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Hits on an Axis by the Simple Random Walk in Three Dimensions

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§ 1

We are concerned with a simple random walk $\boldsymbol{S}_n, n \geqq 0$, on the lattice R in 3-dimensional space E,

$$R = \{\boldsymbol{x} \in E \mid \boldsymbol{x} \text{ has integer coordinates } x_1, x_2, x_3\}.$$

The random walk starts at $\boldsymbol{x} \in R$, so that

$$\boldsymbol{S}_0 = \boldsymbol{x}; \quad \boldsymbol{S}_n = \boldsymbol{S}_{n-1} + \boldsymbol{X}_n, \quad n \geqq 1,$$

where $\boldsymbol{X}_n$ are independent identically distributed random variables such that

$$\begin{aligned} P_{\boldsymbol{x}}\{\boldsymbol{X}_n = \boldsymbol{y}\} &= 1/6 \quad \text{for} \quad \boldsymbol{y} \in R \text{ with } |\boldsymbol{y}| = 1, \forall \boldsymbol{x} \in R, \\ P_{\boldsymbol{x}}\{\boldsymbol{X}_n = \boldsymbol{y}\} &= 0 \quad \text{for} \quad \boldsymbol{y} \in R \text{ with } |\boldsymbol{y}| \neq 1, \forall \boldsymbol{x} \in R, \end{aligned} \tag{1.1}$$

where $P_{\boldsymbol{x}}\{\cdot\}$ means $P\{\cdot \mid \boldsymbol{S}_0 = \boldsymbol{x}\}$.

If A is the subset of R

$$A = \{\boldsymbol{x} \in R \mid \boldsymbol{x}_1 = 0, \boldsymbol{x}_2 = 0\}$$

it is known that the random walk is almost certain to hit A, i.e.

$$P_{\boldsymbol{x}}\{\boldsymbol{S}_n \in A \text{ for some } n > 0\} = 1, \quad \forall \boldsymbol{x} \in R.$$

We can therefore define a random variable, $\boldsymbol{F}(\boldsymbol{x}, A)$, whose value is the position at which the first hit on A occurs. If $\boldsymbol{F}(\boldsymbol{x}, A) = (0, 0, x_3 + D_{\boldsymbol{x}})$, $D_{\boldsymbol{x}}$ measures the displacement of the random walk parallel to A up to the time of the first hit. Our object is the study of the random variables $D_{\boldsymbol{x}}$ as $\|\boldsymbol{x}\|$, the perpendicular distance from $\boldsymbol{x}$ onto A, tends to infinity.

We begin by calculating the characteristic function of $D_{\boldsymbol{x}}$ (Theorem I) and finding an asymptotic estimate for it, (Lemma 2.11). Whereas in the analogous 2-dimensional situation $D_{\boldsymbol{x}}/\|\boldsymbol{x}\|$ has a non-degenerate limiting distribution as $\|\boldsymbol{x}\| \to +\infty$, it follows from this estimate that in 3 dimensions $D_{\boldsymbol{x}}/d(\|\boldsymbol{x}\|)$ cannot have such a limit, whatever norming function $d(\|\boldsymbol{x}\|)$ is chosen (Theorem II). A limiting distribution for $\log|D_{\boldsymbol{x}}|/\log\|\boldsymbol{x}\|$ is found, however, in Theorem III.

§ 2

Plainly we may assume that the starting point $\boldsymbol{x}$ of the random walk lies in the $x_1 x_2$ plane, and it is convenient to denote it by $(-a, -b, 0)$, and $D_{\boldsymbol{x}}$ by D_{ab}. If D_{ab} has characteristic function $\varphi_{ab}(\theta)$, then for all real θ

$$\varphi_{ab}(\theta) = \sum_{k=-\infty}^{\infty} f_{ab}^{k} e^{ik\theta}, \tag{2.1}$$

where

$$f_{ab}^{k} = P(D_{ab} = k) = P_x\{S_n = (0,0,k) \text{ for some } n \geqq 1, S_{n-r} \notin A \text{ for } 1 \leqq r < n\}.$$

It is easy to calculate $\varphi_{ab}(\theta)$ in terms of the characteristic function $\boldsymbol{\varphi}(\boldsymbol{\theta})$ of X_n, which, by (1.1) is given for $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$ by

(2.2) $$\boldsymbol{\varphi}(\boldsymbol{\theta}) = \sum_{y \in R} P(X_n = y)\, e^{i y \cdot \theta} = \tfrac{1}{3}(\cos\theta_1 + \cos\theta_2 + \cos\theta_3),$$

the result being

Theorem I. *Provided that one of a, b is non-zero,*

$$\varphi_{ab}(\theta) = \begin{cases} g_{ab}(\theta)/g_{00}(\theta) & \text{for } \theta \neq 0 \pmod{2\pi}, \\ 1 & \text{for } \theta = 0 \pmod{2\pi}, \end{cases}$$

where

$$g_{ab}(\theta) = \frac{3}{(2\pi)^2} \iint_{-\pi}^{\pi} \frac{e^{-i(a\alpha + b\beta)}\,d\alpha\,d\beta}{3 - (\cos\alpha + \cos\beta + \cos\theta)},$$

$$g_{00}(\theta) = \frac{3}{(2\pi)^2} \iint_{-\pi}^{\pi} \frac{d\alpha\,d\beta}{3 - (\cos\alpha + \cos\beta + \cos\theta)}.$$

Proof of Theorem I. Write $P_0\{S_n = (i,j,k)\} = p_{ijk}^{n}$, note that $P_x\{S_n = (0,0,c)\} = p_{abc}^{n}$ and define q_{abc}^{n} by

(2.3) $$q_{abc}^{0} = 0, \quad q_{abc}^{n} = P_x\{S_n = (0,0,c), S_{n-r} \notin A \text{ for } 1 \leqq r < n\} \text{ for } n \geqq 1.$$

Then plainly

(2.4) $$\begin{aligned} p_{abc}^{0} &= q_{abc}^{0}, \\ p_{abc}^{n} &= \sum_{r=1}^{n} \sum_{k=-\infty}^{\infty} q_{abk}^{r}\, p_{00\,c-k}^{n-r} \quad \text{for} \quad n \geqq 1. \end{aligned}$$

Writing for all real θ and real s with $|s| < 1$

$$P_{ab}^{n}(\theta) = \sum_{c=-\infty}^{\infty} p_{abc}^{n} e^{ic\theta}, \qquad Q_{ab}^{n}(\theta) = \sum_{c=-\infty}^{\infty} q_{abc}^{n} e^{ic\theta},$$

$$P_{ab}(s,\theta) = \sum_{n=0}^{\infty} P_{ab}^{n}(\theta)\, s^n, \qquad Q_{ab}(s,\theta) = \sum_{n=0}^{\infty} Q_{ab}^{n}(\theta)\, s^n,$$

it follows from (2.4) that

(2.5) $$P_{ab}(s,\theta) = Q_{ab}(s,\theta)\, P_{00}(s,\theta).$$

Now

$$\begin{aligned} P_{ab}(s,\theta) &= \sum_{n=0}^{\infty} \left\{ \sum_{c=-\infty}^{\infty} p_{abc}^{n} e^{ic\theta} \right\} s^n \\ &= \sum_{c=-\infty}^{\infty} e^{ic\theta} \left\{ \sum_{n=0}^{\infty} p_{abc}^{n} s^n \right\}, \end{aligned}$$

the interchange of order of summation being justified for $|s| < 1$ by the absolute convergence of the sum, since $\sum_{c=-\infty}^{\infty} p_{abc}^{n} \leqq 1$. Recalling the definition (2.2) of

$\boldsymbol{\varphi}(\boldsymbol{\theta})$, a straightforward argument shows that for $|s|<1$

$$\sum_{n=0}^{\infty} p_{abc}^{n} s^{n} = \frac{1}{(2\pi)^3} \iiint_{-\pi}^{\pi} \frac{e^{-i(a\theta_1+b\theta_2+c\theta_3)}}{1-s\boldsymbol{\varphi}(\boldsymbol{\theta})} d\theta_1\, d\theta_2\, d\theta_3 .$$

If we now write

$$\psi_{ab}(s,\theta_3) = \frac{1}{(2\pi)^2} \iint_{-\pi}^{\pi} \frac{e^{-i(a\theta_1+b\theta_2)}}{1-s\boldsymbol{\varphi}(\boldsymbol{\theta})} d\theta_1\, d\theta_2$$

we have

$$P_{ab}(s,\theta) = \sum_{c=-\infty}^{\infty} e^{ic\theta} \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} \psi_{ab}(s,\theta_3)\, e^{-ic\theta_3}\, d\theta_3 . \tag{2.6}$$

For each s with $|s|<1$, $\psi_{ab}(s,\theta_3)$ is everywhere differentiable with respect to θ_3, so that DINI's convergence theorem applies to (2.6) to yield

$$P_{ab}(s,\theta) = \psi_{ab}(s,\theta) . \tag{2.7}$$

From (2.7) and (2.5) we have an explicit expression for $Q_{ab}(s,\theta)$. Moreover, since $\sum_{c=-\infty}^{\infty} q_{abc}^{n} \leqq 1$,

$$Q_{ab}(s,\theta) = \sum_{n=0}^{\infty} s^n \sum_{c=-\infty}^{\infty} q_{abc}^{n} e^{ic\theta} = \sum_{c=-\infty}^{\infty} e^{ic\theta} \sum_{n=0}^{\infty} q_{abc}^{n} s^n, \tag{2.8}$$

and as $s \uparrow 1$

$$\sum_{n=0}^{\infty} q_{abc}^{n} s^n \uparrow \sum_{n=0}^{\infty} q_{abc}^{n} = f_{ab}^{c} \leqq 1 .$$

Since $\sum_{c=-\infty}^{\infty} f_{ab}^{c} = P\{\text{particle starting at } \boldsymbol{x} \text{ hits } A\} = 1$, we can let s increase to one in (2.8) and apply the theorem of dominated convergence to get

$$\lim_{s\uparrow 1} Q_{ab}(s,\theta) = \sum_{c=-\infty}^{\infty} e^{ic\theta} f_{ab}^{c} = \varphi_{ab}(\theta) . \tag{2.9}$$

But if $\theta_3 \neq 0 \pmod{2\pi}$

$$\lim_{s\uparrow 1} \psi_{ab}(s,\theta_3) = \frac{1}{(2\pi)^2} \iint_{-\pi}^{\pi} \frac{e^{-i(a\theta_1+b\theta_2)}}{1-\boldsymbol{\varphi}(\boldsymbol{\theta})} d\theta_1\, d\theta_2$$
$$= g_{ab}(\theta_3)$$

so that, by virtue of (2.5), (2.7), and (2.9),

$$\varphi_{ab}(\theta) = \begin{cases} g_{ab}(\theta)/g_{00}(\theta) & \text{if} \quad \theta \neq 0 \pmod{2\pi}, \\ 1 & \text{if} \quad \theta = 0 \pmod{2\pi}. \end{cases} \tag{2.10}$$

Thus the behaviour of the characteristic functions $\varphi_{ab}(\theta)$ is completely determined by the behaviour of the functions $g_{ab}(\theta)$, some of whose properties are the content of:

Lemma 2.11. *For all* (a,b) *and* $\theta \neq 0 \pmod{2\pi}$

$$0 < g_{ab}(\theta) \leqq g_{00}(\theta) < +\infty . \tag{2.12}$$

There exist constants k_1 and k_2 independent of θ, a, b such that

$$(2.13)\qquad \left|g_{ab}(\theta)-\frac{3}{\pi}K_0(r|\theta|)\right|<k_1 \quad \textit{for} \quad 0<|\theta|\leqq\pi \textit{ and } (a,b)\neq(0,0),$$

$$(2.14)\qquad \left|g_{00}(\theta)-\frac{3}{\pi}\log\frac{1}{|\theta|}\right|<k_2 \quad \textit{for} \quad 0<|\theta|\leqq\pi,$$

where K_0 is the Bessel coefficient of zero order and imaginary argument and $r=(a^2+b^2)^{\frac{1}{2}}$.

Corollary to Lemma 2.11. *For all (a, b) and all $\theta \neq 0$* (mod 2π)

$$(2.15)\qquad 0<\varphi_{ab}(\theta)\leqq 1.$$

Proof of Lemma 2.11. If we write, for $\theta \neq 0$ (mod 2π),

$$g_{ab}(\theta)=\frac{3}{(2\pi)^2}\iint_{-\pi}^{\pi}e^{-i(a\alpha+b\beta)}\int_0^\infty e^{-t(3-\cos\theta-\cos\alpha-\cos\beta)}dt\,d\alpha\,d\beta$$

the fact that

$$\iint_{-\pi}^{\pi}\int_0^\infty e^{-t(3-\cos\theta-\cos\alpha-\cos\beta)}dt\,d\alpha\,d\beta<+\infty$$

allows us to interchange the order of integration to get,

$$(2.16)\qquad g_{ab}(\theta)=3\int_0^\infty e^{-t(3-\cos\theta)}\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-ia\alpha}e^{t\cos\alpha}d\alpha\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-ib\beta}e^{t\cos\beta}d\beta\,dt,$$

$$=3\int_0^\infty e^{-t(3-\cos\theta)}I_a(t)\,I_b(t)\,dt,$$

where $I_a(t)$ is the modified Bessel coefficient of order a, and the first assertion follows.

Noting that $g_{ab}(\theta)$ is an even function of θ, take $0<\theta<\pi$ and write

$$g_{ab}(\theta)=\frac{3}{2\pi^2}\iint_0^\pi\frac{\cos a\alpha\cos b\beta}{\sin^2\frac{1}{2}\alpha+\sin^2\frac{1}{2}\beta+\sin^2\frac{1}{2}\theta}d\alpha\,d\beta$$

$$=\frac{6}{\pi^2}\iint_0^\pi\frac{\cos a\alpha\cos b\beta}{\alpha^2+\beta^2+\theta^2}d\alpha\,d\beta+h_{ab}(\theta).$$

The inequalities

$$\theta/\pi\leqq\sin\tfrac{1}{2}\theta\leqq\tfrac{1}{2}\theta,$$
$$0\leqq(\tfrac{1}{2}\theta)^2-\sin^2\tfrac{1}{2}\theta=\tfrac{1}{2}(\cos\theta-1+\tfrac{1}{2}\theta^2)\leqq\theta^4/48,$$

both hold in the range $0\leqq\theta\leqq\pi$ and lead to

$$(2.17)\qquad |h_{ab}(\theta)|\leqq\frac{1}{8}\iint_0^\pi\frac{\alpha^4+\beta^4+\theta^4}{(\alpha^2+\beta^2+\theta^2)^2}d\alpha\,d\beta\leqq\frac{\pi^2}{8}.$$

If we now note that for $r>0$

$$(2.18)\qquad \frac{2}{\pi}\iint_0^\infty\frac{\cos a\alpha\cos b\beta}{\alpha^2+\beta^2+\theta^2}d\alpha\,d\beta=K_0(r|\theta|)$$

we see that (2.14) will follow from (2.17) if we can show that the error involved in replacing the region of integration, $(0 \leqq \alpha \leqq \pi, 0 \leqq \beta \leqq \pi)$, by the region $(0 \leqq \alpha, 0 \leqq \beta)$ is bounded for all $0 < \theta \leqq \pi$ uniformly in a and b. Since $g_{ab}(\theta)$ is symmetric in a and b and $a^2 + b^2 > 0$, we can take $|a| \geqq 1$ and apply the second mean value theorem for double integrals [*4*, p. 572], to show that for each $R > \pi$ there exists $A \in \langle \pi, R \rangle, B \in \langle 0, \pi \rangle$ such that

(2.19) $$\int_{\pi}^{R\pi}\int_{0} \frac{\cos a\alpha \cos b\beta}{\alpha^2 + \beta^2 + \theta^2} \, d\alpha \, d\beta = \frac{1}{\pi^2 + \theta^2} \int_{\pi}^{A} \cos a\alpha \, d\alpha \int_{0}^{B} \cos b\beta \, d\beta .$$

Now

$$\left| \int_{0}^{B} \cos b\beta \, d\beta \right| \leqq \pi, \left| \int_{\pi}^{A} \cos a\alpha \, d\alpha \right| \leqq 2,$$

and as $1/(\alpha^2 + \beta^2 + \theta^2)$ is integrable in $(\pi \leqq \alpha, 0 \leqq \beta \leqq \pi)$,

$$\lim_{R \to +\infty} \int_{\pi}^{R}\int_{0}^{\pi} \frac{\cos a\alpha \cos b\beta}{\alpha^2 + \beta^2 + \theta^2} \, d\alpha \, d\beta$$

exists. Using the fact [*2*, p. 7] that

$$\int_{0}^{\infty} \frac{\cos a\alpha \, d\alpha}{\alpha^2 + \beta^2 + \theta^2} = \frac{1}{2} \pi \frac{e^{-a(\beta^2 + \theta^2)^{\frac{1}{2}}}}{(\beta^2 + \theta^2)^{\frac{1}{2}}} ,$$

we can therefore let $R \to +\infty$ in (2.19) to get

(2.20) $$\left| \frac{1}{2} \pi \int_{0}^{\pi} \frac{e^{-a(\beta^2 + \theta^2)^{\frac{1}{2}}} \cos b\beta}{(\beta^2 + \theta^2)^{\frac{1}{2}}} \, d\beta - \int_{0}^{\pi}\int \frac{\cos a\alpha \cos b\beta}{\alpha^2 + \beta^2 + \theta^2} \, d\alpha \, d\beta \right| < k_3$$

where k_3 is a finite constant independent of θ, a, and b. Since, [*2*, p. 17],

$$\int_{0}^{\infty} \frac{e^{-a(\beta^2 + \theta^2)^{\frac{1}{2}}} \cos b\beta}{(\beta^2 + \theta^2)^{\frac{1}{2}}} \, d\beta = K_0(r|\theta|) ,$$

and

$$\left| \int_{\pi}^{\infty} \frac{e^{-a(\beta^2 + \theta^2)^{\frac{1}{2}}} \cos b\beta}{(\beta^2 + \theta^2)^{\frac{1}{2}}} \, d\beta \right| \leqq \int_{\pi}^{\infty} \frac{e^{-\beta}}{\beta} \, d\beta < +\infty ,$$

(2.13) follows from (2.17) and (2.20)

It is easily seen that

$$g_{00}(\theta) = \frac{3}{(2\pi)^2} \int_{-\pi}^{\pi}\int \frac{d\alpha \, d\beta}{3 - \cos\theta - \cos\alpha - \cos\beta}$$

$$= \frac{3k}{\pi} \int_{0}^{\infty} \frac{dt}{(1 + t^2)^{\frac{1}{2}}(1 + k'^2 t^2)^{\frac{1}{2}}}$$

where

$$k = 1/(1 + \sin^2 \tfrac{1}{2}\theta)$$

and $k^2 + k'^2 = 1$. Thus we have

(2.21) $$g_{00}(\theta) = \frac{3k}{\pi} K(k) ,$$

K denoting the complete elliptic integral of the first kind. Now

$$k' = \frac{\sin\frac{1}{2}\theta\,(2+\sin^2\frac{1}{2}\theta)^{\frac{1}{2}}}{1+\sin^2\frac{1}{2}\theta} \sim \frac{\theta}{\sqrt{2}} \quad \text{as} \quad \theta \to 0\,,$$

and it is not difficult to see that $|\log 1/k' - \log 1/\theta|$ is bounded for $0 \leqq \theta \leqq \pi$. Since it is known [3, p. 318] that $|K(k) - \log 1/k'|$ is bounded for all k, this is sufficient to establish (2.14).

§ 3

An obvious question to ask about the random variables D_{ab} is whether or not there exists a norming function $d(r)$ (where $r^2 = a^2 + b^2 = \|\boldsymbol{x}\|^2$ in the notation of § 1) such that $D_{ab}/d(r)$ has a non-degenerate limiting distribution as $r \to +\infty$. According to the continuity theorem for characteristic functions (see e.g. [5, p. 54]) $D_{ab}/d(r)$ has a limiting distribution if and only if $\varphi_{ab}(\theta/d(r))$ converges for each θ to a function which is continuous at $\theta = 0$. This fact, together with the results of Lemma 2.11 leads to

Theorem II. *If the sequence of random variables $D_{ab}/d(r)$ converges in distribution as $r \to +\infty$, then its limit is degenerate and has distribution function $G_0(y)$ given by*

$$G_0(y) = \begin{cases} 1, & y \geqq 0\,, \\ 0, & y < 0\,. \end{cases}$$

Proof of Theorem II. Suppose that $\varrho(\theta) = \lim_{r\to+\infty} \varphi_{ab}(\theta/d(r))$ exists for all real θ, assume, with no loss of generality, that $d(r)$ is positive for all r and note that $\varrho(0) = 1$. If $\liminf_{r\to+\infty} d(r) < +\infty$ it is easy to check that $\varrho(\theta) = 0$ for all $\theta \neq 0$, so that $\varrho(\theta)$ is discontinuous at 0. If $\lim_{r\to+\infty} d(r) = +\infty$ it follows from (2.13) and (2.14) that for all $\theta > 0$

$$\varrho(\theta) = \lim_{r\to+\infty} \{K_0(r\,\theta/d(r))/\log d(r)\}\,.$$

In order that $\varrho(\theta)$ be non zero for $\theta > 0$, it is therefore necessary that $\lim_{r\to+\infty} \{d(r)/r\} = +\infty$. But in that case, since $K_0(z) \sim \log z^{-1}$ as $z \to 0$, we have, for all $\theta > 0$,

$$\varrho(\theta) = \lim_{r\to+\infty} \{\log(d(r)/r\,\theta)/\log d(r)\} = \lim_{r\to+\infty} \{1 - \log r/\log d(r)\} = \gamma\,,$$

where $\gamma \in (0, 1)$, and is independent of θ. When $\gamma < 1$, $\varrho(\theta)$ is again discontinuous at 0; when $\gamma = 1$, $\varrho(\theta) \equiv 1$. Since this is the characteristic function of $G_0(y)$, the theorem is established.

In particular, when $d(r) = r^\beta$ this argument shows that

$$\lim_{r\to+\infty} \varphi_{ab}(\theta/r^\beta) = \begin{cases} 0 \text{ for } \theta > 0 \text{ and } 0 < \beta \leqq 1\,, \\ 1 - 1/\beta \text{ for } \theta > 0 \text{ and } \beta > 1\,. \end{cases} \tag{3.1}$$

This suggests that the distribution of D_{ab} is too spread out to lie completely within the interval $(-r^\beta, r^\beta)$ for large values of r however large β is. Moreover, if $L_{ab}^\beta = P\{|D_{ab}| < r^\beta\}$ and we write N_r for $[r^\beta] + \frac{1}{2}$, where $[r^\beta]$ denotes the integral part of r^β,

$$L^\beta_{ab} = \sum_{|k|<r^\beta} f^k_{ab} = \sum_{|k|<r^\beta} \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi_{ab}(\theta) \cos k\theta \, d\theta$$

$$(3.2) \qquad = \frac{1}{\pi} \int_0^\pi \varphi_{ab}(\theta) \frac{\sin N_r \theta}{\sin \frac{1}{2}\theta} \, d\theta$$

$$= \frac{2}{\pi} \int_0^{\pi r^\beta} \varphi_{ab}(\theta/r^\beta) \frac{\sin(N_r \theta/r^\beta)}{2 r^\beta \sin(\theta/2 r^\beta)} \, d\theta .$$

According to (3.1) the integrand in (3.2) tends, as $r \to +\infty$, for each $\theta > 0$ to 0 or $\frac{1-\beta}{\beta} \frac{\sin\theta}{\theta}$ according as $\beta \leqq 1$ or $\beta > 1$.

Since $\int_0^\infty \frac{\sin\theta}{\theta} \, d\theta = \frac{1}{2}\pi$, the obvious conjecture is

$$(3.3) \qquad \lim_{r \to +\infty} P\{|D_{ab}| < r^\beta\} = \begin{cases} 1 - 1/\beta & \text{if} \quad \beta > 1, \\ 0 & \text{if} \quad \beta \leqq 1, \end{cases}$$

which is equivalent to

Theorem III. *If the random variables D'_{ab} are defined by*

$$D'_{ab} = \begin{cases} \log|D_{ab}| & \textit{when} \quad D_{ab} \neq 0, \\ 0 & \textit{when} \quad D_{ab} = 0, \end{cases}$$

then

$$\lim_{r \to +\infty} P\{D'_{ab}/\log r < \beta\} = \begin{cases} 1 - 1/\beta & \text{if} \quad \beta > 1, \\ 0 & \text{if} \quad \beta \leqq 1. \end{cases}$$

Since the characteristic function of D'_{ab} is not readily accessible, the usual methods of proving such a theorem do not apply. A straightforward, but laborious argument is therefore used to establish Theorem III, via (3.3), in the next section.

§ 4

We will assume throughout this section that $a \geqq b \geqq 0$; we can do this without loss of generality, since $\varphi_{ab}(\theta) = \varphi_{AB}(\theta)$, where $A = \max(|a|, |b|)$, $B = \min(|a|, |b|)$. Two preliminary lemmas are required.

Lemma 4.1. *A function $\delta(r)$ exists such that, when $r^{-1}\log r \leqq \theta \leqq \pi$,*

$$0 < \varphi_{ab}(\theta) \leqq \delta(r), \quad \textit{and} \quad \delta(r)\log r \to 0 \quad \textit{as} \quad r \to +\infty.$$

Proof. The relation [*1*, p. 207],

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\cos n\alpha \, d\alpha}{z - \cos\alpha} = (z^2-1)^{-\frac{1}{2}} \{z - (z^2-1)^{\frac{1}{2}}\}^n \quad \text{for} \quad z > 1$$

gives

$$g_{ab}(\theta) = \frac{3}{\pi} \int_0^\pi (\eta^2 - 1)^{-\frac{1}{2}} \{\eta - (\eta^2-1)^{\frac{1}{2}}\}^a \cos b\beta \, d\beta \quad \text{for} \quad \theta \neq 0 \,(\operatorname{mod} 2\pi),$$

where $\eta = 3 - \cos\beta - \cos\theta$, so that

$$g_{ab}(\theta) \leqq \frac{3}{\pi} \int_0^\pi (\eta^2-1)^{-\frac{1}{2}} \mathrm{d}\beta \cdot \sup_{0\leqq\beta\leqq\pi} \{\eta - (\eta^2-1)^{\frac{1}{2}}\}^a$$
$$= g_{00}(\theta)\{2-\cos\theta - ((2-\cos\theta)^2-1)^{\frac{1}{2}}\}^a.$$

Since $a \geqq \frac{1}{2} r > 0$, it follows that

$$\sup_{r^{-1}\log r \leqq \theta \leqq \pi} \varphi_{ab}(\theta) \leqq \{2-\cos\theta - ((2-\cos\theta)^2-1)^{\frac{1}{2}}\}^{\frac{1}{2}r}_{\theta=r^{-1}\log r} = \delta(r)$$

and $\delta(r) \sim r^{-\frac{1}{2}}$ as $r \to +\infty$.

Lemma 4.2. *If the total variation of $\varphi_{ab}(\theta)$ in the interval $(r^{-\beta}, r^{-1}\log r)$ is V^β_{ab}, then*

$$\limsup_{r\to+\infty} V^\beta_{ab} \leqq 2\log\beta \quad \textit{for} \quad \beta > 1.$$

Proof. For $\theta \neq 0 \pmod{2\pi}$

$$g'_{ab}(\theta) = \frac{-3\sin\theta}{(2\pi)^2} \int_{-\pi}^{\pi}\!\!\int \frac{\cos a\alpha \cos b\beta \,\mathrm{d}\alpha\,\mathrm{d}\beta}{(3-\cos\alpha-\cos\beta-\cos\theta)^2},$$

so for $\theta \in \langle 0,\pi)$ $|g'_{ab}(\theta)| \leqq -g'_{00}(\theta)$. In this range we also have, from (2.12), $0 < g_{ab}(\theta) \leqq g_{00}(\theta)$, and therefore

$$|\varphi'_{ab}(\theta)| = \frac{1}{g^2_{00}(\theta)} |g_{00}(\theta) g'_{ab}(\theta) - g'_{00}(\theta) g_{ab}(\theta)| \leqq \frac{-2g'_{00}(\theta)}{g_{00}(\theta)}.$$

Thus

$$V^\beta_{ab} = \int_{r^{-\beta}}^{r^{-1}\log r} |\varphi'_{ab}(\theta)|\,\mathrm{d}\theta \leqq 2\log\{g_{00}(r^{-\beta})/g_{00}(r^{-1}\log r)\}$$

and the lemma follows from (2.14).

Proof of Theorem III. Take $\beta > 1$ and consider, in the notation of §3,

$$L^\beta_{ab} = \frac{1}{\pi}\int_0^\pi \varphi_{ab}(\theta)\frac{\sin N_r\theta}{\sin\frac{1}{2}\theta}\mathrm{d}\theta. \tag{4.3}$$

Now $\sin N_r\theta/\sin\frac{1}{2}\theta$ for $\theta \in (0,\pi)$ is less in absolute value than π/θ. Therefore, by Lemma 4.1

$$\frac{1}{\pi}\left|\int_{r^{-1}\log r}^{\pi} \varphi_{ab}(\theta)\frac{\sin N_r\theta}{\sin\frac{1}{2}\theta}\mathrm{d}\theta\right| \leqq \delta(r)\log\frac{\pi r}{\log r} \to 0 \quad \text{as} \quad r\to+\infty. \tag{4.4}$$

Since $|N_r - r^\beta| \leqq 1$ for all r,

$$|\sin(N_r\theta/r^\beta) - \sin\theta| = 2|\cos\tfrac{1}{2}(N_r/r^\beta+1)\theta \sin\tfrac{1}{2}(N_r/r^\beta-1)\theta|$$
$$\leqq 2\sin(\theta/2r^\beta) \quad \text{for} \quad \theta\in(0,\pi),$$

and in this range we also have

$$0 \leqq \frac{1}{2r^\beta\sin(\theta/2r^\beta)} - \frac{1}{\theta} \leqq \frac{\pi\theta}{48r^{2\beta}}.$$

If $\beta = 1+\alpha$

$$\int_0^{r^{-1}\log r} \varphi_{ab}(\theta)\frac{\sin N_r\theta}{\sin\frac{1}{2}\theta}\mathrm{d}\theta = r^{-\beta}\int_0^{r^\alpha\log r}\varphi_{ab}(\theta/r^\beta)\frac{\sin(N_r\theta/r^\beta)}{\sin(\theta/2r^\beta)}\mathrm{d}\theta$$

and if we write

$$\frac{\sin(N_r\,\theta/r^\beta)}{\sin(\theta/2\,r^\beta)} = \frac{\sin\theta}{\theta} + \frac{\sin(N_r\,\theta/r^\beta) - \sin\theta}{2\,r^\beta\sin(\theta/2\,r^\beta)} + \sin\theta\left\{\frac{1}{2r^\beta\sin(\theta/2r^\beta)} - \frac{1}{\theta}\right\}$$

the above estimates show that

$$\lim_{r\to+\infty}\left\{\int_0^{r^{-1}\log r}\varphi_{ab}(\theta)\,\frac{\sin N_r\,\theta}{\sin\frac{1}{2}\theta}\,\mathrm{d}\theta - 2\int_0^{r^\alpha\log r}\varphi_{ab}(\theta/r^\beta)\,\frac{\sin\theta}{\theta}\,d\theta\right\} = 0\,. \tag{4.5}$$

Now take any $R > 1$ and recall that $0 \leqq \varphi_{ab}(\theta) \leqq 1$ for all θ (Corollary to Lemma 2.11) and that $\lim\limits_{r\to+\infty}\varphi_{ab}(\theta/r^\beta)$ is $1 - 1/\beta$ for $\beta > 1$ and $\theta > 0$ [(3.1)]. Then, by the theorem of dominated convergence,

$$\lim_{r\to+\infty}\int_0^R\varphi_{ab}(\theta/r^\beta)\,\frac{\sin\theta}{\theta}\,\mathrm{d}\theta = (1 - 1/\beta)\int_0^R\frac{\sin\theta}{\theta}\,\mathrm{d}\theta\,,$$

and since

$$\int_0^\infty\frac{\sin\theta}{\theta}\,\mathrm{d}\theta = \frac{1}{2}\,\pi\,,$$

$$\lim_{R\to+\infty}\ \lim_{r\to+\infty}\left\{\int_0^R\varphi_{ab}(\theta/r^\beta)\,\frac{\sin\theta}{\theta}\,\mathrm{d}\theta - \frac{1}{2}\,\pi\,(1 - 1/\beta)\right\} = 0\,. \tag{4.6}$$

Since

$$\left|\int_A^B\frac{\sin\theta}{\theta}\,\mathrm{d}\theta\right| \leqq 2/A \quad\text{for}\quad \forall B > A > 0\,,$$

it follows from the second mean value theorem for functions of bounded variation, [4, p. 570], that for each R and all large enough r

$$\left|\int_R^{r^\alpha\log r}\varphi_{ab}(\theta/r^\beta)\,\frac{\sin\theta}{\theta}\,\mathrm{d}\theta\right| \leqq 2/R\,\{\varphi_{ab}(R/r^\beta) + V_{ab}^\beta\}$$

so that, by Lemma 4.1,

$$\lim_{R\to+\infty}\ \limsup_{r\to+\infty}\left|\int_R^{r^\alpha\log r}\varphi_{ab}(\theta/r^\beta)\,\frac{\sin\theta}{\theta}\,\mathrm{d}\theta\right| = 0\,. \tag{4.7}$$

It follows from (4.6) and (4.7) that

$$\lim_{R\to+\infty}\ \limsup_{r\to+\infty}\left|\int_0^{r^\alpha\log r}\varphi_{ab}(\theta/r^\beta)\,\frac{\sin\theta}{\theta}\,\mathrm{d}\theta - \frac{1}{2}\,\pi\,(1 - 1/\beta)\right| = 0\,. \tag{4.8}$$

Since the left hand side of (4.8) is independent of R, (4.8) implies

$$\lim_{r\to+\infty}\int_0^{r^\alpha\log r}\varphi_{ab}(\theta/r^\beta)\,\frac{\sin\theta}{\theta}\,\mathrm{d}\theta = \frac{1}{2}\,\pi\,(1 - 1/\beta)\,. \tag{4.9}$$

This, together with (4.4) and (4.5), says that $\lim\limits_{r\to+\infty} L_{ab}^\beta = 1 - 1/\beta$ so that (3.3) holds for $\beta > 1$.

However, if $\beta \leqq 1$ and $\varepsilon > 0$, $0 \leqq L_{ab}^\beta \leqq L_{ab}^{1+\varepsilon}$, whence $0 \leqq \liminf\limits_{r\to+\infty} L_{ab}^\beta \leqq \limsup\limits_{r\to+\infty} L_{ab}^\beta \leqq \varepsilon/(1+\varepsilon)$ for every $\varepsilon > 0$, so that $\lim\limits_{r\to+\infty} L_{ab}^\beta = 0$, (3.3) holds for $\beta \leqq 1$, and Theorem III is established.

§ 5

An alternative approach to this problem is to investigate the time at which the random walk first hits the axis, and then make deductions about the position of the first hit. This programme has been carried out by RIDLER-ROWE [6], and he shows that Theorem III actually holds for a wide class of 3-dimensional random walks.

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