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On the Optimal Filtering of Diffusion Processes

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Summary. Let $x(t)$ be a diffusion process satisfying a stochastic differential equation and let the observed process $y(t)$ be related to $x(t)$ by $dy(t) = g(x(t)) + dw(t)$ where $w(t)$ is a Brownian motion. The problem considered is that of finding the conditional probability of $x(t)$ conditioned on the observed path $y(s)$, $0 \leqq s \leqq t$. Results on the Radon-Nikodym derivative of measures induced by diffusions processes are applied to derive equations which determine the required conditional probabilities.

1. Introduction

Let x_t satisfy the stochastic differential equation $dx_t = a(x_t)\,dt + b(x_t)\,d\tilde{w}_t$ where $\tilde{w}_t$ is a Brownian motion, and let y_t be related to x_t by $dy_t = g(x_t)\,dt + dw_t$, where w_t is a Brownian motion independent of $\tilde{w}_t$ (conditions on $a(x)$, $b(x)$ and $g(x)$ will be imposed in the next section). The x_t process can be considered as the motion of a noise-perturbed dynamical system and y_t as a noisy observation on x_t. This suggests the problem of determining the conditional probability of x_t, conditioned on the observed path y_s, $0 \leqq s \leqq t$. This problem was considered by STRATONOVICH [1], KUSHNER [2], BUCY [3], SHIRYAEV [4] and others.[1]

Previous work on this problem concentrated mainly on finding equations satisfied by $p(u, y_0^t)$, the conditional density of x_t given the observed path y_s, $0 \leqq s \leqq t$, (assuming it exists) and the derivations were formal. Recently KUSHNER proved that conditional expectations of functions of x_t satisfy (under certain restrictions) a stochastic equation [2].

Let $\Phi(u, y_0^t)$ be related to $p(u, y_0^t)$ by

$$p(u, y_0^t) = \Phi(u, y_0^t) \cdot \Big(\int_{E_y} \Phi(u, y_0^t)\,du\Big)^{-1},$$

the results of this paper deal with the unnormalized density $\Phi(u, y_0^t)$. It turns out that this leads to considerably simpler equations as compared to the equations for $p(u, y_0^t)$. The results of this paper are based on results of SKOROHOD and GIRSANOV for the Radon-Nikodym derivaties of measures induced by solutions of stochastic differential equations. Two stochastic equations for $\Phi(u, y_0^t)$ are derived. The first equation is derived in section 3 (theorem 1 and corollary 1).

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[1] After this paper was written we learned of the work of R. E. MORTENSEN (University of California, Berkeley, Elektronics Research Laboratory Report ERL-66-1, August 1966), and T. E. DUNCAN (Stanford University, Center for Systems Research, Technical Report 7001-4, May 1967), which contain results similar to some of the results of this paper. In particular, MORTENSEN derived, under some additional restrictions, the results of corollary 1 and DUNCAN obtained, formally, the results of theorem 3. We wish to thank T. KAILATH for calling our attention to these references.

The uniqueness of the solution to this equation is considered in section 4. The second equation is derived in section 5. The assumptions of theorem 3 (section 5) include some smoothness properties of the solution to equation (18). Conditions under which the solution has these properties are unknown at present. A similar remark applies to theorem 4 (section 5). It will be obvious from the proofs that it is possible to trade assumptions on the transition density of the x_t process for assumptions on the solution to Eq. (18) [or Eq. (30)]. The results of section 5 should therefore be considered to be of exploratory nature.

2. Some General Relations

Let $(\Omega, \mathscr{A})$ be a measurable space and μ_0, μ_1, two equivalent probability measures on $(\Omega, \mathscr{A})$. Let

$$\Lambda(\omega) = \frac{d\mu_1}{d\mu_0}(\omega)$$

be the Radon-Nikodym derivative of μ_1 with respect to μ_0. Let $X(\omega)$ be a random variable on $(\Omega, \mathscr{A})$ and let the expectation of $|X(x)|$ with respect to μ_1 be finite. Let $\mathscr{B}$ be a sub σ-field of $\mathscr{A}$ and $E_{(0)}(E_{(1)})$ denote expectations and conditional expectation with respect to $\mu_0(\mu_1)$. Then, a.s. μ_1 and μ_0, ([5], section 24.4):

$$E_{(1)}(X(\omega)\,|\,\mathscr{B}) = \frac{E_{(0)}(X(\omega)\,\Lambda(\omega)\,|\,\mathscr{B})}{E_{(0)}(\Lambda(\omega)\,|\,\mathscr{B})}. \tag{1}$$

Let x_t be the solution to the stochastic equation

$$x_t = x_0 + \int_0^t a(x_s)\,ds + \int_0^t b(x_s)\,d\tilde{w}_s \tag{2}$$

$$y_t = \int_0^t g(x_s)\,ds + \int_0^t dw_s,$$

where x and $a(x)$ are vectors in the Euclidean r-space E_r, $b(x)$ is an $r \times r$ matrix, $\tilde{w}_t$ is the standard r-dimensional Brownian motion, $g(w)$ is scalar valued and w_s is a standard one dimensional Brownian motion, independent of the $\tilde{w}_s$ process. Let $h(x)$ stand for $g(x)$ or any of the entries of $a(x)$ and $b(x)$, we assume that $h(x)$ satisfies the Lipschitz condition

$$|h(x) - h(y)| \leqq k\,\|x - y\|, \qquad x, y \in E_r.$$

The initial condition for the x_t process, x_0, will be assumed to be a random variable independent of the $\tilde{w}_t$, w_t processes. Let μ_1 be the measure induced on the space of E_{r+1} valued continuous functions on $[0, T]$ by Eq. (2), and let μ_0 be the measure induced on the same space by:

$$x_t = x_0 + \int_0^t a(x_s)\,ds + \int_0^t b(x_s)\,d\tilde{w}_s$$

$$y_t = \int_0^t dw_s. \tag{3}$$

It is known that under these conditions, μ_1 and μ_0 are equivalent measures and the Radon-Nikodym derivative of μ_1 with respect to μ_0 is given a. s. by ([6], Ch. 4, section 4; [7]):

$$\Lambda(x_0^T, y_0^T) = \frac{d\mu_1}{d\mu_0}(x_0^T, y_0^T) = \operatorname{Exp}\left[-\frac{1}{2}\int_0^T g^2(x_s)\,ds + \int_0^T g(x_s)\,dy_s\right], \tag{4}$$

where x_0^T (y_0^T) stands for the path x_s (y_s), $0 \leqq s \leqq T$. Since μ_1 and μ_0 are equivalent, any a.s. property with respect to one measure is also a.s. with respect to the other measure. From here on all equalities between conditional expectations and conditional probabilities are to be understood in the a.s. sense, even when this is not stated explicitly. Also, all the σ-fields considered will be assumed to be complete with respect to the involved measures (the Lebesgue measure on E_r and $[0, T]$, and μ_0).

Let $\mathscr{B}(x_t, y_0^s)$ denote the sub σ-field induced by the family of random variables x_t, y_θ, $0 \leqq \theta \leqq s$ (similarly, we will use $\mathscr{B}(x_s, x_t, y_0^s)$, $\mathscr{B}(y_0^s)$ etc.). Let $f(x)$ be a real valued, bounded Borel function of x ($x \in E_r$;) then, by (1) and the smoothing property for conditional expectations we have:

$$E_{(1)}(f(x_t) \mid \mathscr{B}(y_0^t)) = \frac{E_{(0)}\{E_{(0)}(\Lambda(x_0^t, y_0^t) \mid \mathscr{B}(x_t, y_0^t))\, f(x_t) \mid \mathscr{B}(y_0^t)\}}{E_{(0)}\{E_{(0)}(\Lambda(x_0^t, y_0^t) \mid \mathscr{B}(x_t, y_0^t)) \mid \mathscr{B}(y_0^t)\}}. \tag{5}$$

Let $\tilde{\Lambda}(u, y_0^t)$ be the value of some version of $E_{(0)}(\Lambda(x_0^t, y_0^t) \mid \mathscr{B}(x_t, y_0^t))$ at $x_t = u$ and the path y_0^t. Let

$$P(u, t) = \operatorname{Prob}\{x_t \leqq u\}, \qquad u \in E_r \tag{6}$$

(where $x_t \leqq u$ means that each component of x_t satisfies the inequality with respect to the corresponding component of u). Since, under μ_0, $\mathscr{B}(y_0^t)$ and $\mathscr{B}(x_0^t)$ are independent, it follows by Fubini's theorem that:

$$E_{(0)}\{E_{(0)}(\Lambda(x_0^t, y_0^t) \mid \mathscr{B}(x_t, y_0^t))\, f(x_t) \mid \mathscr{B}(y_0^t)\} = \int_{E_r} f(u)\, \tilde{\Lambda}(u, y_0^t)\, P(du, t) \tag{7}$$

and

$$E_{(1)}(f(x_t) \mid \mathscr{B}(y_0^t)) = \frac{\int_{E_r} \tilde{\Lambda}(u, y_0^t)\, f(u)\, P(du, t)}{\int_{E_r} \tilde{\Lambda}(u, y_0^t)\, P(du, t)}. \tag{8}$$

Since $f(x)$ was arbitrary, a version of the conditional probability of the random variable x_t conditioned on y_0^t, with respect to μ_1 is given by:

$$\operatorname{Prob}\{x_t \in \Gamma \mid \mathscr{B}(y_0^t)\} = \frac{\int_{\Gamma} \tilde{\Lambda}(u, y_0^t)\, P(du, t)}{\int_{E_r} \tilde{\Lambda}(u, y_0^t)\, P(du, t)} \tag{9}$$

where Γ is any Borel set in E_r. Furthermore, the conditional probability obtained by (9) is a conditional probability distribution of x_t relative to $\mathscr{B}(y_0^t)$ ([8], Ch. I, § 9).

In particular, if $P(u, t)$ is absolutely continuous with respect to the Lebesgue measure and $P(du, t) = p(u, t)\,du$ then a version of $\operatorname{Prob}\{x_t \in \Gamma \mid \mathscr{B}(y_0^t)\}$ is also absolutely continuous with respect to the Lebesgue measure and the density satisfies:

$$p(u, t \mid \mathscr{B}(y_0^t)) = \frac{\tilde{\Lambda}(u, y_0^t)\, p(u, t)}{\int_{E_r} \tilde{\Lambda}(u, y_0^t)\, p(u, t)\, du}. \tag{10}$$

In view of Eqs. (9) and (10), the problem of finding the conditional probability of x_t conditioned on y_0^t (with respect to the measure induced by Eq. (2)) has been transformed to the problem of finding $E_{(0)}(\Lambda(x_0^t, y_0^t) | \mathscr{B}(x_t, y_0^t))$ where the conditional expectation is with respect to the measure induced by (3) and $\Lambda(x_0^t, y_0^t)$ is given by (4). The numerators of (9) and (10) will be called the unnormalized conditional probability and density, respectively. Equations for expectations of multiplicative functionals of Markov processes, including expectations of the form $E(\Lambda(x_0^t, y_0^t) | \mathscr{B}(x_t))$ (and $E_{(0)}(\Lambda(x_0^t, y_0^t) | \mathscr{B}(x_t, y_0^t))$ with y_t differentiable) were derived by Kac, Fortet and others [9]. These results, though not applicable to the present case, motivated the results of sections 3 and 5.

3. An Integral Equation for the Unnormalized Conditional Probability

Theorem 1. *For the* (x_t, y_t) *process defined by Eq.* (2), $P(x_t \in \Gamma | \mathscr{B}(y_0^t))$ *satisfies Eq.* (9) and $\tilde{\Lambda}(u, y_0^t)$ *satisfies a.s. the equation*

$$\tilde{\Lambda}(u, y_0^t) = 1 + \int_0^t \int_{E_r} g(z)\, \tilde{\Lambda}(z, y_0^s)\, P(dz, s; u, t)\, dy_s, \tag{11}$$

where $P(dz, s; u, t)$ *is the conditional distribution:*

$$P(\Gamma, s; u, t) = \operatorname{Prob}\{x_s \in \Gamma | x_t = u\}, \qquad s \leqq t. \tag{12}$$

Proof. Applying Itô's formula ([10], [11]) to Eq. (4) we get:

$$\Lambda(x_0^t, y_0^t) = 1 + \int_0^t \Lambda(x_0^s, y_0^s)\, g(x_s)\, dy_s. \tag{13}$$

Therefore:

$$E_{(0)}(\Lambda(x_0^t, y_0^t) | \mathscr{B}(x_t, y_0^t)) = 1 + E_{(0)}\left\{\int_0^t \Lambda(x_0^s, y_0^s)\, g(x_s)\, dy_s \,|\, \mathscr{B}(x_t, y_0^t)\right\}. \tag{13a}$$

It will be proved later (starting with Eq. (19)) that (13a) implies that:

$$E_{(0)}(\Lambda(x_0^t, y_0^t) | \mathscr{B}(x_t, y_0^t)) = 1 + \int_0^t E_{(0)}\{\Lambda(x_0^s, y_0^s)\, g(x_s) | \mathscr{B}(x_t, y_0^t)\}\, dy_s. \tag{14}$$

Let $t \geqq s \geqq 0$ and $\mathscr{B}_1 = \mathscr{B}(x_0^s, y_0^s)$, $\mathscr{B}_2 = \mathscr{B}(y_\eta - y_s,\ s \leqq \eta \leqq t)$ and $\mathscr{B} = \mathscr{B}(x_t, y_0^s)$. Since, under μ_0, $\mathscr{B}(x_0^t)$ is independent of $\mathscr{B}(y_0^t)$ and y_0^t is a Brownian motion, it follows that $\mathscr{B}_1$ and $\mathscr{B}_2$ are conditionally independent given $\mathscr{B}$ ([5], 25.3A p. 351). Applying again theorem 25.3A of [5] (with subscripts 1 and 2 interchanged) it follows that for any $B \in \mathscr{B}(x_0^s, y_0^s)$:

$$P(B | \mathscr{B}(x_t, y_0^t) = P(B | \mathscr{B}(x_t, y_0^s)).$$

By [8] (theorem 8.4 chapter I) and by the smoothing property of conditional expectations:

$$\begin{aligned} E_{(0)}\{\Lambda(x_0^s, y_0^s)\, g(x_s) | \mathscr{B}(x_t, y_0^t)\} &= E_{(0)}\{\Lambda(x_0^s, y_0^s)\, g(x_s) | \mathscr{B}(x_t, y_0^s)\} \\ &= E_{(0)}\{E_{(0)}(\Lambda(x_0^s, y_0^s) | \mathscr{B}(x_s, x_t, y_0^s))\, g(x_s) | \mathscr{B}(x_t, y_0^s)\}. \end{aligned}$$

By the Markov property of (x_s, y_s), $\mathscr{B}(x_0^s, y_0^s)$ and $\mathscr{B}(x_t)$ are conditionally independent given $\mathscr{B}(x_s, y_s)$. Therefore ([5], 25.3A) $\mathscr{B}(x_0^s, y_0^s)$ and $\mathscr{B}(x_t)$ are also conditionally independent given $\mathscr{B}(x_s, y_0^s)$. Applying again 25.3A of [5] with interchanged subscripts) we have from the last equation,

$$E_{(0)}\{\Lambda(x_0^s, y_0^s)\, g(x_s) \mid \mathscr{B}(x_t, y_0^t)\} = E_{(0)}\{E_{(0)}(\Lambda(x_0^s, y_0^s) \mid \mathscr{B}(x_s, y_0^s))\, g(x_s) \mid \mathscr{B}(x_t, y_0^s)\}. \tag{15}$$

By the same argument as used for Eq. (7), the right hand side of (15) is given by ([8], chapter I, theorem 9.5):

$$\int_{E_r} g(z)\, \tilde{\Lambda}(z, y_0^s)\, P(dz, s; u, t) \tag{16}$$

where $P(dz, s;\, u, t)$ is defined by Eq. (12). Eq. (11) follows now by substituting (16) and (15) into (14).

Corollary 1. *If the transition probability* $\operatorname{Prob}\{x_t \in \Gamma \mid x_s = z\}$, $s < t$, *is absolutely continuous with respect to the Lebesgue measure with the density* $p_z(u, t - s)$, *then a version of* $\operatorname{Prob}\{x_t \in \Gamma \mid \mathscr{B}(y_0^t)\}$, $t > 0$, *is also absolutely continuous with respect to the Lebesgue measure. This density,* $p(u, t \mid \mathscr{B}(y_0^t))$, *satisfies*

$$p(u, t \mid \mathscr{B}(y_0^t)) = \frac{\Phi(u, t)}{\int_{E_r} \Phi(u, t)\, du} \tag{17}$$

where $\Phi(u, t)$, *satisfies a.s. the stochastic integral equations:*

$$\Phi(u, t) = p(u, t) + \int_0^t \int_{E_r} g(z)\, \Phi(z, s)\, p_z(u, t - s)\, dz\, dy_s \tag{18}$$

(where $p(u, t)$ *is the density of* $\operatorname{Prob}\{x_t \in \Gamma\}$*), and* $(0 < s \leqq t)$:

$$\Phi(u, t) = \int_{E_r} \Phi(z, s)\, p_z(u, t - s)\, dz + \int_s^t \int_{E_r} g(z)\, \Phi(z, \eta)\, p_z(u, t - \eta)\, dz\, dy_\eta. \tag{18a}$$

Proof. Since

$$\operatorname{Prob}\{x_t \in \Gamma\} = \int_{E_r} \operatorname{Prob}\{x_t \in \Gamma \mid x_0 = z\}\, dF(z)$$

where $F(z)$ is the probability distribution of x_0, it follows that for $t > 0$ $\operatorname{Prob}\{x_t \in \Gamma\}$ is also absolutely continuous with respect to the Lebesgue measure with density $p(u, t)$. By Eq. (10), we set: $\Phi(u, t) = \tilde{\Lambda}(u, y_0^t)\, p(u, t)$. Since x_t is a Markov process $p(u,t)\, P(dz, s; u, t) = p(z, s)\, p_z(u, t - s)\, dz$. Therefore, multiplying Eq. (11) by $p(u, t)$, (18) follows. Following the same arguments as used to derive (11), it also follows that

$$\tilde{\Lambda}(u, y_0^t) = \int_{E_r} \tilde{\Lambda}(z, y_0^s)\, P(dz, s; u, t)\, dz + \int_s^t \int_{E_r} g(z)\, \tilde{\Lambda}(z, y_0^\eta) P(dz, \eta; u, t)\, dy_\eta,$$

multiplying this equation by $p(u, t)$ gives (18a).

It remains, now, to prove that, a.s.,

$$E_{(0)}\left\{\int_0^t \Lambda(x_0^s, y_0^s)\, g(x_s)\, dy_s \mid \mathscr{B}(x_t, y_0^t)\right\} = \int_0^t E_{(0)}\{\Lambda(x_0^s, y_0^s)\, g(x_s) \mid \mathscr{B}(x_t, y_0^t)\}\, dy_s \tag{19}$$

where $\Lambda(x_0^s, y_0^s)$ is given by Eq. (4). Throughout the proof we will write $\Lambda(s)$ for $\Lambda(x_0^s, y_0^s)$ and $\mathscr{B}(s)$ for $\mathscr{B}(x_s, y_0^s)$.

Let us first assume that $|g(x)|$ is bounded. It follows, then, from theorem 7.3 of [11] that all the moments of $\Lambda(s)$ are bounded in any finite s interval and $\Lambda(s)g(x_s)$ is continuous in quadratic mean. Therefore, there exists a sequence of partitions

$$0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_i^{(n)} \cdots t_n^{(n)} = t$$
$$\delta_n = \max_{0 \leqq j \leqq n-1} (t_{i+1}^{(n)} - t_i^{(n)}); \quad \lim_{n\to\infty} \delta_n = 0,$$

such that the sequence of partial sums

$$I_n = \sum_{i=0}^{n-1} \Lambda(t_i^{(n)}) g(x_{t_i^{(n)}}) (y_{t_{i+1}^{(n)}} - y_{t_i^{(n)}})$$

converges in q.m. to $\int_0^t \Lambda(s) g(x_s) dy_s$. Let

$$J_n = E_{(0)}\{I_n \mid \mathscr{B}(t)\}$$
$$= \sum_{i=0}^{n-1} E_{(0)}\{\Lambda(t_i^{(n)}) g(x_{t_i^{(n)}}) \mid \mathscr{B}(t)\} (y_{t_{i+1}^{(n)}} - y_{t_i^{(n)}}).$$

Since convergence in q.m. and conditional expectations commute,

$$E_{(0)}\{\Lambda(s) g(x_s) \mid \mathscr{B}(t)\}$$

is q.m. continuous (in s) and:

$$E_{(0)}\left\{\int_0^t \Lambda(s) g(x_s) dy_s \mid \mathscr{B}(t)\right\} = \lim_{n\to\infty} E_{(0)}\{I_n \mid \mathscr{B}(t)\}$$
$$= \lim_{n\to\infty} J_n$$
$$= \int_0^t E_{(0)}\{\Lambda(s) g(x_s) \mid \mathscr{B}(t)\} dy_s,$$

which proves (19) for $|g(x)|$ bounded. It follows, by the same argument, that (19) holds whenever $\Lambda(s)g(x_s)$ is continuous in q.m.

From this point up to Eq. (24) we follow Dynkin ([11], proof of theorem 7.3). Let

$$f_n(v) = \begin{cases} v & \text{for } v \leqq n \\ v + \frac{1}{2}(v - n - 2)(v - n)^3 & \text{for } n < v < n+1 \\ n + \frac{1}{2} & \text{for } n + 1 \leqq v. \end{cases} \tag{20}$$

It follows by a direct calculation that:

a) for all $x \in [0, \infty)$, $f_n(x)$, $f_n'(x)$, $f_n''(x)$ are continuous and $0 \leqq f_n'(x) \leqq 1$; $-\frac{3}{2} \leqq f_n''(x) \leqq 0$,

b) $f_n'(x) = 0$ for $x \geqq n + 1$; $f_n''(x) = 0$ for $x \notin (n, n+1)$,

c) for any $x \in [0, \infty)$, $0 \leqq f_n(x) \uparrow x$ as $n \to \infty$.

Applying Itô's formula to $f_n(\Lambda(s))$ we have by Eq. (4)

$$f_n(\Lambda(s)) - 1 = \int_0^s f_n'(\Lambda(\theta))\,\Lambda(\theta)\,g(x_\theta)\,dy_\theta + \tfrac{1}{2}\int_0^s f_n''(\Lambda(\theta))\,\Lambda^2(\theta)\,g^2(x_\theta)\,d\theta\,. \tag{21}$$

By property c and monotone convergence for conditional expectations we have

$$\begin{aligned} E_{(0)}\{\Lambda(t)\,|\,\mathscr{B}(t)\} - 1 &= \lim_{n\to\infty} E_{(0)}\{f_n(\Lambda(t)\,|\,\mathscr{B}(t))\} - 1 \\ &= P\text{-}\lim_{n\to\infty} E_{(0)}\left\{\int_0^t f_n'(\Lambda(s))\,\Lambda(s)\,g(x_s)\,dy_s\,|\,\mathscr{B}(t)\right\} \\ &\quad + P\text{-}\lim_{n\to\infty} E_{(0)}\left\{\int_0^t f_n''(\Lambda(s))\,\Lambda^2(s)\,g^2(x_s)\,ds\,|\,\mathscr{B}(t)\right\} \end{aligned} \tag{22}$$

provided that any one of the last two limits (in probability) exists. Similarly we have from Eq. (21):

$$\begin{aligned} E_{(0)}\{\Lambda(t)\} - 1 &= \lim_{n\to\infty} E_{(0)}\left\{\int_0^t f_n'(\Lambda(s))\,\Lambda(s)\,g(x_s)\,dy_s\right\} \\ &\quad + \lim_{n\to\infty} E_{(0)}\left\{\int_0^t f_n''(\Lambda(s))\,\Lambda^2(s)\,g^2(x_s)\,ds\right\}. \end{aligned} \tag{23}$$

Since $\Lambda(t)$ is the $R-N$ derivative of two equivalent probability measures, $E_{(0)}\{\Lambda(t)\} = 1$. Also, since

$$E_{(0)}\left\{\int_0^t (f_n'(\Lambda(s)))^2\,\Lambda^2(s)\,g^2(x_s)\,ds\right\} \leqq (n+1)^2 \int_0^t E_{(0)}\,g^2(x_s)\,ds < \infty\,,$$

the first term is the r.h.s. of Eq. (23), being the expectation of a stochastic integral, is zero and; therefore:

$$\lim_{n\to\infty} E_{(0)}\left\{\int_0^t f_n''(\Lambda(s))\,\Lambda^2(s)\,g^2(x_s)\,ds\right\} = 0\,. \tag{24}$$

Since $f_n''(\Lambda) \leqq 0$, it follows that the last term in the r.h.s. of Eq. (22) converges in L_1 and, therefore, in probability to zero, and

$$E_{(0)}\{\Lambda(t)\,|\,\mathscr{B}(t)\} - 1 = P\text{-}\lim_{n\to\infty} E_{(0)}\left\{\int_0^t f_n'(\Lambda(s))\,\Lambda(s)\,g(x_s)\,dy_s\,|\,\mathscr{B}(t)\right\}.$$

Since

$$|f_n'(\Lambda(s))\,\Lambda(s)| \leqq n+1 \quad \text{and} \quad E_{(0)}\left\{\int_0^t g^2(x_s)\,ds\right\} < \infty$$

it follows that $f_n'(\Lambda(s))\Lambda(s)g(x_s)$ is continuous in q.m., therefore:

$$E_{(0)}\{\Lambda(t)\,|\,\mathscr{B}(t)\} - 1 = P\text{-}\lim_{n\to\infty} \int_0^t E_{(0)}\{f_n'(\Lambda(s))\,\Lambda(s)\,g(x_s)\,|\,\mathscr{B}(t)\}\,dy_s\,. \tag{25}$$

Since $E_{(1)}|g(x_t)| < \infty$, it follows from (1) that

$$E_{(0)}(\Lambda(t)|g(x_t)|) < \infty. \tag{26}$$

Now, $|f'(\Lambda(s))\Lambda(s)g(x_s)| \leqq |\Lambda(s)g(x_s)|$ and $f_n'(\Lambda(s))\Lambda(s)\cdot|g(x_s)|$ converges to $\Lambda(s)\cdot|g(x_s)|$ as $n\to\infty$. Therefore, a.s., $|E_{(0)}\{f_n'(\Lambda(s))\Lambda(s)g(x_s)|\mathscr{B}(t)\}| \leqq E_{(0)}\{\Lambda(s)\cdot|g(x_s)|\|\mathscr{B}(t)\}$ and $E_{(0)}\{f_n'(\Lambda(s))\Lambda(s)g(x_s)|\mathscr{B}(t)\}$ converges a.s. to $E_{(0)}\{\Lambda(s)g(x_s)|\mathscr{B}(t)\}$. Applying, now, ITô's dominated convergence lemma for stochastic integrals ([12] property $G-2$ p. 14, or [3] property 5 Ch. 2 section I), it follows that the order of the limit and the stochastic integration is Eq. (25) may be interchanged which proves Eq. (19).

4. Sufficient Conditions for the Uniqueness of the Solution to Eq. (18)

Lemma 1. *If $p(u,t)$ is bounded on $E_r\times[0,T]$ and $g(u)$ is bounded on E_r then, under the conditions of corollary* 1,

$$E_{(1)}\int_0^T\int_{E_r}(\Phi(u,t))^2\,du\,dt < \infty.$$

Proof. By Eqs. (10) and (17) we have:

$$\begin{aligned}
E_{(1)}\int_{E_r}(\Phi(u,t))^2\,du &\leqq KE_{(1)}\int_{E_r}(\tilde{\Lambda}(u,y_0^t))^2\,p(u,t)\,du\\
&= KE_{(1)}\{E_{(0)}[E_{(0)}^2(\Lambda(x_0^t,y_0^t)|\mathscr{B}(x_t,y_0^t)|\mathscr{B}(y_0^t)]\}\\
&\leqq KE_{(1)}\{E_{(0)}[E_{(0)}(\Lambda^2(x_0^t,y_0^t)|\mathscr{B}(x_t,y_0^t)|\mathscr{B}(y_0^t)]\}\\
&= KE_{(0)}\{\Lambda(x_0^t,y_0^t)\,E_{(0)}[E_{(0)}(\Lambda^2(x_0^t,y_0^t)|\mathscr{B}(x_t,y_0^t))|\mathscr{B}(y_0^t)]\}\\
&\leqq KE_{(0)}^{1/2}(\Lambda^2(x_0^t,y_0^t))\cdot E_{(0)}^{1/2}\{\Lambda^4(x_0^t,y_0^t)\}.
\end{aligned}$$

The result follows now by the boundedness of $g(u)$, Eq. (4) and theorem 7.3 (equation 7.84′) of [10].

Let H be the class of real valued functions $\psi(u,t,\omega)$ on $E_r\times[0,T]\times\Omega$ such that

$$\|\psi\| = \Big(E_{(1)}\int_0^T\int_{E_r}\psi^2(u,t,\omega)\,du\,dt\Big)^{1/2} < \infty,$$

and for each t in $[0,T]$, the collection of random variables $\psi(u,s,\omega)$, $0\leqq s\leqq t$, $u\in E_r$, is measurable with respect to $\mathscr{B}(\tilde{w}_0^t, w_0^t)$. Note that H is Cauchy complete in the norm $\|\psi\|$ defined above. Let U_t denote the operator

$$U_tf(u) = (U_tf(\cdot))(u) = \int_{E_r} f(z)\,p_z(u,t)\,dz \tag{27}$$

for $t>0$ and $U_0f(u)=f(u)$. In this section we will assume that U_t is a bounded transformation from L_2 functions on E_r to L_2 functions on E_r, uniformly in $[0,T]$. Namely, there exists a constant $k<\infty$ such that

$$\int_{E_r}(U_tf(u))^2\,du \leqq k\int_{E_r} f^2(u)\,du \tag{28}$$

for all $t\in[0,T]$ and all $f(u)$ which are L_2 on E_r.

Sufficient conditions for $p_z(u, t)$ to exist and have this property are the following:

(i) There exists a constant $\lambda > 0$ such that for all

$$x \in E_r, \quad v \in E_r, \quad v^T b^T(x) b(x) v \geqq \lambda v^T v .$$

(ii) The functions,

$$a_i(x), \frac{\partial a_i(x)}{\partial x_i}, \quad (b^T(x) b(x))_{ij}, \frac{\partial (b^T(x) b(x))_{ij}}{\partial x_i}, \frac{\partial^2 (b^T(x) b(x))_{ij}}{\partial x_i \partial x_j}$$

are bounded, and satisfy a Hölder condition in E_r.
This follows from the bound 0.24 C_2 of theorem 0.5 [10] and Parceval's theorem.

Theorem 2. *If $p(u, t)$ is bounded on $E_r \times [0, T]$, $g(u)$ is bounded on E_r and U_t satisfies Eq.* (28), *then Eq.* (18) *has a unique solution in H.*

Proof. Let $\psi(u, s, \omega)$ belong to H. By a standard argument (e.g. approximating $U_{\theta-s}\psi(u, s, \omega)$ by functions continuous in θ, u, s as on p. 17 of [6]), it follows that $\int_0^t U_{t-s}\psi(u, s, \omega) dw_s(\omega)$ has a version which for almost all ω is a Borel function in (u, t). Therefore $\int_0^t U_{t-s}\psi(u, s, \omega) dw_s(\omega)$ also belongs to H and

$$\left\| \int_0^t U_{t-s}\psi(u, s, \omega) dw_s(\omega) \right\| \leqq k T \| \psi(u, s, \omega) \| .$$

Define $\Phi_i(u, s, \omega)$ by:

$$\begin{aligned} \Phi_0(u, t, \omega) &= p(u, t) \\ \Phi_{i+1}(u, t, \omega) &= p(u, t) + \int_0^t U_{t-s} \Phi_i(u, t, \omega) g(u) dy_s(\omega) \\ &= p(u, t) + \int_0^t g(x_s(\omega) U_{t-s} g(u) \Phi_i(u, s, \omega) ds \\ &\quad + \int_0^t U_{t-s} g(u) \Phi_i(u, s, \omega) dw_s(\omega) . \end{aligned}$$

It follows now, by the method of successive approximations, that $\Phi_i(u, t, \omega)$ converges in H as $i \to \infty$ to $\Phi(u, t, \omega)$ which is a solution to (18) and the solution is unique. The details are the same as for stochastic differential equations (e.g. [6], [7], [10]), and, therefore, omitted.

5. An Evolution-Type Integral Equation for the Unnormalized Density

Assume that the x_t process possesses a transition density $p_z(u, t)$ for $t > 0$. Let $\mathfrak{L}^+$ denote the (Fokker-Planck or forward Kolmogorov) differential operator

$$\mathfrak{L}^+ = -\sum_{i=1}^{r} \frac{\partial a_i(u)}{\partial u_i} + \frac{1}{2} \sum_{i=1}^{r} \sum_{j=1}^{r} \frac{\partial^2 (b^T(u) b(u))_{ij}}{\partial u_i \partial u_j}$$

where $a_i(u)$ and $(b^T(u) b(u))_{ij}$ denote the i-th component of $a(u)$ and the ij-th component of $b^T(u) b(u)$, respectively.

A real valued function $f(x)$, $x \in E_r$, will be said to belong to $C^{(2,\lambda)}$ if $f(x)$ and its first and second partial derivatives are bounded, continuous and satisfy on E_r a Hölder condition with exponent $\lambda > 0$.

A transition density will be said to be of class A it if satisfies the following conditions:

A.1. If $f(u)$ is real valued, continuous and bounded on E_r and $0 \leqq s \leqq t$ then

$$U_t f(u) = U_s f(u) + \int_s^t \mathfrak{L}^+ U_\theta f(u)\, d\theta\,. \tag{29}$$

A.2. If $f(u, \theta)$, $u \in E_r$, $\theta \in [\theta_1, \theta_2]$ and its first and second partial derivatives with respect to the u variables are bounded and continuous on $E_r \times [\theta_1, \theta_2]$, and $f(u, \theta)$ is $C^{(2,\lambda)}$ in u, uniformly in $[\theta_1, \theta_2]$, then $U_t f(u, \theta)$ and its first and second partial derivatives with respect to the u viariables are continuous on $(0, T) \times E_r \times [\theta_1, \theta_2]$ and bounded on $[0, T] \times G \times [\theta_1, \theta_2]$ where G is any bounded subset of E_r.

Theorem 3. *Assume that the x_t process possesses a transition density which is of class A. Assume that $\Phi(u, t)$ satisfies Eq.* (18) *and a.s. $\Phi(u, t)$ and $g(u)\Phi(u, t)$ together with their first and second derivatives with respect to the u variables bounded and continuous in $E_r \times [t_1, t_2]$ and $\Phi(u, t)$, $g(u)\Phi(u, t)$ are $C^{(2,\lambda)}$ in u, uniformly in $[t_1, t_2]$, then $\Phi(u, t)$ also satisfies the evolution-type equation*

$$\Phi(u, t) = \Phi(u, s) + \int_s^t \mathfrak{L}^+ \Phi(u, \eta)\, d\eta + \int_s^t g(u)\, \Phi(u, \eta)\, dy_\eta \tag{30}$$

$t_1 \leqq s \leqq t \leqq t_2$.

Proof. Rewriting Eq. (18a) in terms of (27) we have:

$$\Phi(u, \theta) = U_{\theta-s}\Phi(u, s) + \int_s^\theta U_{\theta-\eta}(g(u)\, \Phi(u, \eta))\, dy_\eta\,. \tag{31}$$

By Eq. (29):

$$U_{t-s}\Phi(u, s) = \Phi(u, s) + \int_s^t \mathfrak{L}^+ U_{\theta-s}\Phi(u, s)\, d\theta\,, \tag{32}$$

$$\begin{aligned}\int_s^t (U_{t-\eta}\, g(u)\, \Phi(u, \eta))\, dy_\eta &= \int_s^t g(u)\, \Phi(u, \eta)\, dy_\eta \\ &\quad + \int_s^t \int_\eta^t \mathfrak{L}^+ U_{\theta-\eta}(g(u)\, \Phi(u, \eta))\, d\theta\, dy_\eta\,.\end{aligned} \tag{33}$$

Substituting for $U_{\theta-s}$ from (31) to (32):

$$\begin{aligned}U_{t-s}\Phi(u, s) &= \Phi(u, s) + \int_s^t \mathfrak{L}^+ \Phi(u, \theta)\, d\theta \\ &\quad - \int_s^t \mathfrak{L}^+ \int_s^\theta U_{\theta-\eta}(g(u)\, \Phi(u, \eta))\, dy_\eta\, d\theta\,.\end{aligned} \tag{34}$$

Replacing, in (31), θ by t and substituting from (34) and (33) we have:

$$\Phi(u,t) = \Phi(u,s) + \int_s^t \mathfrak{L}^+ \Phi(u,\theta)\,d\theta + \int_s^t g(u)\,\Phi(u,\eta)\,dy_\eta$$
$$+ \int_s^t \int_\eta^t \mathfrak{L}^+ U_{\theta-\eta}(g(u)\,\Phi(u,\eta))\,d\theta\,dy_\eta$$
$$- \int_s^t \mathfrak{L}^+ \int_s^\theta U_{\theta-\eta}(g(u)\,\Phi(u,\eta))\,dy_\eta\,d\theta\,.$$

Comparing the last equation with (30), it follows that in order to prove (30) it remains to show that:

$$\int_s^t \int_\eta^t \mathfrak{L}^+ U_{\theta-\eta}(g(u)\,\Phi(u,\eta))\,d\theta\,dy_\eta = \int_s^t \mathfrak{L}^+ \int_s^\theta U_{\theta-\eta}(g(u)\,\Phi(u,\eta))\,dy_\eta\,d\theta\,. \tag{35}$$

It follows from A.2, by the mean value theorem for derivatives and Itô's dominated convergence lemma ([12] p. 14 or [3] chapter 2 section I) that, in the right hand side of the last equation we may interchange $\mathfrak{L}^+$ with the integration with respect to y_η. Eq. (35), therefore, becomes:

$$\int_s^t \int_\eta^t \mathfrak{L}^+ U_{\theta-\eta}(g(u)\,\Phi(u,\eta))\,d\theta\,dy_\eta = \int_s^t \int_s^\theta \mathfrak{L}^+ U_{\theta-\eta}(g(u)\,\Phi(u,\eta))\,dy_\eta\,d\theta$$
$$= \int_s^t \int_s^\eta \mathfrak{L}^+ U_{\eta-\theta}(g(u), \Phi(u,\theta)\,dy_\theta\,d\eta\,,$$

(and it remains to justify the formal interchange of the order of the integrations).

Let:

$$\psi(\alpha,\beta) = \mathfrak{L}^+ U_{\alpha-\beta}(g(u)\,\Phi(u,\beta)\,, \qquad \alpha \geqq \beta$$

we have, then, to show that:

$$\int_s^t \int_\eta^t \psi(\theta,\eta)\,d\theta\,dy_\eta = \int_s^t \int_s^\eta \psi(\eta,\theta)\,dy_\theta\,d\eta\,. \tag{35a}$$

Let

$$f_1(\eta) = \int_\eta^t \psi_N(\theta,\eta)\,d\theta$$

$$f_2(\eta) = \int_s^\eta \psi_N(\eta,\theta)\,dy_\theta$$

where

$$\psi_N = \begin{cases} \psi, & |\psi| \leqq N \\ 0, & |\psi| > N\,. \end{cases}$$

Consider now:

$$E\left(\int_s^t f_1(\eta)\,dy_\eta - \int_s^t f_2(\eta)\,d\eta\right)^2. \tag{36}$$

The following relations are derived in [11]:

$$E\left(\int_a^b h_1(s)\,dy_s \int_a^b h_2(s)\,dy_s\right) = E\int_a^b h_1(s)\,h_2(s)\,ds$$

and

$$E\left(\int_a^b h_1(s)\,dy_s \int_a^b h_2(s)\,ds\right) = E\int_a^b h_2(s)\int_a^s h_1(v)\,dy_v\,ds\,.$$

Applying these relations repeatedly and using

$$\chi_u(s) = \begin{cases} 0, & s > u \\ \frac{1}{2}, & s = u \\ 1, & s < u \end{cases}$$

we obtain the following equations:

$$E\left(\int_s^t f_1(\eta)\,dy_\eta\right)^2 = E\int_s^t\int_s^t\int_s^t (1-\chi_\xi(\eta))(1-\chi_\xi(\theta))\,\psi_N(\eta,\xi)\,\psi_N(\theta,\xi)\,d\theta\,d\eta\,d\xi$$

$$\begin{aligned} E\left(\int_s^t f_2(\eta)\,d\eta\right)^2 &= E\int_s^t\int_s^t\int_s^{\min(\eta,\theta)} \psi_N(\eta,\xi)\,\psi_N(\theta,\xi)\,d\xi\,d\eta\,d\theta \\ &= E\int_s^t\int_s^t\int_s^t \chi_\eta(\xi)\,\chi_\theta(\xi)\,\psi_N(\eta,\xi)\,\psi_N(\theta,\xi)\,d\xi\,d\eta\,d\theta \end{aligned}$$

$$\begin{aligned} -2E\left(\int_s^t f_2(\eta)\,d\eta\int_s^t f_1(\eta)\,dy_\eta\right) &= -2E\int_s^t f_2(\eta)\int_s^\eta f_1(\xi)\,dy_\xi\,d\eta \\ &= -2E\int_s^t\int_s^\eta \psi_N(\eta,\xi)\,f_1(\xi)\,d\xi\,d\eta \\ &= -2E\int_s^t\int_s^t\int_s^t \chi_\eta(\xi)\,(1-\chi_\xi(\theta))\,\psi_N(\eta,\xi)\,\psi_N(\theta,\xi)\,d\xi\,d\eta\,d\theta \end{aligned}$$

Substituting into (36), collecting the χ terms, and using $\chi_\theta(\xi) = 1-\chi_\xi(\theta)$; $\chi_\eta(\xi) = 1-\chi_\xi(\eta)$ we get that (36) is zero. Let

$$\begin{aligned} I_N = &\left(\int_s^t\int_\eta^t \psi(\theta,\eta)\,d\theta\,dy_\eta - \int_s^t\int_s^\theta \psi(\theta,\eta)\,dy_\eta\,d\theta\right) \\ &- \left(\int_s^t\int_\eta^t \psi_N(\theta,\eta)\,d\theta\,dy_\eta - \int_s^t\int_s^\theta \psi_N(\theta,\eta)\,dy_\eta\,d\theta\right). \end{aligned}$$

Then, for any $\delta > 0$:

$$\mathrm{Prob}\{|I_N| > \delta\} \leqq \mathrm{Prob}\left\{\sup_{\substack{s\leqq\eta\leqq\theta \\ s\leqq\theta\leqq t}} |\psi(\theta,\eta)| \geqq N\right\}.$$

Since under our assumptions $\psi(\theta,\eta)$ is a.s. a bounded function of θ, η in $\eta\in[s,\theta]$, $\theta\in[s,t]$, it follows that I_N converges in probability to zero. Therefore, (35a) and (35) holds a.s. Note that the strong assumptions in theorem 3 were mainly needed for the justification of the interchange of $\mathfrak{L}^+$ with the stochastic integration in the right hand side of (35) (the proof of (35a) can be modified to hold under weaker assumptions).

Theorem 4. *Assume that U_t is of class A and if $f(u)$ and its first and second partial derivatives are continuous and bounded, then $U_t\mathfrak{L}^+ f(u) = \mathfrak{L}^+ U_t f(u)$. Also* *assume that a.s. $\Phi(u,\theta)$, $g(u)\,\Phi(u,\theta)$ and $\mathfrak{L}^+ g(u)\,\Phi(u,\theta)$ and their first and second partial derivatives are continuous and bounded in $E_r\times[t_1,t_2]$ and $\Phi(u,\theta)$, $g(u)\,\Phi(u,\theta)$, $\mathfrak{L}^+ g(u)\,\Phi(u,\theta)$ are $C^{(2,\lambda)}$ in u uniformly in $[t_1,t_2]$. Then, if $\Phi(u,\theta)$ satisfies in $[t_1,t_2]$ Eq. (30), it also satisfies in $[t_1,t_2]$ Eq. (18).*

Proof. Only the formal part of the proof will be given; the justification for the operations are the same as in the proof of theorem 3 and are, therefore, omitted.

$$\begin{aligned}\int_s^t \mathfrak{L}^+ U_{t-\theta}\,\Phi(u,\theta)\,d\theta &= \int_s^t \mathfrak{L}^+ \Phi(u,\theta)\,d\theta + \int_s^t \mathfrak{L}^+ \int_\theta^t U_{t-\eta}\,\mathfrak{L}^+ \Phi(u,\theta)\,d\eta\,d\theta \\ &= \int_s^t \mathfrak{L}^+ \Phi(u,\theta)\,d\theta + \int_s^t\int_s^\eta \mathfrak{L}^+ U_{t-\eta}\,\mathfrak{L}^+ \Phi(u,\theta)\,d\theta\,d\eta \\ &= \int_s^t \mathfrak{L}^+ \Phi(u,\theta)\,d\theta + \int_s^t \mathfrak{L}^+ U_{t-\eta}\int_s^\eta \mathfrak{L}^+ \Phi(u,\theta)\,d\theta\,d\eta\,.\end{aligned}$$

Substituting for $\int_s^\eta \mathfrak{L}^+ \Phi(u,\theta)\,d\theta$ from Eq. (30) yields:

$$\begin{aligned}\int_s^t \mathfrak{L}^+ U_{t-\theta}\,\Phi(u,\theta)\,d\theta &= \int_s^t \mathfrak{L}^+ \Phi(u,\theta)\,d\theta + \int_s^t \mathfrak{L}^+ U_{t-\eta}(\Phi(u,\eta)-\Phi(u,s))\,d\eta \\ &\quad - \int_s^t \mathfrak{L}^+ U_{t-\eta}\left(\int_s^\eta g(u)\,\Phi(u,\xi)\,dy_\xi\right)d\eta\,,\end{aligned}$$

or:

$$\begin{aligned}\int_s^t \mathfrak{L}^+ U_{t-\eta}\left(\int_s^\eta g(u)\,\Phi(u,\xi)\,dy_\xi\right)d\eta &= \int_s^t \mathfrak{L}^+ \Phi(u,\theta)\,d\theta - \int_s^t \mathfrak{L}^+ U_{t-\eta}\,\Phi(u,s)\,d\eta \\ &= \int_s^t \mathfrak{L}^+ \Phi(u,\theta)\,d\theta - U_{t-s}\,\Phi(u,s) + \Phi(u,s).\end{aligned} \tag{37}$$

Applying (27) and interchanging the order of integrations:

$$\begin{aligned}\int_s^t U_{t-\xi}\,g(u)\,\Phi(u,\xi)\,dy_\xi &- \int_s^t g(u)\,\Phi(u,\xi)\,dy_\xi \\ &= \int_s^t \mathfrak{L}^+ \int_\xi^t U_{t-\eta}\,g(u)\,\Phi(u,\xi)\,d\eta\,dy_\xi \\ &= \int_s^t \mathfrak{L}^+ U_{t-\eta}\int_s^\eta g(u)\,\Phi(u,\xi)\,dy_\xi\,d\eta\,.\end{aligned} \tag{38}$$

Eq. (18a) follows now from Eq. (38) and (37).

Remark. Several generalizations of the results of this paper with, essentially, the same proofs are straightforward. In particular, replacing $a(x)$, $b(x)$, $g(x)$ by the time-dependent $a(x,t)$, $b(x,t)$, $g(x,t)$; or, replacing the equation for y_t in Eq. (2) by

$$y_t = \int_0^t g(x_s, y_s)\,ds + \int_0^t dw_s$$

where y_t, $g(x,y)$, $\tilde{w}_t$ take values in the q-dimensional Euclidean space E_q; or replacing w_t and w_t by general processes with independent increments [13], [14].

References

1. STRATONOVICH, R. I.: Conditional Markov processes. Theor. Probab. Appl. **5**, 156—178 (1960).
2. KUSHNER, H. J.: Dynamical equations for optimal nonlinear filtering. J. Differential Equations **2**, 179—190 (1967).
3. BUCY, R. C.: Nonlinear filtering theory. IEEE Trans. Automatic Control **10**, 198—199 (1965).
4. SHIRYAEV, A. N.: On stochastic equations in the theory of conditional Markov processes. Theor. Probab. Appl. **11**, 179—184 (1966).
5. LOÈVE, M.: Probability theory, 3rd edition. Princeton, N. J.: Van Nostrand 1963.
6. SKOROHOD, A. V.: Studies in the theory of random processes. New York: Addison-Wesley 1965.
7. GIRSANOV, I. V.: On transforming a certain class of stochastic processes by absolutely continuous substitutions of measures. Theor. Probab. Appl. **5**, 285—301 (1960).
8. DOOB, J. L.: Stochastic processes. New York: Wiley 1953.
9. BLANC-LAPIERRE, A., and R. FORTET: Théorie des fonctions aléatoires. Paris: Masson et Cie., Ed. 1953; Chapter VII. (Also: D. A. DARLING, and A. J. F. SIEGERT: A systematic approach to a class of problems in the theory of noise and other random phenomena. Part I, IRE Trans. Inform. Theory IT-**3**, 32—37 (1957)) .
10. DYNKIN, E. B.: Markov processes. Berlin-Heidelberg-New York: Springer 1965.
11. ITÔ, K.: On a formula concerning stochastic differentials. Nagoya math. J. **3**, 55—65 (1951).
12. — On stochastic differential equations. Mem. Amer. math. Soc. **4** (1951).
13. WOHNHAM, M.: Some applications of stochastic differential equations to optimal nonlinear filtering. J. Soc. industr. appl. Math. Control **2**, 347—369 (1965).
14. ZAKAI, M.: The optimal filtering of Markov jump processes in additive white noise. Applied Research Lab, Sylvania Elektronic Systems, Waltham, Mass. R. N. 563, June 1965.

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