omega)$ will be in $L_\infty(\Omega, A, P)$ so that for fixed t and τ the products $k(t,\tau;\omega) f(\tau, x(\tau;\omega))$ and $k_0(t,\tau;\omega)\, e(\tau, x(\tau;\omega))$ will be in L_2.

Further, it will be assumed that the stochastic kernels are continuous functions of (t,τ). That is, if we denote the norm of an element of the space $L_\infty(\Omega, A, P)$ by

$$|||\cdot||| = P\text{-ess}\sup_{\omega\in\Omega} |\cdot|,$$

then as $n \to \infty$

$$|||k(t_n, \tau_n) - k(t,\tau)||| \to 0$$

and

$$|||k_0(t_n, \tau_n) - k_0(t,\tau)||| \to 0$$

whenever $(t_n, \tau_n) \to (t,\tau)$ as $n \to \infty$. It also will be assumed that for each $t \in R_+$, $k_0(t,\tau;\omega)$ is such that $|||k_0(t,\tau)|||$ is integrable with respect to $\tau \in R_+$ and

$$|||k_0(t,\tau)||| \cdot \|x(\tau)\|_{L_2}$$

is integrable with respect to $\tau \in R_+$ for every function x under consideration.

We now define several spaces of functions which will be used in this paper.

Definition 2.1. We let $C = C(R_+, L_2)$ denote the space of all continuous and bounded functions defined from R_+ into L_2. That is, C is the space of all second order stochastic processes on R_+ which are bounded and continuous in mean square,

$$E[|x(t+s) - x(t)|^2] \to 0$$

as $s \to 0$, $s > 0$.

Definition 2.2. We will denote by $C_g = C_g(R_+, L_2)$ the space of all continuous functions from R_+ into L_2 such that there exists a positive continuous function

$g(t)$ defined on R_+ and a constant $Z>0$ satisfying

$$\|x(t)\|_{L_2} \leqq Z\,g(t), \qquad t \in R_+ .$$

The norm of a function in C_g is defined by

$$\|x(t)\|_{C_g} = \sup_{t\in R_+} g(t)^{-1}\ \|x(t)\|_{L_2}.$$

Note that for $g(t)=1$, $t\in R_+$, we have $C_g = C$.

Definition 2.3. We define the space $C_c = C_c(R_+, L_2)$ to be the space of all continuous functions from R_+ into L_2 with the topology of uniform convergence on every interval $[0, Q)$, $Q>0$. That is, x_n converges to x in C_c if and only if

$$\lim_{n\to\infty} \|x_n(t) - x(t)\|_{L_2} \to 0$$

uniformly on every compact interval $[0, Q]$, $Q>0$.

Note that C_c is a Fréchet space [15, pp. 24–26] with distance function defined by the Fréchet combination of the following family of semi-norms:

$$\|x(t)\|_n = \sup_{0\leqq t\leqq n} \|x(t)\|_{L_2}, \qquad n = 1, 2, 3, \ldots .$$

Also, $C \subset C_g \subset C_c$.

Let B and D be a pair of Banach spaces such that $B, D \subset C_c$, and let T denote a linear operator from C_c into itself. We now define what is meant by the "*admissibility*" of a pair of Banach spaces.

Definition 2.4. The pair of Banach spaces (B, D) is said to be *admissible* with respect to the operator T: $C_c \to C_c$ if and only if $T(B) \subset D$.

The following lemma concerning the continuity of T is a result of Tsokos [14].

Lemma 2.1. If the topologies of B and D are stronger than C_c and the pair (B, D) is admissible with respect to T, a continuous linear operator from C_c into itself, then T is a continuous operator from B into D.

Hence, it follows from Lemma 2.1 that such an operator T is bounded, and if $K_1 > 0$ is the norm of T, then we have

$$\|Tx\|_D \leqq K_1\, \|x\|_B .$$

Another space of functions that will be used is the space H of all functions in C_c such that

(i) $\|x(t)\|_{L_2}^2$ is integrable on R_+; and

(ii) for any function y satisfying (i), $y \in H$ if the inner product

$$\big(x(t), y(t)\big)_{L_2} = \int_\Omega x(t;\omega)\, \overline{y(t;\omega)}\, dP(\omega)$$

is integrable on R_+, where the bar denotes the complex conjugate.

Let H_1 and H_2 be Hilbert spaces contained in H with norms defined by

$$\|x\|_{H_i} = \left\{ \int_0^\infty \|x(t)\|_{L_2}^2 \, dt \right\}^{\frac{1}{2}}, \quad i = 1, 2.$$

In order to study Eq. (1.2) we shall first consider the following stochastic integral equation, for $M = 1, 2, \dots$:

$$\begin{aligned} x(t;\omega) = h(t;\omega) + \int_0^t k(t,\tau;\omega) f(\tau, x(\tau;\omega)) \, d\tau \\ + \int_0^M k_0(t,\tau;\omega) e(\tau, x(\tau;\omega)) \, d\tau, \end{aligned} \tag{2.1}$$

for $t \in [0, M]$. In connection with the Eqs. (1.1), (1.2), and (2.1), we define the following linear operators from C_c into itself which are continuous as a result of the continuity properties of $k(t, \tau; \omega)$ and $k_0(t, \tau; \omega)$:

$$(Tx)(t;\omega) = \int_0^t k(t,\tau;\omega) x(\tau;\omega) \, d\tau, \tag{2.2}$$

and

$$(Wx)(t;\omega) = \int_0^\infty k_0(t,\tau;\omega) x(\tau;\omega) \, d\tau, \tag{2.3}$$

for $t \in R_+$, and

$$(T_M x)(t;\omega) = \int_0^t k(t,\tau;\omega) x(\tau;\omega) \, d\tau, \tag{2.4}$$

and

$$(W_M x)(t;\omega) = \int_0^M k_0(t,\tau;\omega) x(\tau;\omega) \, d\tau, \tag{2.5}$$

for $t \in [0, M]$, $M = 1, 2, \dots$.

By a *random solution* of a stochastic integral equation such as Eq. (1.2) we shall mean that for each $t \in R_+$, $x(t;\omega)$ satisfies the equation almost surely. In order to investigate the existence and uniqueness of random solutions of the stochastic integral equations given above, we will use the fixed-point theorems of Banach, Schauder, and Krasnosel'skiĭ [9]. Krasnosel'skiĭ's theorem which contains the results of Banach and Schauder will now be stated.

Theorem 2.1. *Let S be a closed, bounded convex subset of a Banach space and let U and V be operators on S satisfying:*

(i) *$U(x) + V(y) \in S$ whenever $x, y \in S$;*

(ii) *U is a contraction operator on S;*

(iii) *V is completely continuous.*

Then there is at least one point $x^* \in S$ such that

$$U(x^*) + V(x^*) = x^*.$$

In order to use Theorem 2.1 in obtaining the results in the next section we will need conditions which guarantee that the operator W_M given by Eq. (2.5) is completely continuous. The following lemma states such conditions.

Lemma 2.2. *If the integral operator W_M defined by Eq.* (2.5) *maps from H_2 into H_1 and the stochastic kernel k_0 is such that*

$$\int_0^\infty \int_0^M |||k_0(t,\tau)|||^2 \, d\tau \, dt$$

exists and is finite, then W_M is bounded and completely continuous.

The proof of Lemma 2.2 follows the technique of Schmeidler [13, p. 45].

3. Existence Theorems for a Random Solution

In this section we shall prove several theorems concerning the existence and uniqueness of random solutions of the Eqs. (1.1), (1.2), and (2.1).

Throughout the remainder of this paper we will assume that the topologies of the Banach spaces B and D and of the Hilbert spaces H_1 and H_2 which were defined in Section 2 are stronger than the topology of C_c. We will also need from time to time some of the following conditions:

(a) The stochastic kernel k_0 will be said to satisfy condition (a) if

$$\int_0^\infty \int_0^\infty |||k_0(t,\tau)|||^2 \, d\tau \, dt$$

exists and is finite.

(b) The function f will be said to satisfy condition (b) with Lipschitz constant λ and spaces B and D if $x(t;\omega) \to f(t, x(t;\omega))$ is a mapping Φ_f from the set

$$S = \{x\colon\ x \in D,\ \|x\|_D \leqq \rho\}$$

into B satisfying

$$\|\Phi_f(x) - \Phi_f(y)\|_B \leqq \lambda\, \|x-y\|_D$$

for $x, y \in S$, where λ and ρ are constants and B and D are Banach spaces.

We now prove the following theorem with respect to the existence of a random solution of Eq. (2.1).

Theorem 3.1. *Consider the stochastic integral equation* (2.1) *subject to the following conditions:*

(i) *H_2 and H_1 are Hilbert spaces, (H_2, H_1) is admissible with respect to each of the linear operators T_M and W_M given by Eqs.* (2.4) *and* (2.5), *respectively, and k_0 satisfies condition* (a);

(ii) *f satisfies condition* (b) *with Lipschitz constant λ and spaces H_2 and H_1;*

(iii) *Φ_e is a continuous mapping of S into H_2 such that $\|\Phi_e(x)\|_{H_2} \leqq \gamma$, for some constant $\gamma > 0$;*

(iv) *$h \in H_1$.*

Then there exists at least one random solution of Eq. (2.1), *provided that $\lambda K_{1M} < 1$ and*

$$\|h\|_{H_1} + K_{1M}\, \|\Phi_f(0)\|_{H_2} + \gamma K_{2M} \leqq \rho(1 - \lambda K_{1M}),$$

where K_{1M} and K_{2M} are the norms of T_M and W_M, respectively.

Proof. To obtain the desired result, we will show that the fixed-point theorem of Krasnosel'skiĭ (Theorem 2.1) is applicable.

By definition, H_1 and H_2 are also Banach spaces, and the set S of condition (b) with D replaced by H_1 is clearly closed, bounded and convex.

Let $x, y \in S$. Define the operators U_M from S into H_1 by

$$(U_M x)(t;\omega) = h(t;\omega) + \int_0^t k(t,\tau;\omega)\, f\big(\tau, x(\tau;\omega)\big)\, d\tau,$$

and V_M from S into H_1 by

$$(V_M x)(t;\omega) = \int_0^M k_0(t,\tau;\omega)\, e\big(\tau, x(\tau;\omega)\big)\, d\tau,$$

for $t \in (0, M]$. We must show that U_M and V_M satisfy the conditions of Theorem 2.1.

To show that the first condition holds, observe that

$$\|U_M x + V_M y\|_{H_1} \leqq \|h\|_{H_1} + K_{1M} \|\Phi_f(x)\|_{H_2} + K_{2M} \|\Phi_e(y)\|_{H_2}$$

since K_{1M} and K_{2M} are the norms of T_M and W_M, respectively. But by condition (ii) of this theorem, we have

$$\|\Phi_f(x)\|_{H_2} \leqq \lambda \|x\|_{H_1} + \|\Phi_f(0)\|_{H_2}. \tag{3.1}$$

Hence, applying condition (iii) of the theorem, we obtain

$$\begin{aligned}\|U_M x + V_M y\|_{H_1} &\leqq \|h\|_{H_1} + K_{1M}\lambda \|x\|_{H_1} + K_{1M} \|\Phi_f(0)\|_{H_2} + K_{2M}\gamma \\ &\leqq \|h\|_{H_1} + K_{1M} \|\Phi_f(0)\|_{H_2} + K_{2M}\gamma + K_{1M}\lambda\rho \\ &\leqq \rho(1 - K_{1M}\lambda) + K_{1M}\lambda\rho = \rho,\end{aligned}$$

from the last hypothesis of the theorem. Hence, $U_M x + V_M y \in S$ for x and y in S.

To show that the second condition of Theorem 2.1 holds, we must show that U_M is a contraction operator on S. We have that

$$\begin{aligned}\|U_M x - U_M y\|_{H_1} &\leqq K_{1M} \|\Phi_f(x) - \Phi_f(y)\|_{H_2} \\ &\leqq K_{1M}\lambda \|x - y\|_{H_1}\end{aligned}$$

using condition (ii) of the theorem. Since $\lambda K_{1M} < 1$ by hypothesis, U_M is a contraction operator on S.

We must now show that the third condition of Theorem 2.1 holds. From condition (i) of the present theorem and Lemma 2.2, the operator W_M is completely continuous from H_2 into H_1, and by condition (iii) above Φ_e is a bounded continuous operator from H_1 into H_2. We may express the operator V_M as the composition of W_M and Φ_e, and therefore, V_M is a completely continuous operator from S into H_1 [9].

Therefore, the conditions of the fixed-point theorem of Krasnosel'skiĭ hold, and there exists at least one random solution of Eq. (2.1) for $M = 1, 2, \ldots$, which completes the proof.

It is clear that the sequence of integral operator W_M from Hilbert space H_2 into H_1 converges as $M \to \infty$ to the operator W from H_2 into H_1 given by Eq. (2.3), and that W is a completely continuous operator from H_2 into H_1 [2, p. 290] provided k_0 satisfies condition (a). Hence, we have the following result.

Theorem 3.2. *Consider the random integral Eq.* (1.1) *subject to the following conditions:*

(i) H_2 *and* H_1 *are Hilbert spaces,* (H_2, H_1) *is admissible with respect to the linear operator W given by Eq.* (2.3), *and* k_0 *satisfies condition* (a);

(ii) *same as condition* (iii) *of Theorem* 3.1; *and*

(iii) *same as condition* (iv) *of Theorem* 3.1.

Then there exists at least one bounded random solution of Eq. (1.1), *provided*

$$\|h\|_{H_1} + \gamma K \leqq \rho,$$

where K is the norm of W.

The proof is similar to that for Theorem 3.1.

We now turn to the problem of existence of a *unique* random solution of Eq. (1.2). Banach's fixed-point theorem will be employed in this case.

Theorem 3.3. *Suppose the stochastic integral equation* (1.2) *satisfies the following conditions:*

(i) *The pair of Banach spaces* (B, D) *is admissible with respect to each of the linear operators T and W defined by Eqs.* (2.2) *and* (2.3), *respectively;*

(ii) f *satisfies condition* (b) *with Lipschitz constant* λ *and spaces B and D*;

(iii) e *satisfies condition* (b) *with Lipschitz constant* ξ *and spaces B and D; and*

(iv) $h \in D$.

Then there exists a unique random solution of Eq. (1.2), *provided* $\lambda K_1 + \xi K_2 < 1$ *and*

$$\|h\|_D + K_1 \|\Phi_f(0)\|_B + K_2 \|\Phi_e(0)\|_B \leqq \rho(1 - \lambda K_1 - \xi K_2),$$

where K_1 *and* K_2 *are the norms of T and W, respectively.*

Proof. Let us define the operators U and V from S into D by

$$(Ux)(t;\omega) = h(t;\omega) + \int_0^T k(t,\tau;\omega)\, f(\tau, x(\tau;\omega))\, d\tau$$

and

$$(Vx)(t;\omega) = \int_0^\infty k_0(t,\tau;\omega)\, e(\tau, x(\tau;\omega))\, d\tau, \qquad t \geqq 0.$$

We will show that the operator $U+V$ satisfies the conditions of Banach's fixed-point theorem; that is, $U+V$ is a contraction operator from S into S.

To show that $U+V$ maps from S into itself, let $x \in S$. We have

$$\|Ux + Vx\|_D \leqq \|h\|_D + K_1 \|\Phi_f(x)\|_B + K_2 \|\Phi_e(x)\|_B$$

from the statement following Lemma 2.1 and the conditions on T and W given in (i). Using inequalities similar to inequality (3.1) and the last hypothesis of the theorem, we obtain that

$$\begin{aligned}\|Ux + Vx\|_D &\leqq \|h\|_D + K_1 \|\Phi_f(0)\|_B + K_2 \|\Phi_e(0)\|_B + (K_1 \lambda + K_2 \xi)\, \|x\|_D \\ &\leqq \rho(1 - K_1 \lambda - K_2 \xi) + (K_1 \lambda + K_2 \xi)\, \rho = \rho.\end{aligned}$$

That is, $Ux + Vx \in S$ whenever $x \in S$.

We must now show that $U+V$ is a contraction. Let x and y be in S. Then we have

$$\begin{aligned}\|Ux+Vx-Uy-Vy\|_D &\leqq K_1\,\|\Phi_f(x)-\Phi_f(y)\|_B+K_2\,\|\Phi_e(x)-\Phi_e(y)\|_B\\ &\leqq (K_1\,\lambda+K_2\,\xi)\,\|x-y\|_D\end{aligned}$$

using conditions (ii) and (iii) of the theorem. By hypothesis, $K_1\lambda+K_2\xi<1$, and we have that $U+V$ is a contraction on S.

Therefore, by Banach's fixed-point theorem, there exists a unique $x^*\in S$ such that

$$Ux^*+Vx^*=x^*;$$

that is, x^* is the unique random solution of Eq. (1.2), completing the proof.

For the case that W is the null operator, we obtain the results of Tsokos [14]. For the case that T is the null operator, we immediately obtain conditions under which the random Fredholm integral equation (1.1) possesses a unique random solution.

Corollary 3.4. *Consider the random integral equation* (1.1) *subject to the following conditions:*

(i) *The pair of Banach spaces* (B, D) *is admissible with respect to the linear operator* W *given by Eq.* (2.3);

(ii) *same as condition* (iii) *of Theorem* 3.4; *and*

(iii) *same as condition* (iv) *of Theorem* 3.4.

Then there exists a unique random solution of (1.1), *provided* $\xi K_2<1$ *and*

$$\|h\|_D+K_2\,\|\Phi_e(0)\|_B\leqq\rho(1-\xi K_2)$$

where K_2 *is the norm of* W.

4. Special Cases

In this section we present some special cases of Corollary 3.4 by taking as the Banach spaces B and D specific spaces such as C_g or C. These special cases are much more useful in practice than the general results given in the previous section.

Theorem 4.1. *Consider the stochastic integral equation* (1.1) *subject to the following conditions:*

(i) *there exists a constant* $Z>0$ *and a positive continuous function* $g(t)$ *on* R_+ *such that*

$$\int_0^\infty |||k_0(t,\tau)|||\,g(\tau)\,d\tau\leqq Z,\qquad t\in R_+;$$

(ii) $e(t,x)$ *is continuous in* $t\in R_+$ *and* $x\in R$ *such that* $|e(t,0)|\leqq\gamma\,g(t)$ *and*

$$|e(t,x)-e(t,y)|\leqq\xi\,g(t)\,|x-y|$$

for $\|x\|_C$, $\|y\|_C\leqq\rho$ *and* γ *and* ξ *constants; and*

(iii) $h(t;\omega)\in C$.

Then there exists a unique random solution, $x \in C$, *of Eq.* (1.1) *such that* $\|x\|_C \leqq \rho$, *provided that* $\|h\|_C$, ξ, *and* γ *are small enough.*

Proof. We must show that under the above conditions, the pair (C_g, C) is admissible with respect to the integral operator W given by (2.3). Let $x \in C_g$. Then, using the definition of the norm in C_g, we have

$$\begin{aligned}\|Wx\|_{L_2} &\leqq \int_0^\infty \|k_0(t,\tau)\, x(\tau)\|_{L_2}\, d\tau \\ &\leqq \int_0^\infty |||k_0(t,\tau)|||\; \|x(\tau)\|_{L_2}\, d\tau \\ &\leqq \|Wx\|_{L_2} \leqq \sup_{t \geqq 0} g(t)^{-1}\, \|x(t;\omega)\|_{L_2} \int_0^\infty |||k_0(t,\tau)|||\, g(\tau)\, d\tau \leqq \|x\|_{C_g} Z,\end{aligned}$$

by condition (i) of the theorem. Thus, W is a bounded operator and $Wx \in C$. Hence, (C_g, C) is admissible with respect to W.

Condition (ii),

$$|e(t, x(t;\omega)) - e(t, y(t;\omega))| \leqq \xi\, g(t)\, |x(t;\omega) - y(t;\omega)|,$$

implies that

$$\|\Phi_e(x) - \Phi_e(y)\|_{C_g} \leqq \xi\, \|x-y\|_C$$

for $\|x\|_C$, $\|y\|_C \leqq \rho$. Likewise, $|e(t,0)| \leqq \gamma\, g(t)$ implies that $\|\Phi_e(0)\|_{C_g} \leqq \gamma$. Therefore, Corollary 3.4 applies with $B = C_g$ and $D = C$, provided that $\|h\|_C$, ξ, and γ are small enough in the sense that

$$\xi K_2 < 1; \qquad \|h\|_C + K_2\gamma \leqq \rho(1 - \xi K_2),$$

completing the proof.

For $g(t) = 1$ for all $t \in R_+$, we see that the Banach spaces in Theorem 4.1 both become C, and the conditions simplify even further.

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W. J. Padgett
Department of Mathematics
University of South Carolina
Columbia, South Carolina 29208
USA

Chris P. Tsokos
Department of Statistics
Virginia Polytechnic Institute
and State University
Blacksburg, Virginia 24061, USA
and
Department of Mathematics
University of South Florida
Tampa, Florida 33620, USA

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