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On the Conservative Parts of the Markov Processes Induced by a Measurable Transformation

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Summary. For the two Markov processes associated with the application of a measurable transformation in a probability space in the forward and backward direction respectively, the equivalent descriptions by kernel functions and by Markov operators in $\mathfrak{L}_1$, $\mathfrak{L}_\infty$, and in the space of absolutely continuous finite signed measures are identified. The connections between the conservative parts of the probability space with respect to these processes and the various conservative parts associated with a measurable transformation in the literature are clarified. Finally the inclusion relations between these various conservative parts are established.

§ 1. Introduction

Let $(X, \mathfrak{R}, \mu)$ be a normed measure space and let T be a measurable transformation in X. With respect to this transformation several authors have described a decomposition of X into a conservative and a dissipative part (cf. E. Hopf [9], [10], Tsûrumi [15], Choksi [1], Parry [13] and the authors [6], [14]).

Our aim in this paper is to clarify the relationship between these decompositions and the two Markov processes associated with the application of the (not necessarily invertible) transformation T in the forward or backward direction respectively. Both processes have already been introduced by E. Hopf in [10] § 6 who also (as did Tsurumi [15]) implicitely considered the conservative part of X induced by the process associated with the transition $x \to T^{-1}\{x\}$.

Here we shall identify the four equivalent analytic descriptions of both processes by means of a kernel function and by means of Markov operators in $\mathfrak{L}_1(X, \mathfrak{R}, \mu)$, in $\mathfrak{L}_\infty(X, \mathfrak{R}, \mu)$, and in the linear space of absolutely continuous finite signed measures. Furthermore we shall show that the decomposition studied by the authors [6], [14] which is in some sense a refinement of a decomposition studied by Choksi [1] coincides with the one induced by the process associated with the transition $x \to Tx$. The decomposition studied by Parry [13] is obtained if the σ-algebra $\mathfrak{R}$ is replaced by the largest σ-subalgebra $\mathfrak{S}$ with the property that T is "essentially invertible" in $(X, \mathfrak{S}, \mu)$.

The conservative parts of X with respect to the decompositions studied by E. Hopf and Tsurumi, Choksi, the authors, and Parry, form an increasing sequence of sets in this order if T satisfies some regularity conditions. In order to show that inequality may also occur between the last two mentioned "conservative" parts of X (for the other inclusions this has already been shown) we shall finally construct a measurable transformation in a suitable measure space which is "purely dissipative" in the one sense and conservative in the other.

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§ 2. Preliminaries

For later reference we restate the four equivalent descriptions of a μ-measurable Markov process as introduced by E. Hopf in [10] and the connections between them.

Definition 1. A μ-measurable Markov process in $(X, \mathfrak{R}, \mu)$ is a non-negative function $P(\cdot, \cdot)$ on $X \times \mathfrak{R}$ with the following properties:

a) For every $A \in \mathfrak{R}$ the function $P(\cdot, A)$ is defined μ-almost everywhere on X and $\mathfrak{R}$-measurable.

b) If $\{A_n\}_{n=1}^{\infty}$ is a sequence of pairwise disjoint sets $A_n \in \mathfrak{R}$, then

$$P\left(x, \bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(x, A_n)$$

for μ-almost all $x \in X$.

c) $P(x, X) = 1$ for μ-almost all $x \in X$.

d) $P(x, A) = 0$ for μ-almost all $x \in X$ for every μ-null set A.

Definition 2. Let Φ_μ be the real linear space of all real-valued finite signed measures on $(X, \mathfrak{R})$ which are absolutely continuous with respect to μ. A Markov operator in Φ_μ is a non-negative linear operator Λ in Φ_μ such that $\Lambda\varphi(X) = \varphi(X)$ for all $\varphi \in \Phi_\mu$.

Definition 3. A Markov operator in $\mathfrak{L}_1(X, \mathfrak{R}, \mu)$ is a non-negative linear operator L in $\mathfrak{L}_1(X, \mathfrak{R}, \mu)$ such that

$$\int_X Lf\, d\mu = \int_X f\, d\mu \quad \text{for all} \quad f \in \mathfrak{L}_1(X, \mathfrak{R}, \mu).$$

Definition 4. A dual Markov operator in $\mathfrak{L}_\infty(X, \mathfrak{R}, \mu)$ is a non-negative linear operator L^* in $\mathfrak{L}_\infty(X, \mathfrak{R}, \mu)$ with the following properties:

a) If $\{g_n\}_{n=1}^{\infty} \subset \mathfrak{L}_\infty(X, \mathfrak{R}, \mu)$ is a non-increasing sequence of functions such that

$$\lim_{n\to\infty} g_n(x) = 0 \quad \text{for } \mu\text{-almost all } x \in X,$$

then

$$\lim_{n\to\infty} L^* g_n(x) = 0 \quad \text{for } \mu\text{-almost all } x \in X.$$

b) $L^* 1 = 1$.

It is partially well-known (cf. [10]) and for the rest easily verified that the following relations establish one-to-one correspondences between μ-measurable Markov processes in $(X, \mathfrak{R}, \mu)$, Markov operators in Φ_μ, Markov operators in $\mathfrak{L}_1(X, \mathfrak{R}, \mu)$, and dual Markov operators in $\mathfrak{L}_\infty(X, \mathfrak{R}, \mu)$ (we shall denote by $d\varphi/d\mu$ the Radon-Nikodym derivative of φ with respect to μ and by c_A the characteristic function of the set A).

(1) $\quad \Lambda\varphi(A) = \int_X P(x, A)\, \varphi(dx)$ for all $\varphi \in \Phi_\mu$ and all $A \in \mathfrak{R}$;

(2) $$L\frac{d\varphi}{d\mu} = \frac{d\Lambda\varphi}{d\mu} \qquad \text{for all } \varphi \in \Phi_\mu;$$

(2′) $$\left.\begin{aligned} \varphi(A) &= \int_A f(x)\,\mu(dx) \\ \Lambda\varphi(A) &= \int_A Lf(x)\,\mu(dx) \end{aligned}\right\} \qquad \begin{aligned} &\text{for all } \varphi \in \Phi_\mu \text{ (or, equivalently,} \\ &\text{for all } f \in \mathfrak{L}_1(X, \mathfrak{R}, \mu)) \text{ and all } A \in \mathfrak{R}; \end{aligned}$$

(3) $$\int_X f L^* g\, d\mu = \int_X g\, Lf\, d\mu \qquad \begin{aligned} &\text{for all } f \in \mathfrak{L}_1(X, \mathfrak{R}, \mu) \\ &\text{and all } g \in \mathfrak{L}_\infty(X, \mathfrak{R}, \mu); \end{aligned}$$

(4) $$P(\cdot, A) = L^* c_A \qquad \text{for all } A \in \mathfrak{R}.$$

Notice in particular that the conditions imposed on a bounded operator L^* in $\mathfrak{L}_\infty(X, \mathfrak{R}, \mu)$ as in definition 4 guarantee that its adjoint L^{**} in $\mathfrak{L}_\infty^*(X, \mathfrak{R}, \mu)$ leaves the subspace $\mathfrak{L}_1(X, \mathfrak{R}, \mu)$ invariant and there induces a Markov operator L. Hence every dual Markov operator L^* in $\mathfrak{L}_\infty(X, \mathfrak{R}, \mu)$ is the adjoint of a uniquely determined Markov operator L in $\mathfrak{L}_1(X, \mathfrak{R}, \mu)$ (this fact has already been mentioned e.g. in [2] p. 2).

Combinations of (2), (3), and (4) lead to the further relations

(5) $$\int_X P(x, A) f(x)\,\mu(dx) = \int_A Lf(x)\,\mu(dx) \qquad \begin{aligned} &\text{for all } A \in \mathfrak{R} \\ &\text{and all } f \in \mathfrak{L}_1(X, \mathfrak{R}, \mu); \end{aligned}$$

$$\int_X L^* g(x)\,\varphi(dx) = \int_X g(x)\,\Lambda\varphi(dx) \qquad \begin{aligned} &\text{for all } \varphi \in \Phi_\mu \\ &\text{and all } g \in \mathfrak{L}_\infty(X, \mathfrak{R}, \mu). \end{aligned}$$

§ 3. The Transition $x \to Tx$

Let T be a measurable transformation in $(X, \mathfrak{R}, \mu)$ which is negatively non-singular (i.e. $\mu(T^{-1}A) = 0$ for all μ null sets $A \in \mathfrak{R}$). As already done in [10] § 6 we define a μ-measurable Markov process $P'(x, A)$ by

$$P'(\cdot, A) = c_{T^{-1}A} \quad \text{for all} \quad A \in \mathfrak{R}.$$

This definition agrees with the fact that under the transition from x to Tx the probability of entering the set $A \in \mathfrak{R}$ is 1 or 0 according to whether x does or does not belong to the set $T^{-1}A$.

By (1) the corresponding Markov operator Λ' in Φ_μ is given by

$$\Lambda'\varphi(A) = \int_X c_{T^{-1}A}(x)\,\varphi(dx) = \varphi(T^{-1}A) \quad \text{for all } \varphi \in \Phi_\mu \text{ and all } A \in \mathfrak{R}.$$

Relation (4) yields that for every $A \in \mathfrak{R}$ the corresponding dual Markov operator L'^* satisfies

$$L'^* c_A = c_A \circ T$$

from which we deduce that the dual Markov operator L'^* is given by

$$L'^* g = g \circ T \quad \text{for all} \quad g \in \mathfrak{L}_\infty(X, \mathfrak{R}, \mu).$$

In order to obtain a satisfactory formula also for the corresponding Markov operator L' in $\mathfrak{L}_1(X, \mathfrak{R}, \mu)$ we have to impose on T the extra condition $TX \in \mathfrak{R}$ which, incidentally, is equivalent to the condition $TT^{-1}\mathfrak{R} \subset \mathfrak{R}$. The following lemma is an easy consequence of the fact that a $T^{-1}\mathfrak{R}$-measurable function is constant on every set $T^{-1}\{x\}$ for $x \in TX$.

Lemma 1. *Let g be a $T^{-1}\mathfrak{R}$-measurable function on X and suppose $TX \in \mathfrak{R}$. If $T^{-1}x$ denotes any element of the set $T^{-1}\{x\}$ for $x \in TX$, then the relation*

$$g \circ T^{-1}(x) = g(T^{-1}x) \quad \textit{for all} \quad x \in TX$$

unambiguously defines a $\mathfrak{R}$-measurable function $g \circ T^{-1}$ on TX.

Defining $s = \dfrac{d\Lambda'\mu}{d\mu}$ we have $s \in \mathfrak{L}_1(X, \mathfrak{R}, \mu)$, $s \geqq 0$, and

$$\mu(T^{-1}A) = \int_X c_A(Tx)\,\mu(dx) = \int_X c_A(x)\,s(x)\,\mu(dx) \quad \text{for all} \quad A \in \mathfrak{R}. \tag{6}$$

In particular, for $A = X \setminus TX$ we get

$$0 = \int_{X \setminus TX} s(x)\,\mu(dx), \tag{7}$$

hence $s(x) = 0$ for μ-almost all $x \in X \setminus TX$. Moreover, from (6) we derive via monotone approximation

$$\int_X g(Tx)\,\mu(dx) = \int_X g(x)\,s(x)\,\mu(dx) \tag{8}$$

for all $\mathfrak{R}$-measurable functions g satisfying $gs \in \mathfrak{L}_1(X, \mathfrak{R}, \mu)$ (or, equivalently, $g \circ T \in \mathfrak{L}_1(X, \mathfrak{R}, \mu)$).

Let $E_{T^{-1}\mathfrak{R}}$ be the conditional expectation operator with respect to the σ-algebra $T^{-1}\mathfrak{R}$ (cf. e.g. [12], IV. 3). In view of (7) for every $f \in \mathfrak{L}_1(X, \mathfrak{R}, \mu)$ we shall define

$$s(x)\,E_{T^{-1}\mathfrak{R}}f(T^{-1}x) = 0 \quad \text{for all} \quad x \in X \setminus TX.$$

For every $f \in \mathfrak{L}_1(X, \mathfrak{R}, \mu)$ and every $A \in \mathfrak{R}$ we then obtain

$$\begin{aligned}
\int_A s(x)\,E_{T^{-1}\mathfrak{R}}f(T^{-1}x)\,\mu(dx) &= \int_X s(x)\,c_A(x)\,E_{T^{-1}\mathfrak{R}}f(T^{-1}x)\,\mu(dx) \\
&= \int_X c_A(Tx)\,E_{T^{-1}\mathfrak{R}}f(x)\,\mu(dx) \qquad \text{(by (8))} \\
&= \int_X c_A(Tx)\,f(x)\,\mu(dx) \quad (\text{since } c_A \circ T \text{ is } T^{-1}\mathfrak{R}\text{-measurable}) \\
&= \int_X P'(x, A)\,f(x)\,\mu(dx).
\end{aligned}$$

Comparing this with (5) we see that the Markov operator L' in $\mathfrak{L}_1(X, \mathfrak{R}, \mu)$ is given by

$$L'f = s[(E_{T^{-1}\mathfrak{R}}f) \circ T^{-1}] \quad \text{for all} \quad f \in \mathfrak{L}_1(X, \mathfrak{R}, \mu).$$

We collect the results in a theorem:

Theorem 1. *Let T be a negatively non-singular measurable transformation in $(X, \mathfrak{R}, \mu)$. Then*

$$P'(\cdot, A) = c_{T^{-1}A} \quad \textit{for all} \quad A \in \mathfrak{R}$$

defines a μ-measurable Markov process in X. The corresponding Markov operator Λ' in Φ_μ and dual Markov operator L'^* in $\mathfrak{L}_\infty(X, \mathfrak{R}, \mu)$ are given by

$$\begin{aligned}
\Lambda'\varphi(A) &= \varphi(T^{-1}A) && \textit{for all} \quad \varphi \in \Phi_\mu \quad \textit{and all} \quad A \in \mathfrak{R}, \\
L'^*g &= g \circ T && \textit{for all} \quad g \in \mathfrak{L}_\infty(X, \mathfrak{R}, \mu).
\end{aligned}$$

If the transformation T moreover satisfies $TX \in \mathfrak{R}$ then the corresponding Markov operator L' in $\mathfrak{L}_1(X, \mathfrak{R}, \mu)$ is given by

$$L'f = s[(E_{T^{-1}\mathfrak{R}}f) \circ T^{-1}] \quad \textit{for all} \quad f \in \mathfrak{L}_1(X, \mathfrak{R}, \mu)$$

where

$$s = \frac{d\Lambda' \mu}{d\mu}.$$

§ 4. The Transition $x \to T^{-1}\{x\}$

Let T be a measurable transformation in $(X, \mathfrak{R}, \mu)$ which is positively non-singular (i.e. $\mu(A) = 0$ for all sets $A \in \mathfrak{R}$ satisfying $\mu(T^{-1}A) = 0$) and satisfies $TX \in \mathfrak{R}$. Since $T^{-1}(X \setminus TX) = \emptyset$ we obtain $\mu(X \setminus TX) = 0$ and therefore by Lemma 1 for every $T^{-1}\mathfrak{R}$-measurable function g the function $g \circ T^{-1}$ is defined μ-almost everywhere on X and $\mathfrak{R}$-measurable.

Our aim in this section is to give explicit formulas for the four objects characterizing the μ-measurable Markov process corresponding to the transition $x \to T^{-1}\{x\}$ as, under slightly stronger conditions, introduced by E. Hopf in [10], § 6.

We define

$$P(\cdot, A) = (E_{T^{-1}\mathfrak{R}}c_A) \circ T^{-1} \quad \text{for all} \quad A \in \mathfrak{R}. \tag{9}$$

Note that for any $A \in T^{-1}\mathfrak{R}$ this formula reduces to

$$P(\cdot, A) = c_{TA}.$$

This agrees with the fact that under the transition $x \to T^{-1}\{x\}$ the probability of entering the set $A \in T^{-1}\mathfrak{R}$ is 1 or 0, corresponding to whether or not x belongs to TA.

It is easily verified that relation (9) indeed defines a μ-measurable Markov process. By (1) the corresponding Markov operator Λ in Φ_μ is then defined by

$$\Lambda\varphi(A) = \int_X E_{T^{-1}\mathfrak{R}}c_A(T^{-1}x)\,\varphi(dx) \quad \text{for all} \quad \varphi \in \Phi_\mu \quad \text{and all} \quad A \in \mathfrak{R}.$$

Relation (4) yields for the dual Markov operator L^* in $\mathfrak{L}_\infty(X, \mathfrak{R}, \mu)$

$$L^* c_A = (E_{T^{-1}\mathfrak{R}}c_A) \circ T^{-1} \quad \text{for all} \quad A \in \mathfrak{R}.$$

Via monotone approximation this implies

$$L^* g = (E_{T^{-1}\mathfrak{R}}g) \circ T^{-1} \quad \text{for all} \quad g \in \mathfrak{L}_\infty(X, \mathfrak{R}, \mu).$$

In order to obtain a formula for the corresponding Markov operator L in $\mathfrak{L}_1(X, \mathfrak{R}, \mu)$ first consider the measurable space $(X, T^{-1}\mathfrak{R})$. Since the restriction ν of $\Lambda\mu$ to $(X, T^{-1}\mathfrak{R})$ is absolutely continuous with respect to μ, the Radon-Nikodym-derivative $r = \frac{d\nu}{d\mu} \in \mathfrak{L}_1(X, T^{-1}\mathfrak{R}, \mu)$ exists. We obtain

$$\int_X r(x)\,c_A(Tx)\,\mu(dx) = \int_{T^{-1}A} r(x)\,\mu(dx) = \Lambda\mu(T^{-1}A) = \mu(A) = \int_X c_A\,\mu(dx) \quad \text{for all} \quad A \in \mathfrak{R}$$

and via monotone approximation

$$\int_X r(x) f(Tx)\,\mu(dx) = \int_X f(x)\,\mu(dx) \quad \text{for all} \quad f \in \mathfrak{L}_1(X, \mathfrak{R}, \mu). \tag{10}$$

The Markov operator L in $\mathfrak{L}_1(X, \mathfrak{R}, \mu)$ defined by

$$Lf = r(f \circ T) \quad \text{for all} \quad f \in \mathfrak{L}_1(X, \mathfrak{R}, \mu)$$

then turns out to be the adjoint operator in $\mathfrak{L}_1(X, \mathfrak{R}, \mu)$ of the dual Markov operator L^* obtained above: for all $f \in \mathfrak{L}_1(X, \mathfrak{R}, \mu)$ and all $g \in \mathfrak{L}_\infty(X, \mathfrak{R}, \mu)$ we have

$$\begin{aligned}
\int_X f(x) L^* g(x)\,\mu(dx) &= \int_X f(x) E_{T^{-1}\mathfrak{R}} g(T^{-1}x)\,\mu(dx) \\
&= \int_X r(x) f(Tx) E_{T^{-1}\mathfrak{R}} g(x)\,\mu(dx) \quad \text{(by (10))} \\
&= \int_X r(x) f(Tx) g(x)\,\mu(dx) \quad (\text{since } r(f \circ T) \text{ is } T^{-1}\mathfrak{R}\text{-measurable}) \\
&= \int_X g(x)\, Lf(x)\,\mu(dx).
\end{aligned}$$

Again we collect the results in a theorem.

Theorem 2. *Let T be a positively non-singular measurable transformation in $(X, \mathfrak{R}, \mu)$ satisfying $TX \in \mathfrak{R}$. Then*

$$P(\cdot, A) = (E_{T^{-1}\mathfrak{R}} c_A) \circ T^{-1} \quad \textit{for all} \quad A \in \mathfrak{R}$$

defines a μ-measurable Markov process in X. Let ν be the measure $\Lambda\mu$ restricted to $(X, T^{-1}\mathfrak{R})$ and let $r = \frac{d\nu}{d\mu} \in \mathfrak{L}_1(X, T^{-1}\mathfrak{R}, \mu)$. Then the corresponding Markov operators Λ in Φ_μ and L in $\mathfrak{L}_1(X, \mathfrak{R}, \mu)$, and the dual Markov operator L^* in $\mathfrak{L}_\infty(X, \mathfrak{R}, \mu)$ are given by

$$\begin{aligned}
\Lambda\varphi(A) &= \int_X E_{T^{-1}\mathfrak{R}} c_A(T^{-1}x)\,\varphi(dx) && \textit{for all} \quad \varphi \in \Phi_\mu \quad \textit{and all} \quad A \in \mathfrak{R}, \\
Lf &= r(f \circ T) && \textit{for all} \quad f \in \mathfrak{L}_1(X, \mathfrak{R}, \mu), \\
L^* g &= (E_{T^{-1}\mathfrak{R}} g) \circ T^{-1} && \textit{for all} \quad g \in \mathfrak{L}_\infty(X, \mathfrak{R}, \mu).
\end{aligned}$$

§ 5. Relations between the Markov Processes Associated with the Transitions $x \to Tx$ and $x \to T^{-1}\{x\}$

Let T be a (positively and negatively) non-singular measurable transformation in $(X, \mathfrak{R}, \mu)$ satisfying $TX \in \mathfrak{R}$. It is to be expected that the transition $x \to T^{-1}\{x\}$ followed by $x \to Tx$ results in the identity transformation. In fact E. HOPF [10] has already pointed out that in this case we have $\Lambda'\Lambda = I$ in Φ_μ and therefore also $L'L = I$ (in $\mathfrak{L}_1(X, \mathfrak{R}, \mu)$) and $L^*L'^* = I$ (in $\mathfrak{L}_\infty(X, \mathfrak{R}, \mu)$).

From (6) and (10) we deduce

$$\mu(T^{-1}A) = \int_A s(x)\,\mu(dx) = \int_{T^{-1}A} r(x) s(Tx)\,\mu(dx) \quad \text{for all} \quad A \in \mathfrak{R}$$

which implies $r(x)s(Tx) = 1$ for μ-almost all $x \in X$ and therefore $r(T^{-1}x)s(x) = 1$ for μ-almost all $x \in X$. A straightforward computation now gives, in addition to the above-mentioned equalities,

$$\begin{aligned}
LL' &= E_{T^{-1}\mathfrak{R}} \quad \text{in} \quad \mathfrak{L}_1(X, \mathfrak{R}, \mu), \\
L'^* L^* &= E_{T^{-1}\mathfrak{R}} \quad \text{in} \quad \mathfrak{L}_\infty(X, \mathfrak{R}, \mu).
\end{aligned}$$

If T is also measure-preserving, then we have $r(x) = s(x) = 1$ μ-almost everywhere on X and for every $g \in \mathfrak{L}_\infty(X, \mathfrak{R}, \mu)$ $(\subset \mathfrak{L}_1(X, \mathfrak{R}, \mu))$ we obtain

$$Lg = L'^*g = g \circ T,$$
$$L'g = L^*g = (E_{T^{-1}\mathfrak{R}}\, g) \circ T^{-1}.$$

Another special situation which we shall meet in the sequel and which is closely related to invertibility of T is considered in the following definition. Recall that every subset $A \subset X$ has a "measurable cover" $\langle A \rangle \in \mathfrak{R}$, uniquely defined up to a μ-null set by the conditions $A \subset \langle A \rangle$ and $\mu(B) = 0$ for every $B \in \mathfrak{R}$ satisfying $B \subset \langle A \rangle \setminus A$ (cf. [4]). The symbol $[\mu]$ after a statement concerning some measurable sets will indicate that the statement holds if these sets are altered by appropriate μ-null sets.

Definition 5. A measurable transformation T in $(X, \mathfrak{R}, \mu)$ is said to be essentially invertible if the following two conditions are satisfied:

a) $\langle TT^{-1}A \rangle = A\,[\mu]$ for all $A \in \mathfrak{R}$,

b) $T^{-1}\langle TA \rangle = A\,[\mu]$ for all $A \in \mathfrak{R}$.

Note that in contrast to the concept of invertibility, the concept of essential invertibility depends on the underlying measure μ. It follows from condition b) that an invertible transformation is essentially invertible if and only if it is negatively non-singular. If T is non-singular, essentially invertible and satisfies $TX \in \mathfrak{R}$, then from $T^{-1}\mathfrak{R} = \mathfrak{R}[\mu]$ we conclude

$$LL' = I \quad \text{in} \quad \mathfrak{L}_1(X, \mathfrak{R}, \mu),$$
$$L'^*L^* = I \quad \text{in} \quad \mathfrak{L}_\infty(X, \mathfrak{R}, \mu).$$

Given any non-singular measurable transformation T in $(X, \mathfrak{R}, \mu)$ it is always possible to enforce essential invertibility by shrinking the original σ-algebra $\mathfrak{R}$ to the σ-algebra $\mathfrak{R}_\infty = \bigcap_{k=0}^{\infty} T^{-k}\mathfrak{R}$ (notice that $T^{-1}\mathfrak{R}_\infty = \mathfrak{R}_\infty$).

Theorem 3. *Let T be a non-singular measurable transformation in $(X, \mathfrak{R}, \mu)$. Then $\mathfrak{R}_\infty$ is modulo μ the largest σ-subalgebra $\mathfrak{S}$ of $\mathfrak{R}$ such that T is essentially invertible on $(X, \mathfrak{S}, \mu)$.*

Proof. Since $T^{-1}(X \setminus TX) = \emptyset$, it follows from the non-singularity of T that $\langle TX \rangle = X\,[\mu]$ and therefore

$$\langle TT^{-1}A \rangle = A \cap \langle TX \rangle = A\,[\mu] \quad \text{for all} \quad A \in \mathfrak{R}_\infty.$$

For every $A \in \mathfrak{R}_\infty$ there exists a set $B \in \mathfrak{R}_\infty$ such that $A = T^{-1}B$, and therefore

$$\langle TA \rangle = \langle TT^{-1}B \rangle = B[\mu],$$
$$T^{-1}\langle TA \rangle = T^{-1}B = A\,[\mu].$$

Hence T is essentially invertible in $(X, \mathfrak{R}_\infty, \mu)$.

On the other hand, if $\mathfrak{S}$ is a σ-subalgebra of $\mathfrak{R}$ such that T is essentially invertible in $(X, \mathfrak{S}, \mu)$, then some easy computations show that

$$\langle T\langle A \rangle \rangle = \langle TA \rangle\,[\mu] \quad \text{for all} \quad A \subset X$$

and (by induction on n)

$$T^{-n}\langle T^n A\rangle = A[\mu] \quad \text{for all} \quad A \in \mathfrak{S}.$$

The set

$$B = \limsup_{n\to\infty} T^{-n}\langle T^n A\rangle = \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} T^{-n}\langle T^n A\rangle$$

belongs to $T^{-k}\,\mathfrak{R}$ for every $k \geqq 1$ and therefore even to $\mathfrak{R}_\infty$. Moreover it satisfies $B = A[\mu]$. This shows $\mathfrak{S} \subset \mathfrak{R}_\infty[\mu]$.

Let T be a non-singular essentially invertible transformation in $(X, \mathfrak{R}, \mu)$ satisfying $TX \in \mathfrak{R}$, and let $A \in \mathfrak{R}$ be given. Then

$$c_A(x) = c_{T^{-1}\langle TA\rangle}(x) \quad \text{for } \mu\text{-almost all} \quad x \in X.$$

By Theorem 2 we obtain

$$L^* c_A = (E_{T^{-1}\mathfrak{R}}\, c_{T^{-1}\langle TA\rangle}) \circ T^{-1} = c_{\langle TA\rangle}$$

and by induction on n

(11) $$L^{*n} c_A = c_{\langle T^n A\rangle}.$$

§ 6. The Decompositions of $(X, \mathfrak{R}, \mu)$ Corresponding to the Transitions $x \to Tx$ and $x \to T^{-1}\{x\}$

The following theorem is due to E. HOPF ([10] § 6; cf. also [12] V.5).

Theorem I. Let L be a Markov operator in $\mathfrak{L}_1(X, \mathfrak{R}, \mu)$. There exists a modulo μ unique set $C \in \mathfrak{R}$, called the conservative part of X with respect to L, such that for every non-negative function $f \in \mathfrak{L}_1(X, \mathfrak{R}, \mu)$ one has

a) $$\sum_{k=0}^{\infty} L^k f(x) = 0 \quad \text{or} \quad \infty \qquad \text{for } \mu\text{-almost all} \quad x \in C,$$

b) $$\sum_{k=0}^{\infty} L^k f(x) < \infty \qquad \text{for } \mu\text{-almost all} \quad x \in X\setminus C.$$

As shown by FELDMAN [3] the conservative part of X with respect to L can also be described in terms of the dual Markov operator L^*:

Theorem II. Let L^* be a dual Markov operator in $\mathfrak{L}_\infty(X, \mathfrak{R}, \mu)$. There exists a modulo μ unique set $C \in \mathfrak{R}$ wit the following properties:

a) For every measurable set $A \subset C$ one has

$$\sum_{k=0}^{\infty} L^{*k} c_A(x) = \infty \quad \text{for } \mu\text{-almost all} \quad x \in A.$$

b) Every measurable set $B \subset X\setminus C$ of positive μ-measure contains a measurable subset B_0 of positive μ-measure such that

$$\sum_{k=0}^{\infty} L^{*k} c_{B_0}(x) < \infty \quad \text{for } \mu\text{-almost all} \quad x \in B_0.$$

The set C coincides modulo μ with the conservative part of X with respect to the corresponding Markov operator L in $\mathfrak{L}_1(X, \mathfrak{R}, \mu)$.

Recall that a set $W \in \mathfrak{R}$ is said to be wandering with respect to a measurable transformation T in $(X, \mathfrak{R}, \mu)$ if $W \cap T^{-n} W = \emptyset$ for all $n \geqq 1$ and T is called conservative if every wandering set is a μ-null set.

In [6] (cf. also [14]) the following theorem has been shown:

Theorem III. Let T be a measurable transformation in $(X, \mathfrak{R}, \mu)$. Then there exists a modulo μ unique set $C_1 \in \mathfrak{R}$, satisfying $T^{-1} C_1 \supset C_1$, with the following properties:

a) The restriction of T to the measure space $(C_1, \mathfrak{R} \cap C_1, \mu)$ is conservative.

b) The set $X \backslash C_1$ is a countable union of wandering sets.

Theorem 4. *Let T be a negatively non-singular measurable transformation in $(X, \mathfrak{R}, \mu)$. The set C_1 as defined in Theorem III coincides modulo μ with the conservative part of X with respect to the Markov process corresponding to the transition $x \to Tx$.*

Proof. Let C be the conservative part of X with respect to the Markov process corresponding to the transition $x \to Tx$. For any measurable set $A \subset C_1$ μ-almost every point returns to A infinitely often. By Theorem 1 we obtain

$$\sum_{k=0}^{\infty} L'^{*k} c_A(x) = \sum_{k=0}^{\infty} c_A(T^k x) = \infty \qquad \text{for } \mu\text{-almost all} \quad x \in A .$$

Theorem II then implies $C_1 \subset C[\mu]$.

Since $X \backslash C_1$ is a countable union of wandering sets, any measurable set $B \subset X \backslash C_1$ of positive μ-measure contains a wandering subset B_0 of positive μ-measure. We obtain

$$\sum_{k=0}^{\infty} L'^{*k} c_{B_0}(x) = \sum_{k=0}^{\infty} c_{B_0}(T^k x) = 1 \quad \text{for all} \quad x \in B_0 .$$

By Theorem II it follows that $X \backslash C_1 \subset X \backslash C[\mu]$, and therefore $C = C_1[\mu]$.

A sharpening of the requirement $T^{-1} C_1 \supset C_1$ at a cost of a weakening of requirement b) in Theorem III leads to a decomposition studied by Choksi [1] who proved the following theorem:

Theorem IV. Let T be a measurable transformation in $(X, \mathfrak{R}, \mu)$. Then there exists a modulo μ unique set $C_2 \in \mathfrak{R}$, satisfying $T^{-1} C_2 = C_2$, with the following properties:

a) The restriction of T to the measure space $(C_2, \mathfrak{R} \cap C_2, \mu)$ is conservative.

b) Every invariant measurable subset of $X \backslash C_2$ of positive μ-measure contains a wandering set of positive μ-measure.

As shown in [6] the set C_2 may be obtained by an exhaustion procedure as the (modulo μ) largest invariant subset of C_1. This at the same time shows the inclusion $C_2 \subset C_1[\mu]$. As demonstrated in [6] (or by the example below) this inclusion may be strict.

Applying Theorem I to the Markov process associated with the transition $x \to T^{-1}\{x\}$ (cf. Theorem 2) a third decomposition is obtained, implicitely already considered in [10] and [15]:

Theorem V. Let T be a positively non-singular measurable transformation in $(X, \mathfrak{R}, \mu)$ satisfying $TX \in \mathfrak{R}$, and let the $T^{-1}\mathfrak{R}$-measurable function r be defined by

$$\mu(TB) = \int_B r(x)\,\mu(dx) \quad \text{for all} \quad B \in T^{-1}\mathfrak{R}. \tag{12}$$

Then there exists a modulo μ unique set $C_3 \in \mathfrak{R}$ such that for every non-negative function $f \in \mathfrak{L}_1(X, \mathfrak{R}, \mu)$ one has

a) $$f(x) + \sum_{k=1}^{\infty} r(T^{k-1}x)\ldots r(x) f(T^k x) = 0 \text{ or } \infty \quad \text{for } \mu\text{-almost all} \quad x \in C_3,$$

b) $$f(x) + \sum_{k=1}^{\infty} r(T^{k-1}x)\ldots r(x) f(T^k x) < \infty \quad \text{for } \mu\text{-almost all} \quad x \in X \setminus C_3.$$

The fact that the hypotheses in the given version of Theorem V are weaker than in the case considered in [1] makes it necessary to reconsider the remaining inclusion relations between the sets C_1, C_2, and C_3.

Theorem 5. *Let T be a positively non-singular measurable transformation in $(X, \mathfrak{R}, \mu)$ satisfying $TX \in \mathfrak{R}$, and let the sets C_1, C_2, and C_3 be defined as in Theorem III, IV, and V. Then $T^{-1}C_3 \supset C_3[\mu]$ and $C_3 \subset C_1[\mu]$. If T is also negatively non-singular, then $T^{-1}C_3 = C_3[\mu]$ and $C_3 \subset C_2[\mu]$.*

Proof. If $W \in \mathfrak{R}$ were a wandering set of positive μ-measure contained in C_3, then we would get

$$c_W(x) + \sum_{k=1}^{\infty} r(T^{k-1}x)\ldots r(x)\, c_W(T^k x) = 1 \quad \text{for all} \quad x \in W$$

which contradicts Theorem II a). We conclude $C_3 \subset C_1[\mu]$.

In order to verify the inclusion $T^{-1} C_3 \supset C_3[\mu]$ consider the function

$$f \equiv 1 \in \mathfrak{L}_1(X, \mathfrak{R}, \mu).$$

By Theorem V we have

$$1 + \sum_{k=1}^{\infty} r(T^{k-1}x)\ldots r(x) = \infty \quad \text{for } \mu\text{-almost all} \quad x \in C_3$$

and since $r(x) < \infty$ μ-a.e.

$$1 + \sum_{k=1}^{\infty} r(T^k x)\ldots r(Tx) = \infty \quad \text{for } \mu\text{-almost all} \quad x \in C_3.$$

By Theorem V this implies $C_3 \subset T^{-1} C_3[\mu]$.

If T is also negatively non-singular, then by (12) $r(x) > 0$ μ-a.e. and consequently

$$1 + \sum_{k=1}^{\infty} r(T^k x)\ldots r(Tx) < \infty \quad \text{for } \mu\text{-almost all} \quad x \in X \setminus C_3.$$

By Theorem V we conclude $X \setminus C_3 \subset X \setminus T^{-1} C_3[\mu]$ and therefore $C_3 = T^{-1}C_3[\mu]$.

Using the non-singularity of T again we get

$$C_3 = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} T^{-k} C_3[\mu]$$

where the set on the right hand side is even strictly invariant. Since C_2 is modulo μ the largest invariant subset of C_1 we conclude $C_3 \subset C_2[\mu]$.

TSURUMI [15] has given an example of a conservative measurable transformation in a normed measure space $(X, \mathfrak{R}, \mu)$ for which $\mu(C_3) = 0$, thereby demonstrating that the inclusion $C_3 \subset C_2[\mu]$ may be strict. This inclusion has been shown by CHOKSI under the tacit assumption of non-singularity of T. That this inclusion may fail to hold if T is only positively non-singular is demonstrated by the following example (cf. [5], p. 12):

Let X be the set of all natural numbers, let $\mathfrak{R}$ be the σ-algebra of all subsets of X, and let the normed measure μ on $(X, \mathfrak{R})$ be defined by

$$\mu(\{n\}) = \begin{cases} \frac{1}{4} & \text{for} \quad n \leqq 3, \\ 0 & \text{for} \quad n = 4, \\ 2^{2-n} & \text{for} \quad n \geqq 5. \end{cases}$$

Define the transformation T by $T1 = 3$ and $Tn = n - 1$ for $n \geqq 2$. Then T is measurable and positively non-singular in $(X, \mathfrak{R}, \mu)$ (the only μ-null set in $T^{-1}\mathfrak{R}$ is $\emptyset$). However, T is not non-singular since $\mu(\{4\}) = 0$ but $\mu(T^{-1}\{4\}) = \frac{1}{8} \neq 0$. Inspecting the atoms of $T^{-1}\mathfrak{R}$ and taking into account that the function r is $T^{-1}\mathfrak{R}$-measurable we deduce from (12)

$$r(n) = \begin{cases} 1 & \text{for} \quad n \leqq 4, \\ 0 & \text{for} \quad n = 5, \\ 2 & \text{for} \quad n \geqq 6. \end{cases}$$

Using the constant function $f \equiv 1$ we conclude that $C_3 = \{1, 2, 3, 4\}[\mu]$ while on the other hand we have $C_2 = \emptyset$ and $C_1 = \{1, 2, 3\}[\mu]$.

Corollary 5.1. *Suppose that T is a non-singular measurable transformation in $(X, \mathfrak{R}, \mu)$ satisfying $TX \in \mathfrak{R}$ which is either measure preserving or essentially invertible. Then $C_1 = C_2 = C_3[\mu]$.*

Proof. If T is measure preserving, then we have $r(x) = 1$ μ-a.e. on X. Choosing $f \equiv 1$ we see by Theorem V that $C_3 = X[\mu]$.

If T is essentially invertible, let A be any measurable subset of C_1. Since T restricted to C_1 is conservative we have

$$A \subset \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \langle T^k A \rangle [\mu]$$

(cf. [8], Satz 6) and therefore by (11)

$$\sum_{k=0}^{\infty} L^{*k} c_A(x) = \sum_{k=0}^{\infty} c_{\langle T^k A \rangle}(x) = \infty \quad \text{for } \mu\text{-almost all} \quad x \in A .$$

By Theorem II we conclude $C_1 \subset C_3[\mu]$ and therefore $C_1 = C_3[\mu]$.

Observe that while the sets C_1 and C_2 are invariant under transition to an equivalent normed measure, the set C_3 is not. This again is demonstrated by TSURUMI's example mentioned above, where $C_3 = \emptyset$ while $C_2 = C_1 = X$. Since TSURUMI's transformation admits an equivalent invariant normed measure μ', for the corresponding set C_3' we get $C_3' = C_2 = C_1 = X$ by Corollary 5.1.

§ 7. The Tail-algebra Decomposition

For a non-singular measurable transformation T in $(X, \mathfrak{R}, \mu)$ satisfying $TX \in \mathfrak{R}$ let $\mathfrak{R}_\infty = \bigcap_{n=0}^{\infty} T^{-n}\mathfrak{R}$, let r be defined as in (12) and for $f \in \mathfrak{L}_1(X, \mathfrak{R}, \mu)$ define

$$f^n(x) = f(x) + \sum_{k=1}^{n} f(T^k x)\, r(T^{k-1} x) \ldots r(x).$$

The following theorem has been shown by Parry [13].

Theorem VI. Let T be a non-singular measurable transformation in $(X, \mathfrak{R}, \mu)$ satisfying $TX = X$. Then there exists an invariant set $C_0' \in \mathfrak{R}_\infty$ such that for every positive function $f \in \mathfrak{L}_1(X, \mathfrak{R}, \mu)$ one has

$$\lim_{n\to\infty} E_{\mathfrak{R}_\infty} f^n(x) = \infty \quad \text{for } \mu\text{-almost all} \quad x \in C_0',$$

$$\lim_{n\to\infty} E_{\mathfrak{R}_\infty} f^n(x) < \infty \quad \text{for } \mu\text{-almost all} \quad x \in X \backslash C_0'.$$

Theorem 6. *Let T and C_0' be as in Theorem VI. The set C_0' coincides modulo μ with the conservative part C_0 of X with respect to the essentially invertible transformation T in $(X, \mathfrak{R}_\infty, \mu)$.*

Proof. Let f be a positive function in $\mathfrak{L}_1(X, \mathfrak{R}, \mu)$. By induction on k we obtain from (10), replacing there f by $f c_A$

$$\int_A f(x)\, \mu(dx) = \int_{T^{-k}A} f(T^k x)\, r(T^{k-1} x) \ldots r(x)\, \mu(dx) \quad \text{for all} \quad A \in \mathfrak{R}.$$

For every $A \in \mathfrak{R}_\infty$ the relation $TX = X$ implies $T^k A \in \mathfrak{R}_\infty$ and $A = T^{-k} T^k A$, hence

$$\int_{T^k A} f(x)\, \mu(dx) = \int_A f(T^k x)\, r(T^{k-1} x) \ldots r(x)\, \mu(dx),$$

$$\int_A \lim_{n\to\infty} E_{\mathfrak{R}_\infty} f^n(x)\, \mu(dx) = \lim_{n\to\infty} \int_A E_{\mathfrak{R}_\infty} f^n(x)\, \mu(dx) = \sum_{k=0}^{\infty} \int_{T^k A} f(x)\, \mu(dx).$$

In particular for $A \in \mathfrak{R}_\infty \cap C_0$ and for every $n \geqq 1$ we have

$$A \subset \bigcup_{k=n}^{\infty} T^k A\, [\mu]$$

(cf. [8], Satz 3) and therefore in case $\mu(A) > 0$

$$\sum_{k=n}^{\infty} \int_{T^k A} f(x)\, \mu(dx) \geqq \int_A f(x)\, \mu(dx) > 0, \qquad \int_A \lim_{n\to\infty} E_{\mathfrak{R}_\infty} f^n(x)\, \mu(dx) = \infty.$$

This implies $C_0 \subset C_0'[\mu]$.

On the other hand, let $W \in \mathfrak{R}_\infty$ be wandering. Note that

$$T^n W \cap T^m W = \emptyset\, [\mu] \tag{13}$$

for all integers n, m $(n \neq m)$ by non-singularity and essential invertibility of T in $(X, \mathfrak{R}_\infty, \mu)$. We obtain

$$\int_W \lim_{n\to\infty} E_{\mathfrak{R}_\infty} f^n(x)\, \mu(dx) = \sum_{k=0}^{\infty} \int_{T^k W} f(x)\, \mu(dx) \leqq \int_X f(x)\, \mu(dx) < \infty.$$

Since $X\backslash C_0$ is a countable union of wandering sets in $\mathfrak{R}_\infty$ we conclude

$$\lim_{n\to\infty} E_{\mathfrak{R}_\infty} f^n(x) < \infty \quad \text{for } \mu\text{-almost all} \quad x \in X\backslash C_0 .$$

This implies $X\backslash C_0 \subset X\backslash C_0'[\mu]$ and completes the proof.

Theorem 7. *Let T be a non-singular measurable transformation in $(X, \mathfrak{R}, \mu)$ satisfying $TX = X$. Let the sets C_0 and C_1 be defined as in Theorem 6 and Theorem III. Then $\bigcup_{k=0}^{\infty} T^{-k} C_1 \subset C_0[\mu]$. If T is essentially invertible or measure preserving, then $C_1 = C_0[\mu]$.*

Proof. Since every wandering set in C_1 is a μ-null set we have $C_1 \subset C_0[\mu]$. The invariance of C_0 implies the first assertion. If T is essentially invertible, then we have $\mathfrak{R}_\infty = \mathfrak{R}[\mu]$ by Theorem 3, hence every wandering set in $\mathfrak{R}$ coincides modulo μ with a wandering set in $\mathfrak{R}_\infty$. If T preserves the normed measure μ, then $C_1 = C_0 = X$.

Theorem 7 together with Theorem 5 and the further remarks made in section 6 show that under the mentioned hypotheses we have

$$C_3 \subset C_2 \subset C_1 \subset \bigcup_{k=0}^{\infty} T^{-k} C_1 \subset C_0[\mu],$$

where the first three inclusions may be strict. In the next section we shall show that also the inclusion $\bigcup_{k=0}^{\infty} T^{-k} C_1 \subset C_0[\mu]$ may be strict even if T admits an equivalent invariant σ-finite measure.

§ 8. An Example

We shall construct a σ-finite measure space $(X, \mathfrak{R}, \nu)$ and a measure preserving transformation T in $(X, \mathfrak{R}, \nu)$ such that

a) X is a countable union of wandering sets,

b) $TX = X$,

c) T is conservative in $(X, \mathfrak{R}_\infty, \nu)$.

Replacing ν by an equivalent normed measure μ we obtain a non-singular transformation T in $(X, \mathfrak{R}, \mu)$ such that $C_1 = \bigcup_{k=0}^{\infty} T^{-k} C_1 = \emptyset[\mu]$ and $C_0 = X[\mu]$.

Define the subset X of the real plane by

$$X = \bigcup_{n=-\infty}^{+\infty} I_n ,$$

$$I_n = \{(x, y) \colon 0 \leqq x < 1, y = n\} \quad \text{for all integers } n .$$

Let $\mathfrak{R}_n$ be the σ-algebra of Borel sets in I_n and let ν_n be the Lebesgue measure on $(I_n, \mathfrak{R}_n)$. Define

$$\mathfrak{R} = \{A \colon A \subset X, A \cap I_n \in \mathfrak{R}_n \text{ for } -\infty < n < +\infty\},$$

$$\nu(A) = \sum_{n=-\infty}^{+\infty} \nu_n(A \cap I_n) \quad \text{for all} \quad A \in \mathfrak{R} .$$

Obviously $(X, \mathfrak{R}, \nu)$ is a σ-finite measure space.

$$T(x, n) = \begin{cases} (2x, n+1) & \text{if } 0 \leqq x < \frac{1}{2}, \\ (2x-1, n+2) & \text{if } \frac{1}{2} \leqq x < 1. \end{cases}$$

The transformation T in $(X, \mathfrak{R}, \nu)$ is measurable, measure preserving, and satisfies $TX = X$ and $T\mathfrak{R} \subset \mathfrak{R}$. Since every I_n is wandering X is a countable union of wandering sets.

Let $(I, \mathfrak{B}, \lambda)$ be the interval $[0, 1[$ together with the Borel sets and Lebesgue measure. The projection P defined by

$$P(x, n) = x \quad \text{for all} \quad (x, n) \in X$$

is a measurable mapping of $(X, \mathfrak{R}, \nu)$ onto $(I, \mathfrak{B}, \lambda)$ satisfying $P\mathfrak{R} = \mathfrak{B}$. Moreover, for every $A \in \mathfrak{R}$ we have $\lambda(PA) = 0$ if and only if $\nu(A) = 0$.

The projection P induces a measurable transformation T_0 in $(I, \mathfrak{B}, \lambda)$ defined by

$$T_0 x = PTP^{-1}x = 2x \,(\mathrm{mod}\, 1). \tag{14}$$

The conservativity of T in $(X, \mathfrak{R}_\infty, \nu)$ will be shown in a series of propositions.

Proposition 1. *If $A \in \mathfrak{R}_\infty$ and $\nu(A) > 0$, then $\nu(A) \geqq 1$.*

Proof. For every $n \geqq 0$ there exists a set $A_n \in \mathfrak{R}$ such that $A = T^{-n}A_n$. From (14) we conclude

$$T_0^{-n}(PA_n) = PT^{-n}P^{-1}(PA_n) = PA,$$

hence $PA \in \mathfrak{B}_\infty$. It is well known that $\mathfrak{B}_\infty = \{\emptyset, I\}\,[\lambda]$, hence $\nu(A) > 0$ implies $\lambda(PA) = 1$ and $\nu(A) \geqq 1$.

Proposition 2. *If $W \in \mathfrak{R}_\infty$ is wandering and $\nu(W) > 0$, then $\nu(W) = 1$.*

Proof. It suffices to show that the assumption $\nu(W) > 1$ leads to a contradiction.

Define $W_n = W \cap I_n$ and let $\{n_k\}_{k=0}^\infty$ be a denumeration of the integers. Inductively we define

$$W_{n_0}^0 = W_{n_0}, \quad W_{n_k}^0 = W_{n_k} \backslash P^{-1}P\left(\bigcup_{l=0}^{k-1} W_{n_l}^0\right), \quad k \geqq 1.$$

The set $W^0 = \bigcup_{n=-\infty}^{+\infty} W_n^0$ satisfies $W^0 \subset W$, $PW^0 = PW$, and P maps W^0 one-to-one onto PW. Since $\nu(W) > 1$ implies $\lambda(PW) = 1$ we have $\nu(W^0) = 1$ and there exists an integer n such that $\nu(W_n \backslash W_n^0) > 0$. Then also the set

$$V = P^{-1}P(W_n \backslash W_n^0) \cap W^0$$

has positive ν-measure and there exists an integer $m \neq n$ such that $V_1 = V \cap I_m$ has positive ν-measure. The two measurable subsets V_1 (lying on level m) and $V_2 = W_n \cap P^{-1}PV_1$ (lying on level n) of W satisfy $\nu(V_1) = \nu(V_2) > 0$ and $PV_1 = PV_2$. Without loss of generality we assume $m < n$.

Almost all points of V_1 and V_2 are points of density of these sets (cf. [11], § 5, Theorem 1). Therefore there exists a dyadic interval

$$D = \left[\frac{p}{2^q}, \frac{p+1}{2^q}\right[\subset I$$

such that
$$\frac{\nu(V_1 \cap P^{-1}D)}{\lambda(D)} = \frac{\nu(V_2 \cap P^{-1}D)}{\lambda(D)} > 1 - \frac{1}{2^{n-m+1}}.$$
Applying T^q to the sets $V_1 \cap P^{-1}D$ and $V_2 \cap P^{-1}D$ we obtain two sets A_1 and A_2, lying on intervals I_{k_1} and I_{k_2} such that $k_2 - k_1 = n - m$ and satisfying
$$\nu(A_1) = \nu(A_2) > 1 - \frac{1}{2^{n-m+1}}.$$
Let
$$B = A_1 \cap P^{-1}\left[0, \frac{1}{2^{n-m}}\right[, \quad \text{then} \quad \nu(B) > \frac{1}{2^{n-m+1}}, \quad T^{n-m}B \subset I_{k_2}$$
and
$$\nu(T^{n-m}B) > \frac{1}{2}, \quad \text{hence} \quad \nu(A_2 \cap T^{n-m}B) > \frac{1}{2} - \frac{1}{2^{n-m+1}} > 0.$$
This implies $\nu(T^q W \cap T^{q+n-m}) > 0$ which contradicts (13).

Proposition 3. *T is conservative in $(X, \mathfrak{R}_\infty, \nu)$.*

Proof. Suppose $W \in \mathfrak{R}_\infty$ were a wandering set of positive ν-measure. Then there exists an integer n such that $\nu(W \cap I_n) > 0$ and a dyadic interval
$$D = \left[\frac{p}{2^q}, \frac{p+1}{2^q}\right[\subset I$$
such that
$$\frac{\nu(W \cap I_n \cap P^{-1}D)}{\lambda(D)} > \frac{1}{2}.$$
This implies $\nu(T^{q+1}(W \cap I_n \cap P^{-1}D)) > 1$ and $\nu(T^{q+1}W) > 1$. This, however, contradicts Proposition 2, since by (13) $T^{q+1}W$ coincides modulo ν with a wandering set in $\mathfrak{R}_\infty$.

In [7], p. 52 and at the colloquium on ergodic theory in Oberwolfach (fall 1965) the following question has been proposed by the first author: Let T be a measurable transformation in a measurable space $(X, \mathfrak{R})$ satisfying $TX = X$, and let X be a countable union of wandering sets. Does there always exist a wandering set $W \in \mathfrak{R}$ with the following properties:
$$(15) \qquad \begin{aligned} &X = \bigcup_{n=-\infty}^{+\infty} T^n W, \\ &T^m W \cap T^n W = \emptyset \quad \text{for} \quad -\infty < m < n < +\infty, \\ &T^n W \in \mathfrak{R} \qquad\qquad\;\, \text{for} \quad -\infty < n < +\infty. \end{aligned}$$

The present example answers this question in the negative. Indeed, as shown in [7], (15) implies that W should belong to $\mathfrak{R}_\infty$, and therefore at least one of the sets $T^n W$ should have positive measure. This, however, is exluded by Proposition 3.

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