Z. Wahrscheinlichkeitstheorie verw. Geb. 5, 118—138 (1966)

Potential Theory and Non-Markovian Chains*

DONALD A. DAWSON

Received December 22, 1964

1. Introduction

In recent years many deep properties of Markov chains have been investigated by FELLER [6], DOOB [2], KEMENY and SNELL [9] and many others. Hitting probabilities for sets, expected sojourn times for sets, recurrent and transient behavior, and the classification of chains have all been studied intensively. It has been shown that these questions are closely related to potential theoretic considerations. Although very general studies of potential theory have been carried out for Markov chains and processes relatively little has been done for non-Markovian chains or processes.

In this paper we investigate generalizations of certain potential theoretic concepts which have somewhat the same relationship to non-Markovian chains as do the corresponding concepts to Markov chains. In particular we consider superharmonic functions, bounded harmonic functions and boundaries in order to investigate some aspects of the tail field of a non-Markovian chain. Needless to say the non-Markovian situation is more complex but many of the basic structural properties are preserved.

2. Notation for Stochastic Chains

Let Q be a countable space and $\mathfrak{B}$ the σ-field of subsets of Q. Let Ω designate the space of paths with a discrete time parameter, that is, Ω is the set of mappings, $w(\cdot)$, from Z^+ to Q where $Z^+ \equiv \{0, 1, 2, \ldots\}$. Let $\mathfrak{F}^{r,\infty}$ be the σ-field generated by sets of the form

$$\{w: w(n) \in B\}, \quad n \geqq r, \ B \in \mathfrak{B},$$

and let $\mathfrak{F}^{r,m}$ be the σ-field generated by sets of the form

$$\{w: w(n) \in B\}, \quad r \leqq n \leqq m \text{ and } B \in \mathfrak{B}.$$

The σ-field $\mathfrak{F}^* \equiv \bigcap_{n=1}^{\infty} \mathfrak{F}^{n,\infty}$ is known as the *tail field.*

A *stochastic chain* is a triple $(\Omega, \mathfrak{F}^{0,\infty}, P)$ where P is a probability measure on the measure space $(\Omega, \mathfrak{F}^{0,\infty})$. The coordinate functions $X_n(w) = w(n)$ are then a sequence of random variables on the space $(\Omega, \mathfrak{F}^{0,\infty}, P)$. According to DOOB [3, p. 31] we may choose a version of the conditional probabilities $P(B \mid \mathfrak{F}^{r,r+p})$, $B \in \mathfrak{F}^{r+p+1,N}$, such that for each w, $P(\cdot \mid \mathfrak{F}^{r,r+p})(w)$ is a probability measure on $\mathfrak{F}^{r+p+1,N}$, $N \geqq r+p+1$.

---

* This research was supported in part by the Summer Research Institute of the Canadian Mathematical Congress.

Moreover it can be shown that these probability measures may be chosen to be consistent for different N and hence the measures may be extended to $\mathfrak{F}^{r+P+1,\infty}$.

The probability measure μ on $(Q, \mathfrak{B})$ defined by

$$\mu(x_0) \equiv P(\{w : X_0(w) = x_0\})$$

is the initial distribution. We assume that the support of μ is the whole space Q.

The shift operator θ is a mapping from Ω into itself defined by

$$(\theta\, w)(n) = w(n+1).$$

We assume that all subfields of $\mathfrak{F}^{0,\infty}$ are completed with respect to P and let $\mathfrak{Z}$ be the sub-σ-field of $\mathfrak{F}^{0,\infty}$ of sets of P-measure zero.

Given a σ-field $\mathfrak{F} \subset \mathfrak{F}^{0,\infty}$ we designate by $L_\infty(\mathfrak{F})$ the class of bounded real valued functions measurable with respect to $\mathfrak{F}$ and by $\mathfrak{L}_\infty(\mathfrak{F})$ the quotient space $L_\infty(\mathfrak{F})/L_\infty(\mathfrak{Z} \cap \mathfrak{F})$. θ induces a mapping

$$\theta : L_\infty(\mathfrak{F}^{0,\infty}) \to L_\infty(\mathfrak{F}^{0,\infty})$$

defined by

$$(\theta\, f)(w) = f(\theta\, w)$$

and a mapping

$$\theta : \mathfrak{L}_\infty(\mathfrak{F}^{0,\infty}) \to \mathfrak{L}_\infty(\mathfrak{F}^{0,\infty})$$

defined by

$$\theta[f] = [\theta\, f],$$

where $[f]$ stands for the equivalence class of f. Note that the latter definition is possible only if $f = g$ a.e. implies that $\theta\, f = \theta\, g$ a.e. Finally, there is a mapping

$$\theta : \mathfrak{F}^{0,\infty} \to \mathfrak{F}^{0,\infty}$$

defined by

$$\theta\, A = A'$$

where $A' \equiv \{w : \theta\, \chi_A(w) = 1\}$. It is easy to verify that $w \in \theta\, A$ if and only if $\theta\, w \in A$, that is,

$$\theta(\{w : X_n(w) \in A\}) = \{w : X_{n+1}(w) \in A\}$$

and that θ is a σ-field isomorphism. Note that if f and g are measurable with respect to $\mathfrak{F}^{0,\infty}$ and $A = \{w : f(w) = g(w)\}$, then $\{w : f(\theta\, w) = g(\theta\, w)\} = \theta\, A$.

3. Invariant and Superinvariant Functions

In this section we introduce invariant functions, superinvariant functions and potentials.

If f is a real-valued function measurable with respect to $\mathfrak{F}^{0,\infty}$, the *k-potential* of f is defined by

$$N_k\, f(w) \equiv \sum_{i=0}^{\infty} f(\theta^{ik}\, w), \quad k = 1, 2, 3, \ldots,$$

whenever the sum is absolutely convergent a.e. If $f(w) = \chi_\Delta(X_0(w))$, $\Delta \in Q$, then $N_1 f(w)$, if it exists, is the number of values of n for which $X_n(w)$ lies in Δ.

A set $\Delta \in \mathfrak{F}^{0,\infty}$ is *k-negligible* if $\bigcup_{i=0}^{\infty} \theta^{ik}\Delta$ has measure zero, that is, if $N_k(\chi_\Delta(w)) = 0$ a.e.

A real-valued function, f, is *n-invariant* if it is measurable with respect to $\mathfrak{F}^{0,\infty}$ and if $\{w : f(w) \neq f(\theta^n w)\}$ is an n-negligible set. The class of n-invariant functions is denoted by I^n and the class of equivalence classes of I^n is denoted by $[I^n]$.

Proposition 3.1. *If $f \in I^n$, then $f \in I^{nm}$ for every positive integer m.*

Proof.

$$\Delta' = \{ : f(w) \neq f(\theta^{nm} w)\}$$
$$\subset \bigcup_{i=0}^{m-1} \{w : f(\theta^{in} w) \neq f(\theta^{(i+1)n} w)\}.$$

Hence

$$\Delta' \subset \bigcup_{i=0}^{\infty} \theta^{in} \Delta = \Delta'',$$

where $\Delta = \{w : f(w) \neq f(\theta^n w)\}$. However by assumption Δ'' is a set of measure zero and $\theta^{imn}\Delta' \subset \theta^{imn}\Delta'' \subset \Delta''$ for $i = 0, 1, 2, \ldots$, and hence, Δ' is a nm-negligible set. Q.e.d.

This proposition implies that any function $f \in I^p$ is measurable with respect to $\mathfrak{F}^{n,\infty}$ for every positive integer n. Thus functions in I^p, $p = 1, 2, 3, \ldots$, are measurable with respect to the tail field $\mathfrak{F}^*$. The sub-σ-field of $\mathfrak{F}^*$ generated by the functions in I^n is designated by $\mathfrak{F}^{n*}$.

A real-valued function measurable with respect to $\mathfrak{F}^{0,\infty}$ is *n-superinvariant* if $\{w : f(w) < f(\theta^n w)\}$ is a n-negligible set. The class of n-superinvariant functions is designated by I_n'. It is easy to verify that the n-potential of a non-negative function, if it exists, is a n-superinvariant function.

In general if a set $\Delta \in \mathfrak{F}^* \cap \mathfrak{Z}$ and $\theta^k \Delta \subset \Delta$, then Δ is a k-negligible set. In a few places we need to assume that if $\Delta \in \mathfrak{F}^* \cap \mathfrak{Z}$ then $\theta \Delta \in \mathfrak{Z}$. A stochastic chain which has this property is said to have a *non-singular tail field.*

We complete this section with a result that closely resembles the Riesz decomposition theorem for ordinary superharmonic functions [2].

Proposition 3.2. *If f is a k-superinvariant function bounded from below, then it can be decomposed uniquely as*

$$f = f^\infty + N_k f^*,$$

where f^∞ is a k-invariant function and $f^* \geqq 0$. f^∞ is actually the largest k-invariant minorant of f.

Proof. If f is a k-superinvariant function, then an argument similar to that used in Proposition 2.1 shows that

$$f(w) \geqq \theta^k f(w) \geqq \theta^{2k} f(w) \geqq \cdots$$

except for a k-negligible set. The limit, f_∞, of this monotone decreasing sequence of functions is k-invariant. Then

$$\begin{aligned} f(w) &= f(w) - \theta^k f(w) + \theta^k f(w) - \theta^{2k} f(w) + \cdots \\ &\qquad + \theta^{nk} f(w) - \theta^{(n+1)k} f(w) + \theta^{(n+1)k} f(w) \\ &= \sum_{i=0}^{n} \theta^{ik} (f(w) - \theta^k f(w)) + \theta^{(n+1)k} f(w). \end{aligned}$$

Hence letting $n \to \infty$,

$$f(w) = \sum_{i=0}^{\infty} \theta^{ik} (f(w) - \theta^k f(w)) + f^\infty(w)$$

except for a k-negligible set. Moreover $f^*(w) = f(w) - \theta^k f(w) \geqq 0$ except for a k-negligible set.

We next show that f^∞ is the largest k-invariant minorant of f. In fact if g is a k-invariant function and $f \geqq g$, then

$$f - g \geqq 0$$

and therefore

$$\theta^{ik}(f - g) \geqq 0$$

which implies that

$$f^\infty - g \geqq 0.$$

If $f = f_1^\infty + N_k f_1^*, f_1^* \geqq 0, f_1^\infty \in I^k$, is another decomposition of f we have

$$\theta^{ik} f = f_1^\infty + \theta^{ik} N_k f_1^*.$$

Then letting $i \to \infty$ we obtain

$$f_1^\infty(w) = f^\infty(w)$$

except for a k-negligible set and therefore the decomposition is unique. Q.e.d.

This means that the study of superinvariant functions for stochastic chains can be broken down into the study of potentials and invariant functions. In the remainder of this paper we will restrict our attention to the invariant functions.

4. Invariant Functions and Harmonic Functions

In this section we study certain subalgebras of the invariant functions. The functions in these subalgebras have some properties in common with the ordinary harmonic functions and are used in Section 6 to obtain a Feller boundary.

In section 5 it is noted that in the case of a Markov chain $\{X_n\}$ the set of harmonic functions is isomorphic to the set of invariant functions. However, for a Markov chain the harmonic functions have another crucial property, namely, if f is a bounded harmonic function then $f(X_n)$ is a martingale and hence converges a.e. as $n \to \infty$. In general this is no longer true but this idea will provide us with a natural way of decomposing the algebra of invariant functions.

For a given positive integer p let $M(Q^p)$ denote the set of bounded real valued functions on Q^p. Let $f \in M(Q^p)$ and consider the sequence of functions on Ω defined by $f_n(w) = \theta^{np} f(X_0(w), \ldots, X_{p-1}(w))$. If $f_n(w)$ converges a.e. to $\tilde{f}(w)$, we write $f \in \hat{M}(Q^p)$. Then $\tilde{f}(w)$ is a p-invariant function. We denote the correspondence between functions $\tilde{f} \in \hat{M}(Q^p)$ and $f \in I^p$ by $f \rightsquigarrow \tilde{f}$. A function in the class

$$I_p^p \equiv \{\tilde{f} : \tilde{f} \in I^p, f \rightsquigarrow \tilde{f}, f \in \hat{M}(Q^p)\}, \quad p = 1, 2, 3, \ldots,$$

is called a *p-harmonic* function.

In this paper the p-harmonic functions, $p = 1, 2, 3, \ldots$, play the role for stochastic chains which the bounded harmonic functions play for Markov chains.

We introduce the following norms in $M(Q^p)$ and $L_\infty(\mathfrak{F}^{0,\infty})$;

$$\|f\| = \sup_{x \in Q^p} |f(x)| \quad \text{for} \quad f \in M(Q^p),$$

and

$$\|\tilde{f}\|_\infty = \inf\{\lambda : |\tilde{f}(x)| \leqq \lambda \text{ a.e.}\}.$$

Two functions f and g in $M(Q^p)$ are *equivalent*, $f \sim g$, if $f \rightsquigarrow \tilde{f}$ and $g \rightsquigarrow \tilde{f}$. This relation is clearly an equivalence relation on $\hat{M}(Q^p)$. It is clear that $f \rightsquigarrow \tilde{f}$ induces an isomorphism between $[\hat{M}(Q^p)]$ and $[I_p^p]$ where as usual $[\cdot]$ designates the set of equivalence classes. In order to simplify notation we write $f(w)$ for

$$f(X_0(w), \dots, X_{p-1}(w)) \quad \text{when} \quad f \in M(Q^p).$$

Proposition 4.1. *I_p^p is a vector lattice containing the constants.*

Proof. If $f \rightsquigarrow \tilde{f}$ and α is a real constant, then $\alpha f \rightsquigarrow \alpha \tilde{f}$. Since $1 \rightsquigarrow 1$ it follows that I_p^p contains the constants. Moreover I_p^p is a vector space since if $f_1 \rightsquigarrow \tilde{f}_1$ and $f_2 \rightsquigarrow \tilde{f}_2$ then $f_1 + f_2 \rightsquigarrow \tilde{f}_1 + \tilde{f}_2$.

We adopt the following notation:

$$f_1 \wedge f_2(x) = \min(f_1(x), f_2(x)),$$
$$f_1 \vee f_2(x) = \max(f_1(x), f_2(x)).$$

Let us now show that if $f_1 \rightsquigarrow \tilde{f}_1$ and $f_2 \rightsquigarrow \tilde{f}_2$, then $f_1 \wedge f_2 \rightsquigarrow \tilde{f}_1 \wedge \tilde{f}_2$. If for example $\tilde{f}_2(w) > \tilde{f}_1(w)$, then for sufficiently large n, $\theta^{np} f_2(w) > \theta^{np} f_1(w)$ and hence $\theta^{np}(f_1 \wedge f_2)(w) = \theta^{np} f_1(w) \to \tilde{f}_1(w) = \tilde{f}_1(w) \wedge \tilde{f}_2(w)$.
Moreover if $\tilde{f}_1(w) = \tilde{f}_2(w)$, then it is obvious that

$$\theta^{np}(f_1 \wedge f_2)(w) \to \tilde{f}_1(w) \wedge \tilde{f}_2(w).$$

A similar result holds for $\tilde{f}_1 \vee \tilde{f}_2$. Q.e.d.

Proposition 4.2.

(i) *If $\tilde{f} \in I_p^p$ and $\|\tilde{f}\|_\infty = K$, then there is a function $f \in \hat{M}(Q^p)$ such that $\|f\| = K$ and $f \rightsquigarrow \tilde{f}$.*

(ii) $\inf\{\|g\| : g \rightsquigarrow \tilde{f}\} = K$.

Proof. (i) If $\tilde{f}(w) < K$ and $g \rightsquigarrow \tilde{f}$, then $\theta^{np} g(w) \to \tilde{f}(w)$ and therefore for sufficiently large n, $\theta^{np} g(w) < K$. It follows that $(\theta^{np} g \wedge K)(w) \to \tilde{f}(w)$. Moreover if $\theta^{np} g(w) \to K$, that is, $\tilde{f}(w) = K$, then clearly $(\theta^{np} g \wedge K)(w) \to K$. Hence for all paths w for which $\tilde{f}(w) \leqq K$, $(\theta^{np} g \wedge K)(w) \to \tilde{f}(w)$. Since the set of paths for which $\tilde{f}(w) > K$ has measure zero, $g \wedge K \rightsquigarrow \tilde{f}$. Similarly $g \wedge K \vee -K \rightsquigarrow \tilde{f}$ which yields the result.

(ii) Assume the contrary, that is, $g \rightsquigarrow \tilde{f}$ with $\|g\| = K - \varepsilon$ for some $\varepsilon > 0$. Then $\theta^{np} g(w) \to \tilde{f}(w) \leqq K - \varepsilon$ for all but a null set of w which contradicts the definition of $\|\tilde{f}\|_\infty$. Q.e.d.

Proposition 4.3. *I_p^p is an algebra of functions.*

Proof. Let $f \rightsquigarrow \tilde{f}$, $g \rightsquigarrow \tilde{g}$ with $\|\tilde{f}\|_\infty \leqq K$ and $\|\tilde{g}\|_\infty \leqq K$. By Proposition 4.2 we can assume that $\|g\| \leqq K$ and $\|f\| \leqq K$. Given $\varepsilon > 0$ there is an $N(w)$ (for all but a null set of w) such that for $n \geqq N(w)$, $|\theta^{np} f(w) - \tilde{f}(w)| < \varepsilon$ and $|\theta^{np} g(w) - \tilde{g}(w)| < \varepsilon$. Then for $n \geqq N(w)$,

$$\begin{aligned}
&|\theta^{np} f(w)\, g(w) - \tilde{f}(w)\, \tilde{g}(w)| \\
&\quad \leqq |\theta^{np} f(w) \cdot \theta^{np} g(w) - \theta^{np} g(w) \cdot \tilde{f}(w) \\
&\qquad + \theta^{np} g(w) \cdot \tilde{f}(w) - \tilde{f}(w) \cdot \tilde{g}(w)| \\
&\quad \leqq |\theta^{np} g(w)| \cdot |\theta^{np} f(w) - \tilde{f}(w)| + |\tilde{f}(w)| \cdot |\theta^{np} g(w) - \tilde{g}(w)| \\
&\quad \leqq 2K\, \varepsilon.
\end{aligned}$$

Hence $(\theta^{np} f \cdot g)(w) \to \tilde{f}(w) \cdot \tilde{g}(w)$ except for a null set of w. Therefore $f \cdot g \rightsquigarrow \tilde{f} \cdot \tilde{g}$. Q.e.d.

Proposition 4.4. *Let Δ be a subset of Q^p. Then $\chi_\Delta \in \hat{M}(Q^p)$ if and only if the set*

$$A \equiv \{w : (X_{rp}(w), \ldots, X_{(r+1)p-1}(w)) \in \Delta \; i.o. \; (r) \text{ and} \in Q^p - \Delta \; i.o. \; (r)\}$$

has measure zero. The notation i.o. (r) stands for „infinitely often in r".

Proof. $\theta^{np} \chi_\Delta(w)$ converges if and only if $\theta^{np} w$ visits only one of the two sets Δ and $Q^p - \Delta$ infinitely often. Hence $\theta^{np} \chi_\Delta(w)$ converges a.e. if and only if A has measure zero. Q.e.d.

Let $\mathfrak{C}_p$ be the class of sets $\Delta \subset Q^p$ such that $\chi_\Delta \in \hat{M}(Q^p)$. The following result is immediate.

Proposition 4.5. *Let $\Delta \in \mathfrak{C}_p$. Then $\chi_\Delta \rightsquigarrow 0$ if and only if $\{w : (X_{rp}(w), \ldots, X_{(r+1)p-1}(w)) \in \Delta$ for all sufficiently large $r\}$ has measure zero.*

The class of *transient sets*, $\mathfrak{Z}_p$, is the class of sets $\Delta \subset Q^p$ such that $\chi_\Delta \rightsquigarrow 0$.

Proposition 4.6. *If $\tilde{\Delta} \in \mathfrak{J}^{p*}$ and $\chi_{\tilde{\Delta}} \in I_p^p$, then there is a set $\Delta \in \mathfrak{C}_p$ such that $\chi_\Delta \rightsquigarrow \chi_{\tilde{\Delta}}$.*

Proof. Since $\chi_{\tilde{\Delta}} \in I_p^p$ there is a function $f \in \hat{M}(Q^p)$ such that $f \rightsquigarrow \chi_{\tilde{\Delta}}$. We will next show that if $\Delta \equiv \{x : f(x) > \varepsilon\}$ for some $0 < \varepsilon < 1$, then $\chi_\Delta \rightsquigarrow \chi_{\tilde{\Delta}}$. If $\theta^{np} f(w) \to 0$, then for sufficiently large n $\theta^{np} f(w) < \varepsilon$ and therefore $\theta^{np} \chi_\Delta(w) \to 0$. On the other hand if $\theta^{np} f(w) \to 1$, then for sufficiently large n $\theta^{np} f(w) > \varepsilon$ and then $\theta^{np} \chi_\Delta(w) \to 1$. Q.e.d.

Proposition 4.7. *If $\chi_{\Delta_1} \rightsquigarrow \chi_{\tilde{\Delta}_1}$ and $\chi_{\Delta_2} \rightsquigarrow \chi_{\tilde{\Delta}_2}$, then $\chi_{\Delta_1 \cap \Delta_2} \rightsquigarrow \chi_{\tilde{\Delta}_1 \cap \tilde{\Delta}_2}$.*

Proof. This follows immediately from the proof of Proposition 4.3.

Note that since I_p^p is algebra which contains the constants the set of $\tilde{\Delta}$ such that $\chi_{\tilde{\Delta}} \in I_p^p$ forms an algebra of subsets of $\mathfrak{J}^{p*}$. In Proposition 4.11 we prove that this algebra of subsets is actually a σ-algebra if the stochastic chain is such that all finite subsets of Q^p belong to $\mathfrak{Z}_p$. A stochastic chain having the latter for some p must have it for all p and is called a *transient chain.*

Proposition 4.8. *If the chain is transient and $f \rightsquigarrow \tilde{f}$, there is a sequence of functions $f_i \in \hat{M}(Q^p)$ such that $f_i \rightsquigarrow \tilde{f}$ and such that* $\operatorname{Spt}(f_i) \searrow \varphi$ *where* $\operatorname{Spt}(f_i)$ *stands for the support of f_i.*

Proof. This follows immediately since if we define $f_i = f$ for all but i given points in $\operatorname{Spt}(f)$ and $= 0$ at the i given points then $f_i \rightsquigarrow \tilde{f}$ and $\operatorname{Spt}(f_i) \searrow \varphi$.

Proposition 4.9. *If $\tilde{\Delta}_1 \cap \tilde{\Delta}_2 = \varphi$, $\chi_{\Delta_1} \rightsquigarrow \chi_{\tilde{\Delta}_1}$ and $\chi_{\Delta_2} \rightsquigarrow \chi_{\tilde{\Delta}_2}$, then $\chi_{\Delta_2 - \Delta_1} \rightsquigarrow \chi_{\tilde{\Delta}_2}$. In other words Δ_1 and Δ_2 can be chosen so that $\Delta_1 \cap \Delta_2 = \varphi$.*

Proof. This follows immediately since the set of paths which visit both Δ_1 and Δ_2 infinitely often must form a p-negligible set.

Proposition 4.10. *Assume that the stochastic chain is transient and let $\tilde{f}$ be a function in I_p^p such that $|\tilde{f}(w)| \geqq \varepsilon$ or $\tilde{f}(w) = 0$ except for a p-negligible set of path. Then for any $\eta > 0$ there is an $f \rightsquigarrow \tilde{f}$ such that the set $\{w : \theta^{np} w \in \operatorname{Spt}(f)$ for some n, $\theta^{np} \chi_{\operatorname{Spt}(f)}(w) \not\to 1\}$ has measure less than η.*

Proof. An argument similar to that used in Proposition 4.6 shows that f can be chosen so that either $|f(x)| \geqq \varepsilon$ or $f(x) = 0$. But then the set of paths which visit both $\operatorname{Spt}(f)$ and $\{x : f(x) = 0\}$ infinitely often has measure zero.

By Proposition 4.8 there is a sequence $f_i \rightsquigarrow \tilde{f}$ such that $f_1 = f$, $\mathrm{Spt}(f_i) = A_i \subset \mathrm{Spt}(f)$ and such that $A_i \searrow \varphi$. Let

$$B_J = \{w : w \text{ visits } A_j \text{ for some } j \geqq J, \chi_{A_j}(w) \nrightarrow 1\}.$$

But

$$B_J \searrow B_\infty = \{w : w \text{ visits infinitely many } A_j, \chi_{A_j}(w) \nrightarrow 1 \text{ for any } j\}$$
$$\subset \{w : w \text{ visits } \mathrm{Spt}(f) \text{ i.o.}, \chi_{\mathrm{Spt}(f)}(w) \nrightarrow 1\}.$$

Hence $P(B_J) \searrow 0$ and therefore there is some J such that $P(B_J) < \eta$. Then

$$P(\{w : w \text{ visits } A_J, \chi_{A_J}(w) \nrightarrow 1\}) < \eta$$

so that f_J satisfies the requirement of the proposition. Q.e.d.

Proposition 4.11. *If the stochastic chain is transient, then* $\mathfrak{S}_p^{p*} \equiv \{\tilde{\Delta} : \tilde{\Delta} \in \mathfrak{S}^{p*}, \chi_{\tilde{\Delta}} \in I_p^p\}$ *is a σ-field.*

Proof. It has already been shown that $\mathfrak{S}_p^{p*}$ is an algebra of sets. Hence it suffices to show that if $\tilde{\Delta}_i \in \mathfrak{S}_p^{p*}$, $i = 1, 2, 3, \ldots$, and $\tilde{\Delta}_i \cap \tilde{\Delta}_j = \varphi$ for $i \neq j$ then $\bigcup_{i=0}^{\infty} \tilde{\Delta}_i \in \mathfrak{S}_p^{p*}$.

By Propositions 4.6 and 4.10 there are sets $\Delta_i \in Q^p$ such that $\chi_{\Delta_i} \rightsquigarrow \chi_{\tilde{\Delta}_i}$ and such that

$$P(\{w : w \text{ visits } \Delta_i, \chi_{\Delta_i}(w) \nrightarrow 1\}) < 2^{-i}.$$

It will suffice to show if the Δ_i are chosen so that the above applies, then $\sum_{i=0}^{\infty} \chi_{\Delta_i} \rightsquigarrow \sum_{i=0}^{\infty} \chi_{\tilde{\Delta}_i}$. If $\chi_{\tilde{\Delta}_i}(w) = 1$, $\chi_{\Delta_j}(w) \to 0$ for $i \neq j$ except for a p-negligible set of w. Hence it suffices to show that if

$$\chi_{\tilde{\Delta}_j}(w) = 0 \quad \text{for} \quad i \neq j, \quad \text{then} \quad \theta^{np} \sum_{j \neq i} \chi_{\Delta_j}(w) \to 0$$

except for a p-negligible set of paths.

But

$$\{w : \chi_{\tilde{\Delta}_j}(w) = 0 \quad \text{for} \quad j \neq i, \theta^{np} \sum_{j \neq i} \chi_{\Delta_j}(w) \nrightarrow 0\}$$
$$= \{w : w \text{ visits } \Delta_j \text{ i.o.}(j), \chi_{\Delta_j}(w) \to 0 \text{ for each } j \neq i\}$$
$$\subset \limsup_{j \to \infty} A_j,$$

where A_j is the event that w visits Δ_j but does not eventually remain in Δ_j. But by the choice of the sets Δ_j,

$$\sum_{j \neq i} P(A_j) < \infty$$

and therefore by the Borel Cantelli lemma,

$$P(\{\limsup_{j \to \infty} A_j\}) = 0.$$

But since $\theta^p\{\limsup_{j \to \infty} A_j\} = \limsup_{j \to \infty} A_j$, this implies that it is p-negligible.

Therefore $\sum_{j=0}^{\infty} \chi_{\Delta_j}(w) \to 1$ if and only if $\chi_{\Delta_j}(w) \to 1$ for some j except for a p-negligible set of w. Q.e.d.

Corollary. *If* $\tilde{\Delta}_j \in \mathfrak{F}_p^{p*}$ *and* $\tilde{\Delta}_i \cap \tilde{\Delta}_j = \varphi$ *for* $i \neq j$, *then there are sets* Δ_j *in* $\mathfrak{C}_p$ *such that* $\Delta_i \cap \Delta_j = \varphi$ *for* $i \neq j$ *and such that* $\chi_{\Delta_i} \rightsquigarrow \chi_{\tilde{\Delta}_i}$.

Proof. Let $\Delta_i' \equiv \Delta_i - \bigcup_{j \neq i} \Delta_j$ where the sets Δ_i have been chosen as in the proof of the proposition. Then it follows easily from the proof of the proposition that $\chi_{\Delta_i'} \rightsquigarrow \chi_{\tilde{\Delta}_i}$.

Proposition 4.12. *If* $\tilde{f} \in I_p^p$, *then* $\tilde{f}$ *is measurable with respect to the σ-field* $\mathfrak{F}_p^{p*}$.

Proof. It suffices to show that $\{w : \tilde{f}(w) > a\} \in \mathfrak{F}_p^{p*}$ for any real number a. Let r_1 and r_2 be any two real numbers such that $r_1 < r_2$ and

$$P(\{w : \tilde{f}(w) = r_1\} \cup \{w : \tilde{f}(w) = r_2\}) = 0.$$

We first prove that $\tilde{A} = \{w : r_1 \leqq \tilde{f}(w) \leqq r_2\} \in \mathfrak{F}_p^{p*}$. If $f \rightsquigarrow \tilde{f}$, let $\Delta = \{x : r_1 \leqq f(x) \leqq r_2\}$. Since $\{w : \theta^{np}\, w \in \Delta$ i.o. (n) and $\theta^{np}\, w \in \Delta^c$ i.o. $(n)\} \subset \{w : \theta^{np} f(w) \to r_1$ or $r_2\} \cup \{w : \theta^{np} f(w)$ does not converge$\}$ which has measure zero, $\chi_\Delta \rightsquigarrow \chi_{\tilde{A}}$ and hence $\tilde{A} \in \mathfrak{F}_p^{p*}$.

Given any real number a there is a countable set of real number r_i, $i = 0, \pm 1, \pm 2, \ldots$, such that

$$r_i < r_{i+1}$$
$$P(w : \tilde{f}(w) = r_i) = 0 \qquad \text{for each } i,$$

and such that

$$\{w : \tilde{f}(w) > a\} = \bigcup_{i=-\infty}^{+\infty} \{w : r_i \leqq \tilde{f}(w) \leqq r_{i+1}\}.$$

Hence $\{w : \tilde{f}(w) > a\} \in \mathfrak{F}_p^{p*}$ since each of the sets on the right hand side belongs to the σ-field $\mathfrak{F}_p^{p*}$. Q.e.d.

Finally we obtain the following result which completes the proof that I_p^p is the set of bounded real functions on Ω which are measurable with respect to $\mathfrak{F}_p^{p*}$, that is, $I_p^p = L_\infty(\mathfrak{F}_p^{p*})$. The equivalence classes $[I_p^p]$ are therefore equal to $\mathfrak{L}_\infty(\mathfrak{F}_p^{p*})$.

Proposition 4.13. *If the stochastic chain is transient and if* $\tilde{f}$ *is a bounded real valued function measurable with respect to* $\mathfrak{F}_p^{p*}$, *then* $\tilde{f} \in I_p^p$.

Proof. Given $\tilde{f}$ there is a monotone increasing sequence of simple functions $\tilde{f}_n \in I_p^p$ such that $\|\tilde{f}_n - \tilde{f}\|_\infty < n^{-1}$. (Recall that a simple function is a finite linear combination of indicator functions of sets.) By Propositions 4.6 and 4.9 there is a monotone increasing sequence of simple functions $f_n \in \hat{M}(Q^p)$ such that $f_n \rightsquigarrow \tilde{f}_n$. If $f = \lim_{n \to \infty} f_n$, then it is easy to verify that $f \rightsquigarrow \tilde{f}$. Q.e.d.

5. Three Examples of Stochastic Chains

In this section the set of invariant functions is studied for three particular types of stochastic chain. The first example provides us with a chain whose invariant field is non-trivial but whose only p-harmonic functions for any finite p are the constants. The second example is that of Markov chains in which case there is an isomorphism between the set of invariant functions and the set of bounded harmonic functions. The last example is that of a chain which satisfies the semi-group property in which case we investigate invariant functions which arise from ergodic averages of functions on the state space.

Example 1. Let $Q = \{0, 1, 2, \ldots, 9\}$ and let $(\Omega, \mathfrak{F}^{0,\infty})$ be defined as usual. The measure P is concentrated on two paths w_1 and w_2, $P(\{w_1\}) = P(\{w_2\}) = 1/2$ where w_1 and w_2 are defined as follows.

$$w_1 = (1, 2, 3, \ldots, 9, 1, 0, 1, 1, 1, 2, 1, 3, \ldots)$$

that is, w_1 is given by the digits in the sequence of positive integers written in decimal notation.

$$w_2 = (2, 1, 4, 3, \ldots)$$

that is, w_2 is given by the digits in the sequence in which the nth even integer is followed by the nth odd integer written in decimal notation.

Since for $\theta^{np} f(X_0(w), \ldots, X_p(w))$ to converge a.e. f must be a constant, I_p^p contains only the constants. However if for $i = 1, 2$,

$$\begin{aligned} f_i(w) &\equiv 1 \quad \text{if} \quad (X_n(w), X_{n+1}(w), \ldots) = (X_m(w_i), X_{m+1}(w_i), \ldots) \\ & \quad \text{for some } m \text{ and } n \\ &\equiv 0 \quad \text{otherwise} \end{aligned}$$

then f_i, $i = 1, 2$, are 1-invariant. Moreover since $f_i(w_j) = \delta_{ij}$, $\{w : f_1(w) = f_2(w)\}$ is a 1-negligible set and therefore I^1 is non-trivial. This provides us with an example of a stochastic chain with non-trivial invariant field but whose p-harmonic functions are the constants for each p.

Note that as it stands this chain has a singular tail field. However it can be changed into a chain with non-singular tail field by modifying P so as to give positive probability to $\theta^n w_i$, $i = 1, 2$, for every positive integer n.

Example 2. Consider a stochastic chain $(\Omega, \mathfrak{F}^{0,\infty}, P)$ for which Q is countable. The chain is said to have *stationary transitions* if

$$\theta P(B \mid \mathfrak{F}^{m,n}) = P(\theta B \mid \mathfrak{F}^{m+1,n+1}), \quad n \geq m, \quad B \in \mathfrak{F}^{n+1,\infty}.$$

The chain is said to be a *Markov chain of order* N if $P(B \mid \mathfrak{F}^{0,n})$ is measurable with respect to $\mathfrak{F}^{(n-N)\vee 0,n}$ for any $B \in \mathfrak{F}^{n+1,\infty}$. A Markov chain of order zero is simply called a Markov chain. It is well known that a Markov chain of order N is defined by a "transition matrix" $\hat{P}$ on Q^N. (See the discussion of transition functions in example 3.)

Proposition 5.1. For a Markov chain of order N,

(i) $I_N^N = I^N$ and

(ii) $[I_N^N]$ is isomorphic to $H^N = \{f : f \in M(Q^N), \hat{P} f = f\}$, that is, there is a one to one linear, order preserving mapping of $[I_N^N]$ onto H^N. (BLACKWELL [*1*].)

Moreover the mapping from $[I_N^N]$ to H^N is given by

$$f(x_0, \ldots, x_{N-1}) = E(\tilde{f} \mid X_0, \ldots, X_{N-1} = x_{N-1})$$

and the inverse of this mapping is given by $f \rightsquigarrow \tilde{f}$.

Proof. This result is essentially Theorem 2 of BLACKWELL [*1*].

Example 3. FELLER [*7*] and ROSENBLATT and SLEPIAN [*12*] have given examples of non-Markovian chains with the semi-group property, that is, chains which satisfy the Chapman-Kolmogorov equation. In this section we study a generalization of this class of stochastic chain.

Consider a stochastic chain $(\Omega, \mathfrak{F}^{0,\infty}, P)$ with stationary transitions. Let the transition functions $\hat{P}(\cdot\ \cdot)$ be defined by

$$\hat{P}(x_n, \ldots, x_{n-r} \mid x_{n-r-1}, \ldots, x_{n-r-p})$$
$$\equiv P(X_n = x_n, \ldots, X_{n-r} = x_{n-r} \mid X_{n-r-1} = x_{n-r-1}, \ldots, X_{n-r-p} = x_{n-r-p}).$$

Note that the expression on the right hand side of this equation stands for the conditional probability with respect to $\mathfrak{F}^{n-r-p,\,n-r-1}$ evaluated on the $\mathfrak{F}^{n-r-p,\,n-r-1}$-measurable set

$$\{w : X_{n-r-1}(w) = x_{n-r-1}, \ldots, X_{n-r-p}(w) = x_{n-r-p}\}.$$

We also set

$$\mu_p(x_0, \ldots, x_p) = P(X_0 = x_0, \ldots, X_p = x_p)$$
$$= \mu(x_0)\,\hat{P}(x_1 \mid x_0)\,\hat{P}(x_2 \mid x_1, x_0) \ldots \hat{P}(x_p \mid x_{p-1}, \ldots, x_0).$$

Since it has been assumed that the chain has stationary transitions, it follows that

$$\hat{P}(x_n \mid x_{n-1}, \ldots, x_{n-r}) = \Sigma_{x_0} \ldots \Sigma_{x_{n-r-1}} \hat{P}(x_n \mid x_{n-1}, \ldots, x_0) \times$$
$$\times\, \mu_{n-r-1}(x_0, \ldots, x_{n-r-1}).$$

The stochastic chain is said to be *n-multiplicative* if

$$P(X_{(k+1)n} = x_{(k+1)n}, \ldots, X_{kn+1} = x_{kn+1} \mid X_n = x_n, \ldots, X_1 = x_1)$$
$$= \Sigma_{x_{n+1}} \ldots \Sigma_{x_{kn}} \hat{P}(x_{(k+1)n}, \ldots, x_{kn+1} \mid x_{kn}, \ldots, x_{(k-1)n+1}) \ldots$$
$$\hat{P}(x_{2n}, \ldots, x_{n+1} \mid x_n, \ldots, x_1). \tag{5.1}$$

This condition is equivalent to saying that the stochastic chain viewed as a chain on the state space Q^n has the semi-group property or, alternately, satisfies the Chapman-Kolmogorov equation.

In general

$$P(X_{(k+1)n} = x_{(k+1)n}, \ldots, X_{kn+1} = x_{kn+1} \mid X_n = x_n, \ldots, X_1 = x_1)$$
$$= \Sigma_{x_{kn}} \ldots \Sigma_{x_{n+1}} \hat{P}(x_{(k+1)n} \mid x_{(k+1)n-1}, \ldots, x_1) \times$$
$$\times\, \hat{P}(x_{(k+1)n-2} \mid x_{(k+1)n-3}, \ldots, x_1) \ldots \hat{P}(x_{n+1} \mid x_n, \ldots x_1).$$

Hence the condition for n-multiplicativity in terms of the $\hat{P}$ is given by

$$\Sigma_{x_{kn}} \ldots \Sigma_{x_{n+1}} \hat{P}(x_{(k+1)n}, x_{(k+1)n-1}, \ldots, x_{kn+1} \mid x_{kn}, \ldots, x_{(k-1)n+1}) \ldots$$
$$\hat{P}(x_{2n}, \ldots, x_{n+1} \mid x_n, \ldots, x_1)$$
$$= \Sigma_{x_{kn}} \ldots \Sigma_{x_{n+1}} \hat{P}(x_{(k+1)n} \mid x_{(k+1)n-1}, \ldots, x_1) \ldots$$
$$\hat{P}(x_{n+1} \mid x_n, \ldots, x_1).$$

A function $f \in M(Q^p)$ is said to be p-regular, $p = 1, 2, 3, \ldots$, for the stochastic chain if

$$f(x_0, \ldots, x_{p-1}) = \Sigma_{x_p} \ldots \Sigma_{x_{2p-1}} f(x_p, \ldots, x_{2p-1})\, \hat{P}(x_{2p-1}, \ldots, x_p \mid x_{p-1}, \ldots, x_0).$$

The class of p-regular functions is designated by $\mathfrak{R}_p$.

Proposition 5.2. *If the stochastic chain is m-multiplicative and $f \in \mathfrak{R}_m$, then the conditional expectation $E(f(X_{pm}, \ldots, X_{(p+1)m-1}) | \mathfrak{I}^{0,m-1})(w)$ is equal to $f(X_0(w), \ldots, X_{m-1}(w))$.*

Proof. If $X_0(w) = x_0, \ldots, X_{m-1}(w) = x_m$, then

$$\begin{aligned} E(f(X_{pm}, \ldots, X_{(p+1)m-1}) | \mathfrak{I}^{0,m})(w) \\ &= \Sigma_{x_{pm}} \ldots \Sigma_{x_{(p+1)m-1}} f(x_{pm}, \ldots, x_{(p+1)m-1}) \times \\ &\times \hat{P}(x_{(p+1)m-1}, \ldots, x_{pm} | x_{pm-1}, \ldots, x_{(p-1)m}) \ldots \\ &\quad \hat{P}(x_{2m-1}, \ldots, x_m | x_{m-1}, \ldots, x_0) \end{aligned}$$

by the m-multiplicative property (5.1). By using the defining property of $\mathfrak{R}_m$ it follows that

$$E(f(X_{pm}, \ldots, X_{(p+1)m-1}) | \mathfrak{I}_0{}^{,m})(w) = f(x_0, \ldots, x_{m-1}). \quad \text{Q.e.d.}$$

Proposition 5.3. *If the stochastic chain is m-multiplicative, then there is a linear mapping from $[I_m^m]$ into $\mathfrak{R}_m$.*

Proof. Let $\tilde{f} \in I_m^m$. Then there is some $f \in \hat{M}(Q^m)$ such that $f \rightsquigarrow \tilde{f}$, that is, $\theta^{nm} f \to \tilde{f}$ a.e. as $n \to \infty$. But if $n > 2$,

$$E(E(\theta^{nm} f | \mathfrak{I}^{m,2m-1}) | \mathfrak{I}^{0,m-1}) = E(\theta^{nm} f | \mathfrak{I}^{0,m})$$

by the m-multiplicative property. Letting $n \to \infty$ and using the bounded convergence theorem we obtain

$$E(E(\tilde{f} | \mathfrak{I}^{m,2m-1}) | \mathfrak{I}^{0,m-1}) = E(\tilde{f} | \mathfrak{I}^{0,m}).$$

In other words if $f'(x_0, \ldots, x_{m-1}) \equiv E(\tilde{f} | X_0 = x_0, \ldots, X_{m-1} = x_{m-1})$, then f' satisfies the equation

$$f'(x_0, \ldots, x_{m-1}) = \sum f'(x_m, \ldots, x_{2m-1}) \hat{P}(x_{2m-1}, \ldots, x_m | x_{m-1}, \ldots, x_0).$$

Finally it is clear that the mapping $\tilde{f} \to f'$ is linear and also order preserving. Q.e.d.

To gain a little more insight into the role of $\mathfrak{R}_m$ for m-multiplicative chains we now show that $E(\cdot | \mathfrak{I}^{0,m-1})$ maps a subset of I^m which contains I_m^m onto $\mathfrak{R}_m$.

A stochastic chain is said to have the *m-ergodic average property* if for every $f \in M(Q^p)$, $\lim_{N\to\infty} N^{-1} \sum_{n=0}^{N-1} f(\theta^{mn} w)$ exists a.e. If the limit exists it must belong to I^m. If the stochastic chain has the m-ergodic average property, let

$$J_m^m = \{\tilde{f} : \tilde{f} \in I^m, \tilde{f} = \lim_{N\to\infty} N^{-1} \sum_{n=0}^{N-1} f(\theta^{mn} w), f \in M(Q^p)\}.$$

It is easy to verify that

$$I_m^m \subset J_m^m \subset I^m.$$

Proposition 5.4. *If the stochastic chain has the m-ergodic average property, then $E(\cdot | \mathfrak{I}^{0,m-1})$ maps J_m^m onto $\mathfrak{R}_m$.*

Proof. If $f \in \mathfrak{R}_m$, then

$$\lim_{N\to\infty} N^{-1} \sum_{n=0}^{N-1} f(\theta^{mn} w) \to \tilde{f} \text{ a.e.}$$

and $\tilde{f} \in J_m^m$. Moreover using the m-multiplicative property it is easy to verify that

$$E(\tilde{f} \mid X_0 = x_0, \ldots, X_{m-1} = x_{m-1}) = f(x_0, \ldots, x_{m-1})$$

which implies that the mapping is onto.

It remains to show that $E(\cdot \mid \mathfrak{F}^{0, m-1})$ maps every function in J_m^m into $\mathfrak{R}_m$. If $f \in M(Q^m)$ then by the m-ergodic average property

$$\lim_{N \to \infty} N^{-1} \sum_{n=0}^{N-1} f(\theta^{mn} w) \to \tilde{f} \text{ a.e.}$$

and $\tilde{f} \in J_m^m$. But

$$\begin{aligned} &E(E(\tilde{f} \mid \mathfrak{F}^{m, 2m-1}) \mid \mathfrak{F}^{0, m-1}) \\ &\qquad = \lim_{N \to \infty} E(E(N^{-1} \sum_{n=0}^{N-1} f(\theta^{mn} w) \mid \mathfrak{F}^{m, 2m-1}) \mid \mathfrak{F}^{0, m-1}) \end{aligned}$$

and

$$\begin{aligned} &E(E(N^{-1} \sum_{n=0}^{N-1} f(\theta^{mn} w) \mid \mathfrak{F}^{m, 2m-1}) \mid \mathfrak{F}^{0, m-1}) \\ &\quad = E(N^{-1} \sum_{n=1}^{N-1} f(\theta^{mn} w) \mid \mathfrak{F}^{0, m-1}) + N^{-1} E(E(f(w) \mid \mathfrak{F}^{m, 2m-1}) \mid \mathfrak{F}^{0, m-1}) \end{aligned}$$

by the m-multiplicative property. Letting $N \to \infty$ it follows that

$$E(E(\tilde{f} \mid \mathfrak{F}^{m, 2m-1}) \mid \mathfrak{F}^{0, m-1}) = E(\tilde{f} \mid \mathfrak{F}^{0, m-1}).$$

Hence if $f'(x_0, \ldots, x_{m-1}) \equiv E(\tilde{f} \mid X_0 = x_0, \ldots, X_{m-1} = x_{m-1})$, then $f' \in \mathfrak{R}_m$. Q.e.d.

6. The Feller Boundaries

In this section a Feller boundary is constructed for I_p^p which plays a role similar to the role played by the Markov chain Feller boundary for the ordinary bounded harmonic functions. Throughout the section it is assumed that the stochastic chain is transient. The method used to construct the Feller boundary is similar to that used by Feller [6] for Markov chains, the difference lying in the fact that we prove the necessary statements by using the properties of I_p^p rather than the ordinary harmonic functions. The Feller boundary for Markov chains has also been studied by Kendall [10] and Feldman [5].

Before beginning the actual construction let us describe what is to be done. A set Γ_p is found and the disjoint set union $Q^p \cup \Gamma_p$ is topologized in such a way that every function in $\hat{M}(Q^p)$ can be extended to a continuous function on $Q^p \cup \Gamma_p$. Moreover two functions f and g in $\hat{M}(Q^p)$ have the same boundary values if and only if $f \rightsquigarrow \tilde{f}$ and $g \rightsquigarrow \tilde{f}$ for some $\tilde{f} \in I_p^p$. Finally the discrete part of the boundary is studied and an integral representation for functions in I_p^p is obtained in terms of the discrete boundary. An excellent reference for the material on Boolean algebras which is used in this section is Halmos [8].

Let S_p be the set of non-negative extreme elements of the unit sphere in $[I_p^p]$. These can be identified with elements in the measure algebra of $\mathfrak{F}_p^{p*}$, that is, an element in S_p can be identified with the equivalence class of an indicator function of a set in $\mathfrak{F}_p^{p*}$. If e is the equivalence class of the function which is identically 1, then S_p is a complete Boolean algebra with unit element e under the operations $\wedge$, $\vee$. Moreover it is easy to show that there is a lattice isomorphism

$\mathfrak{C}_p/\mathfrak{Z}_p \to S_p$. In the case of Markov chains this fact was pointed out by KENDALL [10].

Recall that a subset A of S_p is an *ideal* if

(i) $0 \in A$,

(ii) $s_1 \in A$ and $s_2 \in A$ implies that $s_1 \vee s_2 \in A$,

(iii) $s_1 \in A$ and $s_2 \in S_p$ implies that $s_1 \wedge s_2 \in A$.

The ideal is *maximal* if the only ideal containing A as a proper subset is S_p.

Following FELLER [6], Γ_p is defined to be the Stone space of S_p, that is, Γ_p is the set of maximal ideals topologized by taking sets of the form

$$\{\gamma \in \Gamma_p : s \notin \gamma\}, \qquad s \in S_p,$$

as a subbasis for the open sets. It is well known that Γ_p is an extremely disconnected compact Hausdorff space, [8].

Proposition 6.1. *If $s \in S_p$ let*

$$\begin{aligned} \Phi(s,\gamma) &= 1 \quad \text{if} \quad \gamma \in \Gamma_p \quad \text{and} \quad s \in \gamma \\ &= 0 \quad \text{if} \quad \gamma \in \Gamma_p \quad \text{and} \quad s \notin \gamma. \end{aligned}$$

Then for each $s \in S_p$, $\Phi(s,\cdot)$ is a continuous function on Γ_p. Moreover $\Phi(\cdot,\cdot)$ can be extended uniquely to a mapping if I_p^p onto $C(\Gamma_p)$, the set of continuous real-valued functions on Γ_p, which is a linear lattice isomorphism and which is also norm preserving when $C(\Gamma_p)$ is furnished with the supremum norm.

Proof. This follows immediately since linear combinations of step functions are dense in $C(\Gamma_p)$ and linear combinations of elements of S_p are dense in I_p^p in the essential supremum norm. Q.e.d.

We introduce a topology on $\Gamma_p \cup Q^p$ as follows. A set $A \subset \Gamma_p \cup Q^p$ is open if to each maximal ideal $\gamma \in A$ there is a set $\Delta \in \mathfrak{C}_p$ such that $[\tilde{\chi}_\Delta] \notin \gamma$, $\Delta \subset A$, and such that A contains every maximal ideal β such that $[\tilde{\chi}_{\Delta^c}] \in \beta$. (We write $\chi_{\tilde{\Delta}}$ for $[\chi_{\tilde{\Delta}}]$ in places where it is clear what is meant.)

Alternately we can introduce the topology by letting Γ_p and Q^p have their original topologies and topologizing their union so that

$$x_n \to \gamma \in \Gamma_p \qquad \text{as } n \to \infty$$

if and only if for any $\chi_\Delta \in \hat{M}(Q^p)$ such that $\Phi(\tilde{\chi}_\Delta, \gamma) = 1$, $x_n \in \Delta$ for sufficiently large n.

We use the first definition in the sequel; the fact that the two formulations are equivalent follows from Propositions 6.3 and 6.4.

Notational Remark: The notation $\chi_{\tilde{\Delta}}$ denotes the indicator function of a set $\tilde{\Delta} \in \mathfrak{I}_p^{p*}$, whereas $\tilde{\chi}_\Delta$ stands for the function in I_p^p induced by the indicator function of a set $\Delta \in \mathfrak{C}_p$.

Lemma 6.1. *For any maximal ideal γ and $\tilde{\Delta} \in \mathfrak{I}_p^{p*}$ either $\chi_{\tilde{\Delta}}$ or $\chi_{\tilde{\Delta}^c}$ must belong to γ.*

Proof. See HALMOS [8].

Lemma 6.2. *If γ is a maximal ideal and $\tilde{\chi}_{\Delta_1}, \tilde{\chi}_{\Delta_2} \notin \gamma$, then $\tilde{\chi}_{\Delta_1 \cap \Delta_2} \notin \gamma$.*

Proof. Since γ is maximal, $\tilde{\chi}_{\tilde{\Delta}_1^c} \in \gamma$ and $\tilde{\chi}_{\tilde{\Delta}_2^c} \in \gamma$ by Lemma 6.1. If $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ are the sets of $\mathfrak{I}_p^{p*}$ which correspond to Δ_1 and Δ_2, then $\chi_{\tilde{\Delta}_1^c \cup \tilde{\Delta}_2^c} \in \gamma$ and hence $\chi_{\tilde{\Delta}_1 \cap \tilde{\Delta}_2} \notin \gamma$. But by Proposition 4.7, $\chi_{\Delta_1 \cap \Delta_2} \rightsquigarrow \chi_{\tilde{\Delta}_1 \cap \tilde{\Delta}_2}$ and therefore the result follows. Q.e.d.

Proposition 6.2. *$\Gamma_p \cup Q^p$ is a Hausdorff space and Q^p is a dense subset.*

Proof. We first verify that $\Gamma_p \cup Q^p$ is a topological space. By the definition of an open set it follows that the union of open sets is open. Hence it remains to show that if O_1 and O_2 are open sets then $O_1 \cap O_2$ is open. Since every subset of Q^p is open there is nothing to prove if $O_1 \cap O_2 \subset Q^p$. If $\gamma \in O_1 \cap O_2 \cap \Gamma_p$ let Δ_i, $i = 1, 2$, be the sets contained in O_i, $i = 1, 2$, such that $\tilde{\chi}_{\Delta_1} \notin \gamma$. Then by Lemma 6.2 $\tilde{\chi}_{\Delta_1 \cap \Delta_2} \notin \gamma$ and thus in particular $\tilde{\chi}_{\Delta_1 \cap \Delta_2} \neq 0$; in addition $\Delta_1 \cap \Delta_2 \subset O_1 \cap O_2$. Moreover if γ_0 is a maximal ideal such that $\tilde{\chi}_{(\Delta_1 \cap \Delta_2)^c} \in \gamma_0$, then $\tilde{\chi}_{\Delta_1^c} \in \gamma_0$ and $\tilde{\chi}_{\Delta_2^c} \in \gamma_0$ so that $\gamma_0 \in O_1 \cap O_2$. Hence $O_1 \cap O_2$ satisfies the definition of an open set.

The separation axiom is immediate for two distinct points of Q^p. Moreover if $\gamma \in \Gamma_p$ and $x \in Q^p$, then γ has a neighborhood not containing x by Proposition 4.8. Hence it suffices to prove the separation axiom for two distinct points γ_1 and γ_2 of Γ_p. Since γ_1 and γ_2 are two distinct maximal ideals there is a set $\Delta \in \mathfrak{C}_p$ such that $\tilde{\chi}_\Delta \in \gamma_1$ and $\tilde{\chi}_\Delta \notin \gamma_2$. Moreover no maximal ideal contains both $\tilde{\chi}_\Delta$ and $\tilde{\chi}_{\Delta^c}$. Let

$$O_1 = \Delta^c \cup \{\gamma : \gamma \in \Gamma_p, \tilde{\chi}_\Delta \in \gamma\},$$

and

$$O_2 = \Delta \cup \{\gamma : \gamma \in \Gamma_p, \tilde{\chi}_{\Delta^c} \in \gamma\}.$$

Then O_1 and O_2 are disjoint open sets such that $\gamma_1 \in O_1$ and $\gamma_2 \in O_2$.

The fact that Q^p is dense follows immediately since by definition every neighborhood of a point in Γ_p contains a point of Q^p. Q.e.d.

Proposition 6.3. *Let $\chi_\Delta \in \hat{M}(Q^p)$. If $x_n \to \gamma_0$ as $n \to \infty$ and $x_n \in Q^p$, then*

(i) $\quad \chi_\Delta(x_n) \to 1 \quad$ *if* $\quad \chi_\Delta \notin \gamma_0$

and

(ii) $\quad \chi_\Delta(x_n) \to 0 \quad$ *if* $\quad \tilde{\chi}_\Delta \in \gamma_0$.

Proof. Assume that $x_n \to \gamma_0$ as $n \to \infty$ and that $\tilde{\chi}_\Delta \notin \gamma_0$. Consider the open set $0 \equiv \Delta \cup \{\gamma : \gamma \in \Gamma_p, \tilde{\chi}_{\Delta^c} \notin \gamma\}$. By Lemma 6.1 $\tilde{\chi}_{\Delta^c} \in \gamma_0$ and hence $\gamma_0 \in 0$. In addition $\chi_\Delta(x) = 1$ for all $x \in Q^p \cap 0$ and therefore $\chi_\Delta(x_n) \to 1$ as $x_n \to \gamma_0$. Replacing Δ by Δ^c a similar argument yields (ii). Q.e.d.

Proposition 6.4. *If f is any function in $\hat{M}(Q^p)$ such that $f \rightsquigarrow \tilde{\chi}_\Delta$, then f has the same boundary values as χ_Δ.*

Proof. Assume that $x_n \to \gamma_0$ and $\tilde{\chi}_\Delta \notin \gamma_0$. Then

$$x_n \in \{x : f(x) > 1 - \varepsilon\} \cup \{\gamma : \gamma \in \Gamma_p, \tilde{\chi}_{\Delta^c} \in \gamma\}$$

for sufficiently large n (c.f. Proof of Proposition 4.7). Therefore $\lim_{x_n \to \gamma_0} f(x_n) = 1$. A similar argument holds if $\tilde{\chi}_\Delta \in \gamma_0$. Q.e.d.

Proposition 6.5. *If $f \in \hat{M}(Q^p)$, then f has continuous boundary values on Γ_p.*

Proof. If $f \rightsquigarrow \tilde{f} \in I_p^p$, then there is a simple function $\sum_{i=1}^{N} \alpha_i \chi_{\tilde{\Delta}_i}$ such that $\|\tilde{f} - \sum_{i=1}^{N} \alpha_i \chi_{\tilde{\Delta}_i}\|_\infty < \varepsilon$ and such that the $\tilde{\Delta}_i$ are disjoint. Then by Proposition 4.2,

f and χ_{Δ_i} may be chosen so that $f \rightsquigarrow \tilde{f}$, $\chi_{\Delta_i} \rightsquigarrow \chi_{\tilde{\Delta}_i}$ and $\| f - \sum_{i=1}^{N} \alpha_i \chi_{\Delta_i} \| < \varepsilon$. Since $f(\gamma)$ is then the uniform limit of continuous functions, it is continuous. Q.e.d.

Corollary. *Two functions f and g in $\hat{M}(Q^p)$ have the same boundary values if and only if $f \rightsquigarrow \tilde{f}$ and $g \rightsquigarrow \tilde{f}$ for some $\tilde{f} \in I_p^p$.*

Proof. The first half of the statement follows from the proof of the Proposition and Proposition 6.4. To prove the second half let $f \rightsquigarrow \tilde{f}$, $g \rightsquigarrow \tilde{g}$ and $\tilde{f} - \tilde{g} > 0$ on a set $\tilde{\Delta}$ of positive measure. There is some point $\gamma \in \Gamma_p$ such that $\chi_{\tilde{\Delta}} \notin \gamma$. But then it is easy to verify that $f(\gamma) \neq g(\gamma)$. Q.e.d.

There does not appear to be any direct way to relate the boundaries Γ_p for different values of p. However under certain assumptions it is possible to relate the different discrete boundaries of Feller which we now introduce.

An element $s \in S_p$ is said to be *minimal* if $s' \leqq s$ implies that either $s' = 0$ or $s' = s$. An element $s \in S_p$ is said to be *continuous* if there is not minimal element s' such that $s' \leqq s$.

Proposition 6.6. *S_p contains at most countably many minimal elements.*

Proof. If s_1 and s_2 are distinct minimal elements they correspond to $\chi_{\tilde{\Delta}_1}$ and $\chi_{\tilde{\Delta}_2}$, $\tilde{\Delta}_1, \tilde{\Delta}_2 \in \mathfrak{J}_p^{p*}$, such that $\chi_{\tilde{\Delta}_1} \cdot \chi_{\tilde{\Delta}_2} = 0$ a.e. Moreover if $\chi_{\tilde{\Delta}_1}$ corresponds to a non-zero element of S_p, $\tilde{\Delta}_1$ must have positive measure. But there are at most countably many disjoint sets of positive measure. Q.e.d.

We denote the elements of S_p by $\gamma_1, \gamma_2, \ldots$. By the corollary to Proposition 4.11 it is possible to find $\Delta_i \in \mathfrak{C}_p$ such that the Δ_i are mutually disjoint and $\chi_{\Delta_i} \rightsquigarrow \chi_{\tilde{\Delta}_i}$ where γ_i is the equivalence class of $\chi_{\tilde{\Delta}_i}$.

The discrete Feller boundary Γ_d^p is the set of minimal elements of S_p. $Q^p \cup \Gamma_d^p$ is topologized so that $A \subset Q^p \cup \Gamma_d^p$ is open if for each $\gamma_n \in A \cap \Gamma_d^p$ there is a set $\Delta \subset A \cap Q^p$ such that $\chi_\Delta \rightsquigarrow \chi_{\tilde{\Delta}_n}$.

Proposition 6.7. *$Q^p \cup \Gamma_d^p$ is a Hausdorff space and the relative topologies on both Q^p and Γ_d^p are the discrete topologies.*

Proof. The union of open sets is open by the definition of open set. If $\chi_{\Delta'} \rightsquigarrow \chi_{\tilde{\Delta}_i}$ and $\chi_{\Delta''} \rightsquigarrow \chi_{\tilde{\Delta}_i}$ then $\chi_{\Delta' \cap \Delta''} \rightsquigarrow \chi_{\tilde{\Delta}_i}$. Hence the intersection of finitely many open sets is open. The fact that two distinct points γ_n and γ_m have distinct neighborhoods follows immediately from the above comment that the Δ_i can be chosen to be mutually disjoint. The fact that the relative topologies on Q^p and Γ_d^p are the discrete topologies follows immediately from the definition of the topology. Q.e.d.

Proposition 6.8. $Q^p \cup \Gamma_d^p \subset Q^p \cup \Gamma_p$.

Proof. If $\gamma_i \in \Gamma_d^p$ we define a maximal ideal $\gamma = \{s : s \in S_p, s \wedge \gamma_i = 0\}$. Then $\gamma_i \notin \gamma$ but no other maximal ideal contains $\chi_{\tilde{\Delta}_i^c}$. Hence a point $\gamma \in \Gamma_p$ can be associated with each $\gamma_i \in \Gamma_d^p$. Moreover $\Delta_i \cup \gamma_i$ is open in both $Q^p \cup \Gamma_p$ and $Q^p \cup \Gamma_d^p$. Q.e.d.

Proposition 6.9. *The function $\tilde{f} = \sum_{n=1}^{\infty} \alpha_n \chi_{\tilde{\Delta}_n}$ belongs to the unit ball in $(I_p^p)^+$ if and only if $0 \leqq \alpha_n \leqq 1$. If in addition $f \rightsquigarrow \tilde{f}$, then $f(x) \to \alpha_n$ as $x \to \gamma_n$.*

If S_p contains no continuous elements, then every non-negative function in the unit ball in $(I_p^p)^+$ has this form.

Proof. If $0 \leqq \alpha_n \leqq 1$, then it is clear that $\tilde{f}$ is in the unit ball in $(I_p^p)^+$. If $x_r \to \gamma_n$, then x_r must lie in Δ for any Δ such that $\chi_\Delta \rightsquigarrow \chi_{\tilde{\Delta}_n}$ for sufficiently large r. Hence $f(x_r) \to \alpha_n$. Then if $\tilde{f}$ is a non-negative function in the unit ball in I_p^p having this form it is clear $0 \leqq \alpha \leqq 1$. Finally if $\tilde{f}$ is a non-negative function in the unit ball in I_p^p, $f \rightsquigarrow \tilde{f}$ and $f(\gamma_n) = \alpha_n$, then

$$\tilde{f} - \sum_{n=1}^{\infty} \alpha_n \chi_{\tilde{\Delta}_n}$$

must either be zero or correspond to a continuous element of S_p. Q.e.d.

Corollary. *If S_p contains no continuous elements, then with probability* 1 *all paths converge to some γ_n.*

Proof. This follows immediately since in this case $1 = \sum_{n=1}^{\infty} \chi_{\tilde{\Delta}_n}$.

For the purposes of comparing the boundaries Γ_d^p for different values of p it is convenient to introduce a slightly different topology on $Q^p \cup \Gamma_d^p$. Let Δ_n be a set in $\mathfrak{C}_p$ which corresponds to γ_n and assume as above that the Δ_n are mutually disjoint. Note that it is possible to choose the Δ_n which correspond to points of Γ_d^{pm}, $m = 1, 2, 3, \ldots$, so that each such Δ_n is contained in one of the corresponding sets for Γ_d^p. That is, if we denote by Δ_n^p the sets of $\mathfrak{C}_p$ corresponding to the points in Γ_d^p, then the Δ_n^p can be chosen so that $\Delta_n^{pm} \subset (\Delta_r^p)^m \in \mathfrak{C}_{pm}$ for some r. In the following section we assume that the $p = 1, 2, 3, \ldots$, have been so chosen.

Given a set Δ_n corresponding to a point of Γ_d^p there is a subsequence ${}_i\Delta_n \searrow \varphi$ such that $\chi_{{}_i\Delta_n} \rightsquigarrow \chi_{\tilde{\Delta}_n}$ for each i. In fact ${}_i\Delta_n$ can be chosen to be Δ_n with i given points deleted.

The modified topology on $Q^p \cup \Gamma_d^p$ is defined as follows. A set $A \subset Q^p \cup \Gamma_d^p$ is open if to each $\gamma_n \in \Gamma_d^p$ there is some i such that ${}_i\Delta_n \subset A$. This topology is a weaker topology than the former. To prevent confusion we denote by $(Q^p \cup \Gamma_d^p)^*$ the space with the modified topology.

Proposition 6.10. *$(Q^p \cup \Gamma_d^p)^*$ is a Hausdorff space with a countable base of open sets.*

Proof. The proof of the first part follows in the same way as the proof of Proposition 6.7. The fact that there is a countable base of open sets follows immediately from the definition. Q.e.d.

Note that since the modified topology is weaker than the original topology, the corollary to Proposition 6.9 remains true for the modified topology.

7. Relations between the I_p^p and the Limit Boundary

In the previous sections the I_p^p are studied for a fixed integer p. In this section the relations between the I_p^p for different p are investigated.

The functions in the intersection $I_p^p \cap I^n$, designated by I_p^n, are the bounded real-valued functions measurable with respect to the σ-field $\mathfrak{I}^{n*} \cap \mathfrak{I}_p^{p*}$.

Proposition 7.1. *If the stochastic chain has non-singular tail field then the linear, order preserving mapping*

$$f \to f + \theta^m f + \cdots + \theta^{(n-1)m} f$$

maps I^{nm} onto I^m.

Proof. Since the stochastic chain has non-singular tail field every tail event of measure zero is p-negligible for every p. But

$$\begin{aligned}\theta^m(f+\theta^m f+\cdots+\theta^{(n-1)m}f) &= \theta^m f+\theta^{2m}f+\cdots+\theta^{nm}f\\ &= f+\theta^m f+\cdots+\theta^{(n-1)m}f \qquad \text{a.e.}\end{aligned}$$

and therefore the mapping takes any function in I^{nm} into a function in I^m. If $f\in I^m$, then $f\in I^{nm}$ and hence the mapping is onto. Q.e.d.

Recall that $[I_p^p]$ and $[I_{pn}^p]$ are vector lattices. A mapping of one vector lattice into another is called a *homomorphism* if it is linear and lattice preserving. The *kernel* of a homomorphism is the set of elements mapped into 0. The homomorphism is called a *monomorphism* if the only element in the kernel is 0. An *isomorphism* is a monomorphism which is onto.

Proposition 7.2. *There is a monomorphism from $[I_p^p]$ into $[I_{pn}^p]$ for $n=1,2,3,\ldots$.*

Proof. Let $f\in\mathfrak{C}_p$ and $f \rightsquigarrow \tilde f$. Then

$$g=n^{-1}(f(x_0,\ldots,x_{p-1})+\cdots+f(x_{(n-1)p},\ldots,x_{np-1})) \tag{7.1}$$

belongs to $\mathfrak{C}_{np}$ and $g\rightsquigarrow \tilde f$. Hence $\tilde f\in I_{pn}^p$ and therefore I_p^p is mapped into I_{pn}^p by (7.1). It is easy to verify that this mapping is a monomorphism. Q.e.d.

This allows us to gain a little insight into what is going on. For example if $A\in\mathfrak{C}_1$, then $A\times A\in\mathfrak{C}_2$. On the other hand there can be a subset $B\subset A$ such that $(A-B)\times B$ and $B\times(A-B)$ both belong to $\mathfrak{C}_2$ without having $B\in\mathfrak{C}_1$.

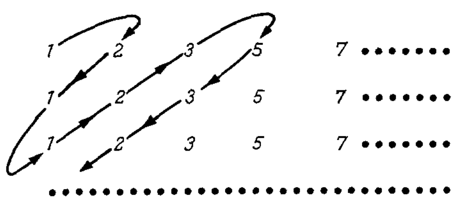

Fig. 1

Let n_i be the subsequence of natural numbers given by the product of the first i numbers in the sequence given by the diagram (Fig. 1) in which each row is the sequence of prime numbers. Then we have

$$I_p^p\to I_{2p}^p\to I_{4p}^p\to I_{12p}^p\to\cdots\to I_{n_ip}^p\cdots$$

in which the arrows represent monomorphisms. It is easy to verify the following proposition.

Proposition 7.3. *For any $m\geqq p$ there is a monomorphism $I_m^p\to I_{n_ip}^p$ for some i.*

We now describe the direct limit

$$\hat I^p=\varinjlim I_{n_ip}^p.$$

To form the direct limit consider the disjoint set union of the $I_{n_ip}^p$ and define the direct limit as the set of equivalence classes under the following equivalence relation. If $n_i<n_j$, $f\in I_{n_ip}^p$ and $g\in I_{n_jp}^p$, then f and g are in the same equivalence class $\langle f\rangle$ if f is mapped into g by the monomorphism from $I_{n_ip}^p$ to $I_{n_jp}^p$. The set of equivalence classes is made into a vector lattice by letting

$$\begin{aligned}\langle f\rangle+\langle g\rangle &= \langle f+g\rangle,\\ \langle\alpha f\rangle &= \alpha\langle f\rangle \quad \text{for a real constant } \alpha,\\ \langle f\rangle\wedge\langle g\rangle &= \langle f\wedge g\rangle\end{aligned}$$

where both f and g are taken to be representatives of $\langle f\rangle$ and $\langle g\rangle$ in the same $I_{n_ip}^p$.

The following proposition suggests that we can think of $\hat{I}^p$ as the set of finite dimensional p-invariant functions.

Proposition 7.4. *Let $\hat{\mathfrak{J}}^{p*}$ be the σ-subfield of $\mathfrak{J}^*$ generated by the σ-fields $\mathfrak{J}^p_{n_i p}$, $i = 1, 2, 3, \ldots$. Then we can imbed $\hat{I}^p$ in I^p so that*

$$\hat{I}^p \subset L_\infty(\hat{\mathfrak{J}}^{p*}) \subset I^p.$$

The inclusions can be proper inclusions.

Proof. If $\tilde{f} \in I^p_{n_i p}$, then $\tilde{f} \in L_\infty(\hat{\mathfrak{J}}^{p*})$. It is easy to verify that the mapping which maps $\langle f \rangle$ into $\tilde{f} \in L_\infty(\hat{\mathfrak{J}}^{p*})$ is independent of the choice of the representative up to sets of measure zero. Finally it is clear that the mapping $\langle f \rangle \to \tilde{f} \in L_\infty(\hat{\mathfrak{J}}^{p*})$ is a monomorphism from $\hat{I}^p$ into $L_\infty(\hat{\mathfrak{J}}^{p*})$. Hence we can identify $\hat{I}^p$ with the image of this monomorphism.

The second inclusion follows immediately since $\hat{\mathfrak{J}}^{p*}$ is a σ-subfield of $\mathfrak{J}^p$. Q.e.d.

The fact that the first inclusion can be proper is due to the fact that I^p need not be closed under the operation of taking limits.

In the remainder of this section it is assumed that none of the S_p contains any continuous elements.

Proposition 7.5. *Γ^{pn}_d is a splitting of Γ^p_d.*

Proof. Let γ^p_i, $i = 1, 2, 3, \ldots$, be the points in Γ^p_d and γ^{pn}_i, $i = 1, 2, 3, \ldots$, be the points in Γ^{pn}_d. For convenience we use the notation γ^p_i for either the point in Γ^p_d or the corresponding minimal function in S_p. Then by Proposition 7.2 $\gamma^p_i \in S_{np}$. If γ^p_i is not minimal in S_{np}, then either $\gamma^{pn}_j \wedge \gamma^p_i = \gamma^{pn}_j$ or $\gamma^{pn}_j \wedge \gamma^p_i = 0$ for each $j = 1, 2, 3, \ldots$. Then we can associate with each γ^{pn}_j the unique γ^p_i such that $\gamma^{pn}_j \wedge \gamma^p_i = \gamma^{pn}_j$ and write this as $\gamma^{pn}_j \rightsquigarrow \gamma^p_i$. Conversely we can consider the point γ^p_i to be split into the set $\{\gamma^{pn}_j : \gamma^{pn}_j \rightsquigarrow \gamma^p_i\}$. Since $\gamma^p_i \wedge \gamma^p_k = 0$ for $i \neq k$ it follows that these classes are disjoint. Q.e.d.

Proposition 7.6. *Consider the mapping*

$$\varphi : (Q^{pn} \cup \Gamma^{pn}_d)^* \xrightarrow{onto} (Q^p \cup \Gamma^p_d)^*$$

defined by

$$\varphi(w_1, \ldots, w_{(n-1)p+1}, \ldots, w_{pn}) = (w_1, \ldots, w_p)$$

for points in Q^{pn}

$$\varphi(\gamma^{pn}_j) = \gamma^p_i \quad \textit{where} \quad \gamma^{pn}_j \rightsquigarrow \gamma^p_i$$

for points in Γ^{pn}_d. Then φ is continuous.

Proof. It suffices to show that if $x_r \to \gamma^{pn}_j$ as $r \to \infty$ and $\gamma^{pn}_j \rightsquigarrow \gamma^p_i$, then $\varphi(x_r) \to \gamma^p_i$ as $r \to \infty$. But this follows immediately from the definitions of the topologies on $(Q^{pn} \cup \Gamma^{pn}_d)^*$ and $(Q^p \cup \Gamma^p_d)^*$. Q.e.d.

If n_i is the subsequence of natural numbers constructed above then Proposition 7.6 implies that

$$(Q^p \cup \Gamma^p_d)^* \leftarrow (Q^{2p} \cup \Gamma^{2p}_d)^* \leftarrow \cdots (Q^{n_i p} \cup \Gamma^{n_i p}_d)^* \leftarrow \cdots$$

is an inverse system of topological spaces in which the arrows represent continuous mappings. Let $(Q^\infty \cup \Gamma^\infty_d)^*$ designate the *inverse limit* of this inverse system. The inverse limit is defined as follows. Consider the set of all sequences $\{x_i\}$ such that $x_i \in (Q^{n_i p} \cup \Gamma_p^{n_i p})^*$ and such that for each i, $\varphi(x_{i+1}) = x_i$. The inverse limit is the topological space formed by topologizing this set of sequences by taking as a base for the neighborhoods the sets of the form $\varphi_i^{-1}(A)$, A open in $(Q^{n_i p} \cup \Gamma_d^{n_i p})^*$. φ_i represents the projection of a sequence onto its ith component.

Note that the relative topology on Q^∞ is the product topology and that $(Q^\infty \cup \Gamma_d^\infty)^*$ is Hausdorff. Note also that it is possible to identify Q^∞ with Ω so that there is induced a measure on Q^∞.

A point $w \in Q^\infty$ is said to *adhere* to the point $\gamma \in \Gamma_d^\infty$ if for every neighborhood $N\gamma$ of γ in $(Q^\infty \cup \Gamma_d^\infty)^*$ there is a positive integer M such that for every $i \geq M$ $\theta^{n_i} w \in N\gamma$. In this case $\theta^{n_i} w$ is said to converge to γ. Since $(Q^\infty \cup \Gamma_d^\infty)^*$ is Hausdorff, a given point $w \in Q^\infty$ can adhere to at most one point in Γ_d^∞.

Proposition 7.7. *The set of points in Q^∞ which do not adhere to points in Γ_d^∞ corresponds to a set of points in Ω of P-measure zero.*

Proof. Since the topological space $(Q^\infty \cup \Gamma_d^\infty)^*$ has a countable base of open sets, it suffices to show that the set of paths which leaves a given open set infinitely often has measure zero. In fact, it suffices to prove this for a set of the form $\varphi_i^{-1}(A)$, A open in $(Q^{n_i p} \cup \Gamma_d^{n_i p})^*$. But this follows from the corollary to Proposition 6.9. Q.e.d.

Since almost every path, w, adheres to the boundary Γ_d^∞, we may associate this boundary point $\gamma(w)$ to the path and we write $w(\infty) = \gamma(w)$. If w does not adhere to the boundary, we write $w(\infty) = \alpha_0$ where α_0 is a new point which we adjoin to $(Q^\infty \cup \Gamma_d^\infty)^*$.

Let $\hat{\mathfrak{F}}^*$ be the subfield of $\mathfrak{F}^*$ generated by the subfields $\mathfrak{F}_p^{p*}$. In the remainder of this section we investigate the relation between the functions on Γ_d^∞ and the functions on Ω which are measurable with respect to $\hat{\mathfrak{F}}^*$.

Consider the σ-field, $\mathfrak{B}^\infty$, of subsets of Γ_d^∞ generated by the open sets. It is clear that P induces a measure on the discrete measure spaces Γ_d^∞. But $(\Gamma_d^\infty, \mathfrak{B}^\infty)$ is also the inverse limit of the discrete measure spaces. Therefore there is induced an inverse limit probability measure P^∞ on $(\Gamma_d^\infty, \mathfrak{B}^\infty)$. (See MEYER [*11*].) Let $\mathfrak{B}^p$ be the subfield of $\mathfrak{B}^\infty$ generated by the inverse images of sets in Γ_d^p.

Proposition 7.8. *The vector lattices $\mathfrak{L}_\infty(\hat{\mathfrak{F}}^*)$ and $\mathfrak{L}_\infty(\mathfrak{B}^\infty)$ are isomorphic.*

Proof. We define a vector-valued measure F on $(\Gamma_d^\infty, \mathfrak{B}^\infty)$ as follows. If $\Delta = \varphi^{-1}(\gamma_n^p)$, then $F(\Delta) = [\chi_{\Delta_n^p}] \in \mathfrak{L}_\infty(\hat{\mathfrak{F}}^*)$. As above this measure can be extended to a vector-valued measure on $(\Gamma_d^\infty, \mathfrak{B}^\infty)$. Moreover the measure is bounded since the measure of any set lies in the unit ball in $\mathfrak{L}_\infty(\hat{\mathfrak{F}}^*)$. (Refer to DUNFORD and SCHWARTZ [*4*] for a discussion of integration with respect to a vector-valued measure.) In addition F is absolutely continuous with respect to the measure P^∞.

Let f be a bounded real-valued measurable function on Γ_d^∞. Then

$$F(f) \equiv \int f(w_\infty) F(dw_\infty) \tag{7.2}$$

belongs to $\mathfrak{L}_\infty(\hat{\mathfrak{F}}^*)$. Moreover if f and g do not belong to the same equivalence class in $\mathfrak{L}_\infty(\mathfrak{B}^\infty)$, then $F(f)$ and $F(g)$ do not belong to the same equivalence class in $\mathfrak{L}_\infty(\hat{\mathfrak{F}}^*)$. It is easy to verify that F is linear and order preserving. Hence F is a monomorphism from $\mathfrak{L}_\infty(\mathfrak{B}^\infty)$ to $\mathfrak{L}_\infty(\hat{\mathfrak{F}}^*)$.

Let $\tilde{f} \in I_p^p$ for some p. By Proposition 6.9 $\tilde{f}$ induces a continuous function on Γ_d^p and thus in turn it induces a continuous function on Γ_d^∞, say $f(w_\infty)$. (This means that there is a monomorphism from $\hat{I}^p$ into $C(\Gamma_d^\infty)$ for each p.) In addition, $f(w) = f(w_\infty(w))$ a.e.

Let $\tilde{f}$ be a representative of $f \in \mathfrak{L}_\infty(\hat{\mathfrak{F}}^*)$. Then the martingale convergence theorem applied to the subfields $\mathfrak{F}_{n_i}^{n_i *}$ of $\hat{\mathfrak{F}}^*$ implies that there exists a sequence of functions $\{\tilde{f}_i\}$ such that $\tilde{f}_i \in I_{n_i}^{n_i}$ and such that $\tilde{f} = \lim_{i \to \infty} \tilde{f}_i$ a.e. Since $\tilde{f}_i(w) = f_i(w_\infty(w))$ a.e., f_i converges a.e. to a measurable function f on Γ_d^∞. Moreover by the bounded convergence theorem for vector-valued measures [*4*, p. 328] and Proposition 6.9.

$$[\tilde{f}(w)] = \int f(w_\infty)\, F(dw_\infty).$$

Thus every element in $\mathfrak{L}_\infty(\hat{\mathfrak{F}}^*)$ has an integral representation of the form 7.2 and therefore $\mathfrak{L}_\infty(\mathfrak{B}^\infty)$ is isomorphic to $\mathfrak{L}_\infty(\hat{\mathfrak{F}}^*)$. Q.e.d.

Proposition 7.9. *If $\tilde{f} \in I_p^p$ for some p, then*

$$\lim_{i \to \infty} \theta^{n_i} \tilde{f}(w) = f(w_\infty(w)) \quad a.e.$$

Proof. By the definition of I_p^p and the sequence n_i, $\theta^{n_i} \tilde{f} = \tilde{f}$ for all sufficiently large i. Since $\tilde{f}(w) = f(w_\infty(w))$ a.e., the result follows immediately. Q.e.d.

Proposition 7.10. *Assume that the stochastic chain has non-singular tail field and that $\tilde{f} \in L_\infty(\hat{\mathfrak{F}}^*)$ is such that*

$$\tilde{f} = \lim_{i \to \infty} \tilde{f}_i \ in \|\cdot\|_\infty \quad with \quad \tilde{f}_i \in I_{n_i}^{n_i}.$$

Then $\lim_{i \to \infty} \theta^{n_i} f(w) = f(w_\infty(w))$ *a.e.*

Proof. Since the tail field is non-singular, $\theta^{n_j} \tilde{f}_i \to \theta^{n_j} \tilde{f}$ in $\|\cdot\|_\infty$ for each $j = 1, 2, 3, \ldots$. Given $\varepsilon > 0$ there is a positive integer i_0 such that for $i \geqq i_0$, $\|\tilde{f} - \tilde{f}_i\|_\infty < \varepsilon$. But then for $i \geqq i_0$,

$$\begin{aligned} \|\tilde{f} - \theta^{n_j} \tilde{f}\|_\infty &\leqq \|\tilde{f} - \tilde{f}_i\|_\infty + \|\tilde{f}_i - \theta^{n_j} \tilde{f}_i\|_\infty + \|\theta^{n_j}(\tilde{f}_i - \tilde{f})\|_\infty \\ &\leqq 2\varepsilon + \|\tilde{f}_i - \theta^{n_j} \tilde{f}_i\|_\infty . \end{aligned}$$

But there is a positive integer j_0 such that for $j \geqq j_0$, $\|\tilde{f}_i - \theta^{n_j} \tilde{f}_i\|_\infty < \varepsilon$ and therefore if $j \geqq j_0$, $\|\tilde{f} - \theta^{n_j} \tilde{f}\| < 3\varepsilon$. Q.e.d.

Corollary. *Let f be a continuous function on Γ_d^∞ which is the uniform limit of a sequence of functions f_i with f_i measurable with respect to $\mathfrak{B}^{n_i}$. Then there is a function $\tilde{f} \in L_\infty(\hat{\mathfrak{F}}^*)$ such that $f(w_\infty(w)) = \lim_{i \to \infty} \theta^{n_i} \tilde{f}(w)$ a.e. In other words one can solve the „Dirichlet problem" for the boundary function f.*

Proof. The proof follows immediately from Propositions 7.8 and 7.10.

Propositions 7.8, 7.9 and 7.10 suggest that the boundary Γ_d^∞ plays somewhat the same role for the set of functions $L_\infty(\hat{\mathfrak{F}}^*)$ as does the boundary Γ_d^p for the set of functions I_p^p. On the other hand there are some differences; for example Proposition 7.10 need not hold for all functions in $L_\infty(\hat{\mathfrak{J}}^*)$. We complete this section by discussing two rather simple examples.

Example 1. Consider a simple Markov chain which is transient. Then using Proposition 5.1 it can be shown that I_p^p is isomorphic to $\{f: f \in M(Q^p),\ \hat{P}^p f = f\}$ and that Γ_1 is the ordinary Feller boundary. In particular if the Markov chain is a random walk, then it is well known that the constant function is a minimal harmonic function, that is the only bounded harmonic function is the constant function. In this case the ordinary Feller boundary consists of a single point. This corresponds to the fact that the random walks have trivial invariant field. If the random walk is an n-dimensional simple random walk, $n \geqq 3$, then Γ_p^p consists of one point if p is odd and two points if p is even. Then Γ_d^∞ consists of two points which correspond to the events that the random walk remains on the even lattice points at even times and the odd lattice points at odd times.

Example 2. Consider the stochastic chain on the positive integers with deterministic transitions given by $X_{n+1}(w) = X_n(w) + 1$. Then Γ_d^p contains p points $\gamma_0^p, \ldots, \gamma_{p-1}^p$ corresponding to $\Delta_i^p = \{m: m = i \bmod p\}$, $i = 0, \ldots, p-1$. Then Γ_d^∞ is the set of all sequences $\{a_i\}$ such that $a_i \in \{1, \ldots, n_i\}$ and

$$a_{i+1} \bmod n_i = a_i.$$

References

[1] BLACKWELL, D.: On transient Markov processes with a countable number of states and stationary transition probabilities. Ann. math. Statistics **26**, 654—658 (1955).

[2] DOOB, J. L.: Discrete potential theory and boundaries. J. Math. Mech. 8, 433—458 (1959).

[3] — Stochastic processes. New York: Wiley 1953.

[4] DUNFORD, N., and J. T. SCHWARTZ: Linear operators. Part I. New York: Interscience Publ. 1958.

[5] FELDMAN, J.: Feller and Martin boundaries for countable sets. Ill. J. Math. **6**, 357—366 (1962).

[6] FELLER, W.: Boundaries induced by non-negative matrices. Trans. Amer. math. Soc. **83**, 19—54 (1956).

[7] — Non-Markovian process with the semi-group property. Ann. math. Statistics **30**, 1252—1253 (1959).

[8] HALMOS, P. R.: Lectures on Boolean algebras. Princeton: D. Van Nostrand 1963.

[9] KEMENY, J. G., and J. L. SNELL: Potentials for denumerable Markov chains. J. math. Analysis Appl. **3**, 196—260 (1961).

[10] KENDALL, D.: Hyperstonian spaces associated with Markov chains. Proc. London math. Soc. III. Ser. **10**, 67—87 (1960).

[11] MEYER, P. A.: Théoremes fondamentaux du calcul des probabilities. Séminaire de Théorie du Potentiel, 5e année, 1957—1958.

[12] ROSENBLATT, M., and D. SLEPIAN: Nth order Markov chains with every N variables independent. J. Soc. industr. appl. Math. **10**, 537—549 (1962).

Department of Mathematics,
McGill University,
Montreal 2, Canada