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The Minimum of an Additive Process with Applications to Signal Estimation and Storage Theory

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1. The Minimum of an Additive Process and Its Location

Let $\{X(t), t \geqq 0\}$ be a real-valued additive process with no negative jumps, right-continuous sample paths with left limits, and such that $P[X(t)<0]>0$ for all $t>0$. Define

$$M_t = \inf_{0 \leqq u \leqq t} X(u),$$

and

$$T_t = \inf\{u: \inf_{0 \leqq y \leqq u} X(y) = M_t\}.$$

In this section we shall determine the joint distribution of (T_t, M_t). [The marginal distributions of T_t and M_t for an additive process with no positive jumps, zero Gaussian component, and with positive drift have already been determined by Shtatland (1965, 1966). The distribution of M_t with no negative jumps is well-known, see e.g. Takacs (1967).

We first recall some well-known properties of the process $\{X(t)\}$. Its Laplace transform is

$$E \exp(-\xi X(t)) = \exp(t A(\xi)), \qquad \operatorname{Re} \xi > 0,$$

where

$$A(\xi) = -m\xi + \frac{\sigma^2 \xi^2}{2} + \int_{(0,\infty)} \left[\exp(-\xi u) - 1 + \frac{\xi u}{1+u^2}\right] \frac{1+u^2}{u^2} G(du)$$

and G is a finite measure. If we denote by $P_{F(-v)}$ the (possibly defective) distribution of the first-passage time $F(-v)$ of $\{X(t)\}$ through the negative level $(-v)$ then, as was shown by Borovkov (1965), the Laplace transform of $P_{F(-v)}$ is

$$E \exp(-\xi F(-v)) = \exp(-v\beta(\xi)), \qquad \xi > 0,$$

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where $\beta(\xi)$ is the unique root with real part greater than 0 of the equation

$$\xi = A(\beta).$$

In particular $F(-v)$ is finite with probability $\exp(-v\beta(0+))$.

Theorem 1. *For all Borel subsets A of $[0,\infty)\times[0,\infty)$,*

$$\int_0^\infty \exp(-\xi t)\, P[(T_t, |M_t|)\in A]\, dt = \xi^{-1}\beta(\xi) \int_A \exp(-\xi u) P_{F(-v)}(du)\, dv, \qquad \xi>0.$$

Proof. The strong Markov property of $\{X(t)\}$ yields the relation

$$P[T_t \leqq u, M_t \leqq -v] = \int_{[0,u]} P[T_{t-s} \leqq u-s]\, P_{F(-v)}(ds), \qquad 0\leqq u\leqq t,\ v\geqq 0,$$

from which it follows after some straightforward manipulations that

$$\int_0^\infty \exp(-\xi t)\, E\exp[-\rho T_t - \theta|M_t|]\, dt = \frac{\beta(\rho+\xi)}{\theta+\beta(\rho+\xi)} \int_0^\infty E\exp(-\xi t - \rho T_t)\, dt,$$

$$\rho\geqq 0,\ \theta\geqq 0,\ \xi>0.$$

The integral on the right is equal to $\lim_{n\to\infty} \psi_n(\xi,\rho)$ where

$$\psi_n(\xi,\rho) = \int_0^\infty E\exp[-\xi t - \rho T_t^{(n)}]\, dt$$

and

$$T_t^{(n)} = \inf\{u: \inf_{0\leqq y\leqq u} X(y) = -n^{-1}[n|M_t|]\}, \qquad n=1,2,\ldots.$$

To determine ψ_n we condition on the time of first passage of $\{X(t)\}$ through level $(-n^{-1})$ to obtain the equation,

$$E\exp[-\rho T_t^{(n)}] = P[F(-1/n)>t] + \int_{[0,t]} E\exp[-u - T_{t-u}^{(n)}]\, P_{F(-1/n)}(du).$$

This equation gives

$$\psi_n(\xi,\rho) = \frac{1-\exp(-\beta(\xi)/n)}{\xi[1-\exp(-\beta(\xi+\rho)/n)]},$$

whence

$$\int_0^\infty E\exp[-\xi t - \rho T_t]\, dt = \xi^{-1}\beta(\xi)/\beta(\xi+\rho).$$

Substituting this in the earlier expression for the joint Laplace transform we obtain

$$\int_0^\infty \exp(-\xi t)\, E\exp[-(\rho T_t + \theta|M_t|)]\, dt = \xi^{-1}\beta(\xi)\,[\theta+\beta(\rho+\xi)]^{-1},$$

$$\rho,\ \theta\geqq 0,\ \xi>0,$$

which is equivalent to the assertion of the theorem.

Corollary 1. *If $\beta(0+)>0$ then $(T_t, M_t) \xrightarrow{\text{a.s.}} (T, M)$ as $t \to \infty$, where*

(a) *$(T, |M|)$ has distribution $\beta(0+)\, P_{F(-v)}(du)\, dv$, $u, v \geqq 0$;*

(b) *$|M|$ has distribution $\beta(0+) \exp(-v\beta(0+))\, dv$, $v \geqq 0$;*

(c) *T has distribution $\beta(0+) \int_{v=0}^{\infty} P_{F(-v)}(du)\, dv$, $u \geqq 0$.*

If $\beta(0+)=0$ then $(T_t, M_t) \xrightarrow{\text{a.s.}} (+\infty, -\infty)$.

Corollary 2. *For Browian motion with $EX(t)=\mu t$ and* $\operatorname{Var} X(t)=t$, *$\beta(\xi)=[(\mu^2+2\xi)^{1/2}+\mu]$ and $F(-v)$ has density*

$$f_{F(-v)}(u)=|v|\, u^{-3/2}\phi(v u^{-1/2}+\mu u^{1/2}), \qquad u>0,$$

where ϕ is the standard normal density. In this case the Laplace transform in Theorem 1 can be inverted explicitly to give the joint density

$$f_{T_t, |M_t|}(u, v)=(2\pi)^{-1/2} f_{F(-v)}(u) \left[2\mu(2\pi)^{1/2} I_{(0,\infty)}(\mu)+\int_{t-u}^{\infty} y^{-3/2} e^{\mu^2 y/2}\, dy\right],$$
$$v>0,\ 0<u<t.$$

If $\beta(0+)>0$, i.e., $\mu>0$, then $(T, |M|)$ has joint density

$$f_{T, |M|}(u, v)=2\mu f_{F(-v)}(u), \qquad u,\ v>0.$$

Corollary 3. *If $X(t)=-ct+A(t)$ where $c>0$ and $\{A(t)\}$ is a stable additive process with $E\exp[-\lambda A(t)]=\exp[-t\lambda^{1/2}]$, $\lambda>0$, then $\beta(\xi)=(2c^2)^{-1}[(1+4c\xi)^{1/2}+1+2c\xi]$ and $F(-v)$ has density*

$$f_{F(-v)}(u)=|v|\,[4\pi(cu-v)^3]^{-1/2} \exp[-u^2/(4(cu-v))], \qquad 0<\frac{v}{c}<u.$$

In this case application of Theorem 1 gives

$$f_{T_t, |M_t|}(u, v)=(4\pi c^3)^{-1/2} f_{F(-v)}(u)[2(\pi/c)^{1/2}+\int_{t-u}^{\infty} y^{-3/2} e^{-y/(4c)}\, dy],$$
$$0<v/c<u<t$$

and

$$P[T_t=t, |M_t| \leqq v]=c^{-1}\int_0^v f_{F(-y)}(t)\, dy.$$

We conclude this section with some remarks on the special case when $X(t)=B(t)+\theta t$ where $B(t)$ is a standard B.M. and $\theta>0$.

Remark 1.1. It follows from the law of iterated logarithm that

$$\liminf_{t\to\infty} X(t)=+\infty \qquad \text{a.s.}$$

and in view of the sample-path continuity properties this implies that almost all paths attain their respective minima on $[0, \infty)$.

Remark 1.2. For each K, let C_K be the set of all ω for which the minimum of $X(t, \omega)$ on $[0, K]$ is nonunique. Then

$$C_K \subset \bigcup_{\substack{0 \leqq r<s \leqq K\\ r, s \text{ rational}}} \{\omega \mid \min_{0 \leqq t \leqq r} X(t, \omega)=\min_{s \leqq t \leqq K} X(t, \omega)\}.$$

Write

$$Z_{rs} = \min_{0 \leqq t \leqq r} X(t) - \min_{s \leqq t \leqq K} X(t) = \min_{0 \leqq t \leqq r} [B(t) - B(r) + \theta t]$$
$$+ [B(r) - B(s)] - \min_{s \leqq t \leqq K} [B(t) - B(s) + \theta t] = Y_1 + Y_2 + Y_3, \quad \text{say.}$$

Then Y_1, Y_2, Y_3 are mutually independent random variables and Y_2 is absolutely continuous. Hence Z_{rs} is absolutely continuous for all rational $r<s$ in $[0, K]$, which makes $P(C_K)=0$. Thus for each K, almost all paths of $X(t)$ have unique minima on $[0, K]$ and therefore, also have unique minima on $[0, \infty)$.

In view of these remarks we can write $M(\omega) = \min_{t \geqq 0} X(t, \omega)$ and define $T(\omega)$ by the equation $X(T(\omega), \omega) = M(\omega)$.

Remark 1.3. For $X(t) = B(t) + \theta t$, the joint distribution of (T, M) given in Corollary 2 can be obtained directly by a completely different method. We give an outline of this alternate derivation omitting the computational details. For t, $u>0$, consider the event $E(t, u) = \{\omega | M(\omega) \leqq -u, T(\omega) \leqq t\}$. Then the distribution function of (T, M) is $F(t, m) = P[E(t, -m)]$. To obtain $P[E(t, u)]$, divide $E(t, u)$ into disjoint events

$$E_{n,j}(t, u) = \{\omega | -u_{n,j+1} < \min_{s \leqq t} X(s, \omega) \leqq -u_{n,j}, \min_{s>t} X(s, \omega) > \min_{s \leqq t} X(s, \omega)\},$$

$j=0, 1, 2, \ldots$ for each positive integer n where $u_{n,j} = u + jn^{-1}$. Each $E_{n,j}(t, u)$ can be approximated from below and from above by

$$A_{n,j}(t, u) = \{\omega | -u_{n,j+1} < \min_{s \leqq t} X(s, \omega) \leqq -u_{n,j}, \min_{s>t} X(s, \omega) > -u_{n,j}\}$$

and

$$B_{n,j}(t, u) = \{\omega | -u_{n,j+1} < \min_{s \leqq t} X(s, \omega) \leqq -u_{n,j}, \min_{s>t} X(s, \omega) > -u_{n,j+1}\}$$

respectively. The conditional probabilities of these events given $X(t) = y$ can be obtained explicitly using the following results due to Doob (1949) and T.W. Anderson (1960):

(i) $P[B(s)$ touches $a + bs$ for some $s] = \exp[-2ab]$, $\quad a, b > 0$.

(ii) The conditional distribution of $\xi(s) = t^{-1}(t+s)\{B(ts(t+s)^{-1}) - s(t+s)^{-1}z\}$, $0 \leqq s < \infty$, given $B(t) = z$ is the same as that of a B.M. on $[0, \infty)$.

These conditional probabilities are then integrated with respect to the distribution of $X(t)$ to yield $P[A_{n,j}(t, u)]$ and $P[B_{n,j}(t, u)]$. As $n \to \infty$, $\sum_{j=0}^{\infty} P[A_{n,j}(t, u)]$ and $\sum_{j=0}^{\infty} P[B_{n,j}(t, u)]$ are seen to converge to the same limit which is $P[E(t, u)]$. The distribution function of (T, M) is obtained in this way and its second mixed partial derivative gives the density function $f_{T,M}$ of Corollary 2.

2. Detection and Estimation of Faint Signals: The Problem

We observe random variables $Y_1^{(n)}, \ldots, Y_n^{(n)}$ where $Y_j^{(n)}$ is the sum of a noise component $X_j^{(n)}$ and a possible signal $a_j^{(n)}$. Suppose that for every n the noise $X_1^{(n)}, \ldots, X_n^{(n)}$

is a sequence of iid random variables with mean μ and variance σ^2, $X_1^{(n)}$ has the same distribution for all n and the signal $a_j^{(n)}$ is either identically 0 (i.e., there is no signal) or there are numbers $0<\lambda_1<\lambda_2<1$ such that

$$a_j^{(n)}=\begin{cases}0, & 1\leqq j\leqq[n\lambda_1],\quad [n\lambda_2]+1\leqq j\leqq n\\ \delta v_n^{-1}, & [n\lambda_1]+1\leqq j\leqq[n\lambda_2]\end{cases}\tag{1}$$

where $\{v_n\}$ is a sequence of positive numbers tending to ∞. The parameters μ, σ^2, δ, λ_1, λ_2 are all unknown; however, physical considerations often dictate δ to be positive if there is a signal. We therefore want to test the null hypothesis $H_0: a_j^{(n)}\equiv 0$ against $H_1: a_j^{(n)}\not\equiv 0$ with the non-zero part being positive and we also want to estimate λ_1 and λ_2 when H_1 holds. Since $v_n\to\infty$, we may call the signal $a_j^{(n)}$ (if it is non-zero) a "faint signal".

In the above problem fix n and consider the degenerate case that arises when $\lambda_2=1$ in the description of $a_j^{(n)}$ under H_1. This problem has received a great deal of attention in the statistical literature. For an extensive treatment of this problem we refer to Chernoff and Zacks (1964) who examined the problem for normally distributed noise from different angles though their main concern was to estimate EY_n. For normally distributed noise, the maximum likelihood estimator (*mle*) of $[n\lambda_1]$ (assuming that H_1 holds) is that integer r for which the absolute value of the statistic

$$\{r^{-1}+(n-r)^{-1}\}^{-1/2}\cdot\left\{r^{-1}\sum_{i=1}^{r}Y_i-(n-r)^{-1}\sum_{i=r+1}^{n}Y_i\right\}$$

is a maximum. If the value of δ in (1) is positive, then with a high probability the maximum absolute value of the above statistic is attained with a negative value when n is large. For this reason, the *mle* of $[n\lambda_1]$ can be defined for asymptotic purposes as that value of r for which this statistic attains its minimum value. Hinkley (1970) has studied the properties of this *mle* and Sen and Srivastava (1975) have studied the properties of tests of the hypothesis H_0 of "no change in mean" based on the *mle* of the change-point assuming that a change occurred somewhere. Hinkley's treatment of the *mle* of the change-point involves random walk arguments. The exact distribution of the *mle* is not given in an explicit form but is expressed in terms of the distribution of the maxima of two random walks. The asymptotic distribution is obtained from approximations of the above distribution and numerical studies are made. It should be noted that Hinkley and Sen and Srivastava have considered the case where the strength of the signal does not diminish with increasing n, i.e., when in (1) we have $\lambda_2=1$ and $v_n=1$. In such a case not much simplification is obtained when n becomes large.

Consider again the non-degenerate case $0<\lambda_1<\lambda_2<1$. When $\delta>0$, the *mle* of $([n\lambda_1],[n\lambda_2])$ for normally distributed noise can again be obtained (for asymptotic purposes) by minimizing a statistic given in Section 3. The asymptotic behaviors of this and a related estimator are understood in terms of stochastic processes indexed by two parameters which are derived from the cumulative sums of the observations. These processes are defined by (5) and (6). For faint signals which are stronger than $O(n^{-1/2})$, an almost sure invariance principle holds for these two-parameter processes on compact sets (Theorem 2). Using this invariance principle, the asymptotic distributions of suitably normalized errors of estimating λ_1 and λ_2 are seen to be independent of each other and each is given in terms of the minimum of two independent B.M.'s with positive drifts (Theorem 3). The final results are then obtained from Corollary 2, Theorem 1. Using a similar invariance principle, we obtain an asymptotic test for the hypothesis H_0 (Remark 6.3). The treatment of the degenerate case differs from the non-degenerate case in actual details but the essential ideas are the same. Results for the degenerate case are stated without proof in Remark 6.2. The study of faint signals has been motivated by a problem of optical image analysis which is discussed in Remark 6.5.

3. Two Estimators

Suppose until further notice that there is a non-degenerate positive signal, i.e., we observe $Y_j^{(n)} = X_j^{(n)} + a_j^{(n)}, j=1, \ldots, n$ where $X_1^{(n)}, \ldots, X_n^{(n)}$ are iid random variables with mean μ and variance σ^2, $X_1^{(n)}$ has the same distribution for all n and $\{a_j^{(n)}\}$ is given by (1) with $\delta > 0$ and $0 < \lambda_1 < \lambda_2 < 1$. We assume that the sequence $\{v_n\}$ in (1) satisfies

(i) $\lim_{n \to \infty} v_n = \infty$ and

(ii) $\lim_{n \to \infty} n^{-1/2} v_n = 0$.

Let $R = R(n) = \{(r_1, r_2) | r_1, r_2$ integers, $1 \leqq r_1 \leqq n-2,\ r_1 + 1 \leqq r_2 \leqq n-1\}$ and for $(r_1, r_2) \in R(n)$, define

$$U_n(r_1, r_2) = \sum_{j=1}^{r_1} Y_j^{(n)} - \sum_{j=1}^{r_2} Y_j^{(n)} + n^{-1}(r_2 - r_1) \sum_{j=1}^{n} Y_j^{(n)}, \tag{2}$$

$$c_n(r_1, r_2) = n\{(r_2 - r_1)(n - r_2 + r_1)\}^{-1/2} \tag{3}$$

and

$$V_n(r_1, r_2) = c_n(r_1, r_2)\, U_n(r_1, r_2). \tag{4}$$

Then the *mle* of $([n\lambda_1], [n\lambda_2])$ for normally distributed noise is obtained by maximizing $|V_n(r_1, r_2)|$ with respect to $(r_1, r_2) \in R(n)$. Since $\delta > 0$, this maximum absolute value is attained with a negative value with high probability when n is large. Thus asymptotically, we are led to an estimator of $([n\lambda_1], [n\lambda_2])$ which is obtained by minimizing $V_n(r_1, r_2)$ with respect to $(r_1, r_2) \in R(n)$. In order to avoid the possibility of undesirable behavior of $V_n(r_1, r_2)$ when either $n^{-1}(r_2 - r_1)$ or $n^{-1}(n - r_2 + r_1)$ is very small, we shall actually restrict V_n to the subset $R' = R'(n)$ of $R(n)$ on which $\min\{r_2 - r_1, n - r_2 + r_1\} \geqq n^{1/2} v_n$. Instead of $n^{1/2} v_n$, we could also have chosen any b_n satisfying $n^{-1} b_n \to 0$ and $v_n^{-2} b_n \to \infty$. This device of slightly restricted minimization is used here in order to avoid a technical difficulty encountered in the proof of Lemma 3. Now V_n may attain its minimum value for several $(r_1, r_2) \in R'(n)$. We take this possibility into account in the following way.

For any function f on the Euclidean plane or part thereof which attains its minimum, let $S(f)$ denote the set of points at which this minimum is attained and let $\psi(f)$ denote an unambiguously defined point in $S(f)$. For example, if $S(f)$ is a closed and bounded set (as will always be in the case in this paper), $\psi(f)$ may be taken to be (t_1^*, t_2^*) where t_1^* is the minimum of the first coordinates of all points in $S(f)$ and t_2^* is the minimum of the second coordinates of all points whose first coordinates are t_1^*.

The estimator of $([n\lambda_1], [n\lambda_2])$ arising in this manner is

$$\tau_n = (\tau_{1n}, \tau_{2n}) = \psi(V_n).$$

We shall examine the asymptotic properties of this estimator. Now the function U_n is much simpler than V_n, and we shall also study the asymptotic properties of the estimator

$$\hat{\tau}_n = (\hat{\tau}_{1n}, \hat{\tau}_{2n}) = \psi(U_n),$$

where ψ denotes the above-mentioned minimization of U_n over $R(n)$. The device of slightly restricted minimization is not needed for U_n. It will be more convenient to work with the following stochastic processes rather than U_n and V_n. Define

$$\xi_n(t_1, t_2) = (v_n \sigma)^{-1} \{U_n([n\lambda_1 + v_n^2 t_1], [n\lambda_2 + v_n^2 t_2]) - U_n([n\lambda_1], [n\lambda_2])\} \tag{5}$$

and

$$\eta_n(t_1, t_2) = (pq)^{1/2} (v_n \sigma)^{-1} \{V_n([n\lambda_1 + v_n^2 t_1], [n\lambda_2 + v_n^2 t_2]) - V_n([n\lambda_1], [n\lambda_2])\} \tag{6}$$

for (t_1, t_2) such that $([n\lambda_1 + v_n^2 t_1], [n\lambda_2 + v_n^2 t_2])$ lies in $R(n)$ in case of $\xi_n(t_1, t_2)$ and in $R'(n)$ in case of $\eta_n(t_1, t_2)$, where

$$p = \lambda_2 - \lambda_1 \quad \text{and} \quad q = 1 - \lambda_2 + \lambda_1.$$

Then the errors of the estimators $\hat{\tau}_n$ and τ_n for $([n\lambda_1], [n\lambda_2])$ and the errors of the estimators

$$\hat{\lambda}_n = (\hat{\lambda}_{1n}, \hat{\lambda}_{2n}) = n^{-1} \hat{\tau}_n \quad \text{and} \quad \lambda_n = (\lambda_{1n}, \lambda_{2n}) = n^{-1} \tau_n$$

for (λ_1, λ_2) are given by

$$v_n^{-2} \{(\hat{\tau}_{1n}, \hat{\tau}_{2n}) - ([n\lambda_1], [n\lambda_2])\} = \psi(\xi_n) + e_n, \tag{7}$$

$$v_n^{-2} \{\tau_{1n}, \tau_{2n}) - ([n\lambda_1], [n\lambda_2])\} = \psi(\eta_n) + e_n, \tag{8}$$

$$n v_n^{-2} \{(\hat{\lambda}_{1n}, \hat{\lambda}_{2n}) - (\lambda_1, \lambda_2)\} = \psi(\xi_n) + e_n, \tag{9}$$

$$n v_n^{-2} \{(\lambda_{1n}, \lambda_{2n}) - (\lambda_1, \lambda_2)\} = \psi(\eta_n) + e_n. \tag{10}$$

Here as well as in subsequent formulas e_n is a generic symbol denoting a discrepancy depending on n. In each of the above four formulas $|e_n| \leqq 2 v_n^{-2}$. Since $v_n \to \infty$, these e_n's can be neglected for the purpose of finding the asymptotic distributions of the above normalized errors of estimation. The problem, therefore, is to obtain the asymptotic distributions of $\xi(\psi_n)$ and $\psi(\eta_n)$. This will be attempted in the next two sections. Though we considered normally distributed noise for motivating these estimators, their asymptotic properties do not depend on such an assumption.

4. Almost Sure Invariance Principles for ξ_n and η_n on Compact Sets

Let B_{11}, B_{12}, B_{21} and B_{22} denote four independent standard B.M.'s on $[0, \infty)$. The stochastic processes

$$B_i(t)=\begin{cases}B_{i1}(-t), & t<0\\ B_{i2}(t), & t\geqq 0\end{cases}, \quad i=1,2 \tag{11}$$

are independent B.M.'s on $(-\infty, \infty)$. We shall refer to these as *two-sided standard B.M.'s* so as to distinguish them from standard B.M.'s on $[0, \infty)$. Define two functions

$$g_1(t)=\begin{cases}-pt, & t<0\\ qt, & t\geqq 0\end{cases}, \quad g_2(t)=g_1(-t)=\begin{cases}-qt, & t<0\\ pt, & t\geqq 0\end{cases} \tag{12}$$

where $p=\lambda_2-\lambda_1$ and $q=1-\lambda_2+\lambda_1$ as before. In this section almost sure invariance principles will be established for ξ_n and η_n. It will be shown that on compact sets, ξ_n can be uniformly approximated by

$$\{B_1(t_1)+\theta g_1(t_1)\}+\{B_2(t_2)+\theta g_2(t_2)\}$$

and η_n can be uniformly approximated by $\{B_1(t_1)+\frac{1}{2}\theta|t_1|\}+\{B_2(t_2)+\frac{1}{2}\theta|t_2|\}$ where B_1, B_2 and g_1, g_2 are given by (11) and (12) and $\theta=\delta\sigma^{-1}$.

We first compute the limiting values of $E\xi_n(t_1, t_2)$ and $E\eta_n(t_1, t_2)$ in the following two lemmas.

For each n, let $I_1(n)$, $I_2(n)$ and $I_3(n)$ denote the sets of integers $\{1, \ldots, [n\lambda_1]\}$, $\{[n\lambda_1]+1, \ldots, [n\lambda_2]\}$ and $\{[n\lambda_2]+1, \ldots, n\}$ respectively and divide $R(n)$ into disjoint parts

$$\begin{aligned}R_{i_1 i_2}&=R_{i_1 i_2}(n)\\ &=\{(r_1, r_2)\in R(n)\,|\,r_1\in I_{i_1}(n),\ r_2\in I_{i_2}(n)\}, \quad 1\leqq i_1\leqq i_2\leqq 3\end{aligned}$$

and divide R' into $R'_{i_1 i_2}$ similarly.

Lemma 1. *For U_n given by* (2),

$$\sigma^{-1}EU_n(r_1, r_2)=v_n^{-1}\theta\{h_n(r_1, r_2)+e_n(r_1, r_2)\}$$

where $\theta=\delta\sigma^{-1}$,

$$h_n(r_1, r_2)=\begin{cases}(r_2-r_1)p, \quad (r_1, r_2)\in R_{11}UR_{33}\\ -npq+|r_1-[n\lambda_1]|\,p+|r_2-[n\lambda_2]|\,q, & (r_1, r_2)\in R_{12}\\ -npq+|r_1-[n\lambda_1]|\,p+|r_2-[n\lambda_2]|\,p, & (r_1, r_2)\in R_{13}\\ -npq+|r_1-[n\lambda_1]|\,q+|r_2-[n\lambda_2]|\,q, & (r_1, r_2)\in R_{22}\\ -npq+|r_1-[n\lambda_1]|\,q+|r_2-[n\lambda_2]|\,p, & (r_1, r_2)\in R_{23},\end{cases}$$

and $|e_n(r_1, r_2)|\leqq 3$ *for all* (r_1, r_2).

Proof. Since $|([a]-[b])-(a-b)|\leqq 1$, $\sigma^{-1}\left(E\sum_{j=1}^{r} Y_j^{(n)}-r\mu\right)$ equals 0 for $r\in I_1(n)$, $\theta v_n^{-1}\{(r-n\lambda_1)+e_n(r)\}$ for $r\in I_2(n)$ and $\theta v_n^{-1}\{n(\lambda_2-\lambda_1)+e_n(r)\}$ for $r\in I_3(n)$ where $|e_n(r)|\leqq 1$. The lemma now follows by computation.

Lemma 2. *For ξ_n and η_n given by* (5) *and* (6), *the following limits hold uniformly in* $(t_1, t_2)\in[-K, K]^2$.

(a) $\lim_{n\to\infty} E\xi_n(t_1, t_2)=\theta\{g_1(t_1)+g_2(t_2)\}$,

(b) $\lim_{n\to\infty} E\eta_n(t_1, t_2)=\frac{1}{2}\theta(|t_1|+|t_2|)$.

Proof. Fix K. Make n large enough so that for $|t_1|\leqq K$, $[n\lambda_1+v_n^2 t_1]$ is either in $I_1(n)$ or in $I_2(n)$ and for $|t_2|\leqq K$, $[n\lambda_2+v_n^2 t_2]$ is either in $I_2(n)$ or in $I_3(n)$. Now apply Lemma 1 for (r_1, r_2) in R_{12}, R_{13}, R_{22} and R_{23} to obtain

$$(v_n\sigma)^{-1} EU_n([n\lambda_1+v_n^2 t_1], [n\lambda_2+v_n^2 t_2])$$
$$=\theta\{-n v_n^{-2} pq+g_1(t_1)+g_2(t_2)+e_n(t_1, t_2)\} \tag{13}$$

where $|e_n(t_1, t_2)|\leqq K_1 v_n^{-2}$ for all $(t_1, t_2)\in[-K, K]^2$. This proves (a). Recall $c_n(r_1, r_2)$ given by (3). It is easy to see that

$$c_n([n\lambda_1+v_n^2 t_1], [n\lambda_2+v_n^2 t_2])$$
$$=(pq)^{-1/2}\{1+\tfrac{1}{2} v_n^2 n^{-1}(q^{-1}-p^{-1})(t_2-t_1)+e_n(t_1, t_2)\} \tag{14}$$

where $|e_n(t_1, t_2)|\leqq K_2 n^{-1}+K_3 v_n^4 n^{-2}$ for all $(t_1, t_2)\in[-K, K]^2$. From (13) and (14),

$$(pq)^{1/2}(v_n\sigma)^{-1} EV_n([n\lambda_1+v_n^2 t_1], [n\lambda_2+v_n^2 t_2])$$
$$=\theta\{-n v_n^{-2} pq+\tfrac{1}{2}(|t_1|+|t_2|)+e_n(t_1, t_2)\}$$

where $|e_n(t_1, t_2)|\leqq K_4 v_n^2 n^{-1}+K_5 v_n^{-2}$ for all $(t_1, t_2)\in[-K, K]^2$. This proves part (b).

We now state and prove the main result of this section.

Theorem 2. *There exist processes $\{\hat{\xi}_n(t_1, t_2)\}$ and $\{\hat{\eta}_n(t_1, t_2)\}$ defined on a common probability space along with independent two-sided B.M.'s $\{B_1(t)\}$ and $\{B_2(t)\}$ such that*

(a) *for each n, $\{\xi_n(t_1, t_2\}$ and $\{\hat{\xi}_n(t_1, t_2)\}$ have the same distribution, $\{\eta_n(t_1, t_2)\}$ and $\{\hat{\eta}_n(t_1, t_2)\}$ have same distribution, and*

(b) *for each $0<K<\infty$ and each sufficiently rapidly increasing subsequence $\{n(i)\}$,*

$$\sup_{(t_1, t_2)\in[-K, K]^2} |\hat{\xi}_{n(i)}(t_1, t_2)-\{B_1(t_1)+\theta g_1(t_1)\}-\{B_2(t_2)+\theta g_2(t_2)\}|\to 0 \quad \text{a.s.}$$

$$\sup_{(t_1, t_2)\in[-K, K]^2} |\hat{\eta}_{n(i)}(t_1, t_2)-\{B_1(t_1)+\tfrac{1}{2}\theta|t_1|\}-\{B_2(t_2)+\tfrac{1}{2}\theta|t_2|\}|\to 0 \quad \text{a.s.}$$

where g_1 and g_2 are given by (12) *and $\theta=\delta\sigma^{-1}$.*

Proof. Let $Z_j^{(n)}=\sigma^{-1}(Y_j^{(n)}-EY_j^{(n)})$. Then

$$(v_n\sigma)^{-1} U_n([n\lambda_1+v_n^2 t_1], [n\lambda_2+v_n^2 t_2])$$
$$=v_n^{-1}\sum_{j=1}^{[n\lambda_1+v_n^2 t_1]} Z_j^{(n)}-v_n^{-1}\sum_{j=1}^{[n\lambda_2+v_n^2 t_2]} Z_j^{(n)}+p v_n^{-1}\sum_{j=1}^{n} Z_j^{(n)}$$
$$+v_n n^{-1}\{(t_2-t_1)+e_n(t_1, t_2)\}\sum_{j=1}^{n} Z_j^{(n)}$$
$$+(v_n\sigma)^{-1} EU_n([n\lambda_1+v_n^2 t_1], [n\lambda_2+v_n^2 t_2]) \tag{15}$$

where $|e_n(t_1, t_2)| \leqq v_n^{-2}$. Hence

$$\xi_n(t_1, t_2) = v_n^{-1}\left\{\sum_{j=1}^{[n\lambda_1 + v_n^2 t_1]} Z_j^{(n)} - \sum_{j=1}^{[n\lambda_1]} Z_j^{(n)}\right\}$$
$$- v_n^{-1}\left\{\sum_{j=1}^{[n\lambda_2 + v_n^2 t_2]} Z_j^{(n)} - \sum_{j=1}^{[n\lambda_2]} Z_j^{(n)}\right\} + E\xi_n(t_1, t_2)$$
$$+ v_n n^{-1}\{(t_2 - t_1) + e_n(t_1, t_2) - e_n(0, 0)\} \sum_{j=1}^{n} Z_j^{(n)}. \tag{16}$$

Since $n^{-1/2} \sum_{j=1}^{n} Z_j^{(n)} = O_p(1)$ and $|e_n(t_1, t_2)| \leqq v_n^{-2}$,

$$\sup_{(t_1, t_2) \in [-K, K]^2} \left| v_n n^{-1}\{(t_2 - t_1) + e_n(t_1, t_2) - e_n(0, 0)\} \sum_{j=1}^{n} Z_j^{(n)} \right| = o_p(1) \tag{17}$$

and by Lemma 2(a),

$$\sup_{(t_1, t_2) \in [-K, K]^2} |E\xi_n(t_1, t_2) - \theta\{g_1(t_1 + g_2(t_2))\}| = o(1). \tag{18}$$

Finally, define $W_{11j}^{(n)} = -Z_{[n\lambda_1]-j}^{(n)}$, $W_{12j}^{(n)} = Z_{[n\lambda_1]+j}^{(n)}$, $W_{21j}^{(n)} = Z_{[n\lambda_2]-j}^{(n)}$ and $W_{22j}^{(n)} = -Z_{[n\lambda_2]+j}^{(n)}$, $j = 1, 2, \ldots$. For each n, each of these four is a sequence of iid random variables with 0 mean and unit variance. Furthermore, the early parts of these four sequences are independent of each other. Now note that

$$v_n^{-1}\left\{\sum_{j=1}^{[n\lambda_1 + v_n^2 t_1]} Z_j^{(n)} - \sum_{j=1}^{[n\lambda_1]} Z_j^{(n)}\right\} = \begin{cases} v_n^{-1} \sum_{j=1}^{-[v_n^2 t_1 + n\lambda_1 - [n\lambda_1]]} W_{11j}^{(n)}, & t_1 < 0 \\ v_n^{-1} \sum_{j=1}^{[v_n^2 t_1 + n\lambda_1 - [n\lambda_1]]} W_{12j}^{(n)}, & t_1 \geqq 0 \end{cases}$$

and

$$v_n^{-1}\left\{\sum_{j=1}^{[n\lambda_2 + v_n^2 t_2]} Z_j^{(n)} - \sum_{j=1}^{[n\lambda_2]} Z_j^{(n)}\right\} = \begin{cases} v_n^{-1} \sum_{j=1}^{-[v_n^2 t_2 + n\lambda_2 - [n\lambda_2]]} W_{21j}^{(n)}, & t_2 < 0 \\ v_n^{-1} \sum_{j=1}^{[v_n^2 t_2 + n\lambda_2 - [n\lambda_2]]} W_{22j}^{(n)}, & t_2 \geqq 0. \end{cases}$$

It therefore follows by a well-known almost sure invariance principle for random walks (see Breiman (1968)) that there exist stochastic processes $B_1^{(n)}$, $B_2^{(n)}$ and independent two-sided B.M.'s B_1, B_2 on an appropriate probability space such that for each n,

$$\left\{v_n^{-1}\left[\sum_{j=1}^{[n\lambda_i + v_n^2 t_i]} Z_j^{(n)} - \sum_{j=1}^{[n\lambda_i]} Z_j^{(n)}\right], |t_i| \leqq K\right\}$$

has the same distribution as $\{B_i^{(n)}(t_i), |t_i| \leqq K\}$, $i = 1, 2$, and along all sufficiently rapidly increasing subsequences

$$\sup_{|t_i| \leqq K} |B_i^{(n)}(t_i) - B_i(t_i)| \to 0 \quad \text{a.s.} \tag{19}$$

From (16), (17), (18) and (19), the theorem is now proved for ξ_n. For η_n, we use (14), (15) and the fact that $\max_{1\leq r\leq n} n^{-1/2}\left|\sum_{j=1}^{r} Z_j^{(n)}\right| = O_p(1)$ to obtain

$$(pq)^{1/2}(v_n\sigma)^{-1} V_n([n\lambda_1+v_n^2 t_1],[n\lambda_2+v_n^2 t_2]) = v_n^{-1}\sum_{j=1}^{[n\lambda_1+v_n^2 t_1]} Z_j^{(n)}$$
$$-v_n^{-1}\sum_{j=1}^{[n\lambda_2+v_n^2 t_1]} Z_j^{(n)} + p v_n^{-1}\sum_{j=1}^{n} Z_j(n) + e_n(t_1,t_2)$$
$$+(pq)^{1/2}(v_n\sigma)^{-1} E V_n([n\lambda_1+v_n^2 t_1],[n\lambda_2+v_n^2 t_2])$$

where

$$\sup_{(t_1,t_2)\in[-K,K]^2} |e_n(t_1,t_2)| = o_p(1).$$

Hence

$$\eta_n(t_1,t_2) = v_n^{-1}\left\{\sum_{j=1}^{[n\lambda_1+v_n^2 t_1]} Z_j^{(n)} - \sum_{j=1}^{[n\lambda_1]} Z_j^{(n)}\right\} - v_n^{-1}\left\{\sum_{j=1}^{[n\lambda_2+v_n^2 t_2]} Z_j^{(n)} - \sum_{j=1}^{[n\lambda_2]} Z_j^{(n)}\right\}$$
$$+E\eta_n(t_1,t_2)+e_n(t_1,t_2)-e_n(0,0),$$

and the proof is completed in exactly the same way as for ξ_n by using Lemma 2(b).

From now on,

$$\xi(t_1,t_2) = B_1(t_1)+\theta g_1(t_1)+B_2(t_2)+\theta g_2(t_2), \tag{20}$$
$$\eta(t_1,t_2) = B_1(t_1)+\tfrac{1}{2}\theta|t_1|+B_2(t_2)+\tfrac{1}{2}\theta|t_2| \tag{21}$$

where B_1, B_2 are independent two-sided B.M.'s and g_1, g_2 are given by (12).

5. Asymptotic Distributions of $\psi(\xi_n)$ and $\psi(\eta_n)$

Theorem 2 suggests that $n v_n^{-2}\{(\hat{\lambda}_{1n},\hat{\lambda}_{2n})-(\lambda_1,\lambda_2)\}$ or $\psi(\xi_n)$ converges in law to $\psi(\xi)=(\hat{T}_1,\hat{T}_2)$ where $\hat{T}_i$ is the time at which $\{B_i(t_i)+\theta g(t_i)\}$ attains its a.s. unique minimum and that a similar convergence in law holds for $n v_n^{-2}\{(\lambda_{1n},\lambda_{2n})-(\lambda_1,\lambda_2)\}$ or $\psi(\eta_n)$. Since B_1 and B_2 are independently distributed, this would mean that for both estimators, the errors in estimating λ_1 an λ_2 are asymptotically independent. In order to actually obtain these results from Theorem 2, a key step is to prove that the probabilities of $\{U_n(r_1,r_2)\}$ and $\{V_n(r_1,r_2)\}$ attaining their minimum values anywhere outside the set of (r_1,r_2) for which $|r_i-[n\lambda_i]|\leq K v_n^2$, $i=1,2$, approaches 0 as $K\to\infty$ and then $n\to\infty$. The following lemma gives the required result.

Let $S=S_1\cup S_2\cup S_3$ where

$$S_1=\{(r_1,r_2)\,\|r_1-[n\lambda_1]|\geq K v_n^2,\ |r_2-[n\lambda_2]|<K v_n^2\},$$
$$S_2=\{(r_1,r_2)\,\|r_1-[n\lambda_1]|<K v_n^2,\ |r_2-[n\lambda_2]|\geq K v_n^2\},$$

and

$$S_3=\{(r_1,r_2)\,\|r_1-[n\lambda_1]|\geq K v_n^2,\ |r_2-[n\lambda_2]|\geq K v_n^2\}.$$

Lemma 3. (a) $P\big[\min_{(r_1, r_2)\in R\cap S} U_n(r_1, r_2) \leqq U_n([n\lambda_1], [n\lambda_2])\big] \leqq C(K^{-1} + n^{-1} v_n^2)$,

(b) $P\big[\min_{(r_1, r_2)\in R'\cap S} V_n(r_1, r_2) \leqq V_n([n\lambda_1], [n\lambda_2])\big] \leqq C(K^{-1} + n^{-1/2} v_n)$

where C is a constant depending only on θ, λ_1 and λ_2.

Proof. Let $E_n(r_1, r_2)$ and $E_n'(r_1, r_2)$ denote the events $\{U_n(r_1, r_2) \leqq U_n([n\lambda_1], [n\lambda_2])\}$ and $\{V_n(r_1, r_2) \leqq V_n([n\lambda_1], [n\lambda_2])\}$ respectively. Treating $n\lambda_1$ and $n\lambda_2$ as integers in order to avoid unnecessary complications, the event $E_n(r_1, r_2)$ is seen to be equivalent to

$$\sum_{j=1}^{|r_1 - n\lambda_1|} W_{1k_1 j}^{(n)} + \sum_{j=1}^{|r_2 - n\lambda_2|} W_{2k_2 j}^{(n)} + (p-d) \sum_{j=1}^{n} Z_j^{(n)} \geqq v_n^{-1} \theta \{h_n(r_1, r_2) + npq\} \tag{22}$$

and the event $E_n'(r_1, r_2)$ is seen to be equivalent to

$$\begin{aligned} &\sum_{j=1}^{|r_1 - n\lambda_1|} W_{1k_1 j}^{(n)} + \sum_{j=1}^{|r_2 - r\lambda_2|} W_{2k_2 j}^{(n)} + (p-d) \sum_{j=1}^{n} Z_j^{(n)} \\ &\quad + [\{d(1-d)/(pq)\}^{1/2} - 1] \cdot \Big\{p \sum_{j=1}^{n} Z_j^{(n)} - \sum_{j=n\lambda_1+1}^{n\lambda_2} Z_j^{(n)}\Big\} \\ &\quad \geqq v_n^{-1} \theta [h_n(r_1, r_2) + n\{d(1-d)pq\}^{1/2}] \end{aligned} \tag{23}$$

where k_1 is 1 or 2 as r_1 is in $I_1(n)$ or not, k_2 is 2 or 1 as r_2 is in $I_3(n)$ or not, $d = d_n(r_1, r_2) = n^{-1}(r_2 - r_1)$, $Z_j^{(n)} = \sigma^{-1}(Y_j^{(n)} - EY_j^{(n)})$ and $W_{11j}^{(n)}$ etc. are the random variables introduced in course of the proof of

Theorem 2. To prove the lemma, we shall decompose R into $R_{i_1 i_2}$, $1 \leqq i_1 \leqq i_2 \leqq 3$ and R' into $R'_{i_1 i_2}$, $1 \leqq i_1 \leqq i_2 \leqq 3$ and obtain suitable bounds for the probability that (22) holds for some $(r_1, r_2) \in R_{i_1 i_2}$ and for the probability that (23) holds for some $(r_1, r_2) \in R'_{i_1 i_2}$. First consider (22). Since $R_{11} \cap S = R_{11}$ and $|p-d| \leqq \lambda_2$ for $(r_1, r_2) \in R_{11}$,

$$\begin{aligned} P\Big[\bigcup_{R_{11}\cap S} E_n(r_1, r_2)\Big] &= P\Big[\bigcup_{R_{11}} E_n(r_1, r_2)\Big] \leqq P\Big[\max_{1 \leqq r \leqq n\lambda_1} \Big|\sum_{j=1}^{r} W_{11j}^{(n)}\Big| \geqq 3^{-1} n v_n^{-1} \theta pq\Big] \\ &\quad + P\Big[\max_{1 \leqq r \leqq n\lambda_2} \Big|\sum_{j=1}^{r} W_{21j}^{(n)}\Big| \geqq 3^{-1} n v_n^{-1} \theta pq\Big] \\ &\quad + P\Big[\Big|\sum_{j=1}^{n} Z_j^{(n)}\Big| \geqq 3^{-1} n v_n^{-1} \theta pq \lambda_2^{-1}\Big] \\ &\leqq 9 n^{-2} v_n^2 \theta^{-2} p^{-2} q^{-2} (n\lambda_1 + n\lambda_2 + n\lambda_2^2) \leqq C n^{-1} v_n^2. \end{aligned}$$

The case of $R_{33} \cap S$ is treated in exactly the same way. To obtain the bounds for $R_{12} \cap S$, $R_{13} \cap S$, $R_{22} \cap S$ and $R_{23} \cap S$, each of these regions is further subdivided into $R_{i_1 i_2} \cap S_1$, $R_{i_1 i_2} \cap S_2$ and $R_{i_1 i_2} \cap S_3$. Of these twelve cases, we shall only treat $R_{12} \cap S_1$ and $R_{12} \cap S_3$ here. The treatments of the other cases vary only in details but the essential arguments are the same.

Since $\{h_n(r_1, r_2)+npq\}/|p-d| \geqq p \wedge q$ for all (r_1, r_2) and since $h_n(r_1, r_2)+npq \geqq |r_1 - n\lambda_1| p \geqq pKv_n^2$ for all $(r_1, r_2) \in R_{12} \cap S_1$, it follows that

$$\begin{aligned}
P\Big[\bigcup_{R_{12}\cap S_1} E_n(r_1, r_2)\Big] &\leqq P\Big[\max_{1 \leqq r_1 \leqq n\lambda_1 - Kv_n^2} |r_1 - n\lambda_1|^{-1} \cdot \Big|\sum_{j=1}^{|r_1 - n\lambda_1|} W_{11j}^{(n)}\Big| \geqq 3^{-1}\theta p v_n^{-1}\Big] \\
&\quad + P\Big[\max_{n\lambda_2 - Kv_n^2 \leqq r_2 \leqq n\lambda_2} \Big|\sum_{j=1}^{|r_2 - n\lambda_2|} W_{21j}^{(n)}\Big| \geqq 3^{-1}\theta pKv_n\Big] \\
&\quad + P\Big[\Big|\sum_{j=1}^{n} Z_j^{(n)}\Big| \geqq 3^{-1}\theta(pq)\, n v_n^{-1}\Big] \\
&\leqq P\Big[\max_{Kv_n^2 \leqq r \leqq n\lambda_1} r^{-1}\Big|\sum_{j=1}^{r} W_{11j}^{(n)}\Big| \geqq 3^{-1}\theta p v_n^{-1}\Big] \\
&\quad + P\Big[\max_{1 \leqq r \leqq Kv_n^2} \Big|\sum_{j=1}^{r} W_{21j}^{(n)}\Big| \geqq 3^{-1}\theta pKv_n\Big] \\
&\quad + P\Big[\Big|\sum_{j=1}^{n} Z_j^{(n)}\Big| \geqq 3^{-1}\theta(pq)\, n v_n^{-1}\Big] \\
&\leqq 9\theta^{-2}\Big[p^{-2} v_n^2 \Big(K^{-1} v_n^{-2} + \sum_{r > Kv_n^2} r^{-2}\Big) + p^{-2}K^{-1} + (pq)^2 n^{-1} v_n^2\Big] \\
&\leqq C(K^{-1} + n^{-1} v_n^2)
\end{aligned}$$

by Kolmogorov's inequality and its generalization due to Hájek and Rényi (1955). Again, for $(r_1, r_2) \in R_{12} \cap S_3$, $h_n(r_1, r_2)+npq \geqq \max\{|r_1 - n\lambda_1| p, |r_2 - n\lambda_2| q\}$. Hence

$$\begin{aligned}
P\Big[\bigcup_{R_{12}\cap S_3} E_n(r_1, r_2)\Big] &\leqq P\Big[\max_{Kv_n^2 \leqq r \leqq n\lambda_1} r^{-1}\Big|\sum_{j=1}^{r} W_{11j}^{(n)}\Big| \geqq 2^{-1}\theta p v_n^{-1}\Big] \\
&\quad + P\Big[\max_{Kv_n^2 \leqq r \leqq np} r^{-1}\Big|\sum_{j=1}^{r} W_{21j}^{(n)}\Big| \geqq 2^{-1}\theta q v_n^{-1}\Big] \\
&\quad + P\Big[\Big|\sum_{j=1}^{n} Z_j^{(n)}\Big| \geqq 2^{-1}\theta(pq)\, n v_n^{-1}\Big] \\
&\leqq 4\theta^{-2}\Big[(p^{-2}+q^{-2})\, v_n^2\Big(K^{-1} v_n^{-2} + \sum_{r > Kv_n^2} r^{-2}\Big) + (pq)^{-2} n^{-1} v_n^2\Big] \\
&\leqq C(K^{-1} + n^{-1} v_n^2)
\end{aligned}$$

and that concludes the proof of part (a) of the lemma.

Let $T_1 = \{n^{1/2} v_n \leqq \min(r_2 - r_1, n - r_2 + r_1) \leqq \varepsilon n\} = \{n^{-1/2} v_n \leqq \min(d, 1-d) \leqq \varepsilon\}$ and $T_2 = \{\min(r_2 - r_1, n - r_2 + r_1) > \varepsilon n\} = \{\varepsilon < d < 1 - \varepsilon\}$. To prove part (b), we shall treat the regions $R'_{i_1 i_2} \cap T_1$ and $R'_{i_1 i_2} \cap T_2$ separately. For sufficiently large n, $R'_{i_1 i_2} \cap T_1 \subset S$. Now for $R'_{11} \cap T_1$, R.H.S. of (23)

$$\geqq n v_n^{-1}\theta\{d(1-d)pq\}^{1/2} \geqq 2^{-1} n v_n^{-1}\theta(dpq)^{1/2} \geqq 2^{-1} n^{3/4} v_n^{-1/2}\theta(pq)^{1/2}$$

for sufficiently small ε. Hence

$$P[\bigcup_{R'_{11}\cap T_1} E'_n(r_1,r_2)] \leqq P\left[\max_{1\leqq r\leqq n\lambda_1}\left|\sum_{j=1}^{r} W^{(n)}_{1k_1j}\right| \geqq 8^{-1} n^{3/4} v n^{-1/2}\theta(pq)^{1/2}\right]$$
$$+P\left[\max_{1\leqq r\leqq n\lambda_2}\left|\sum_{j=1}^{r} W^{(n)}_{2k_1j}\right| \geqq 8^{-1} n^{3/4} v_n^{-1/2}\theta(pq)^{1/2}\right]$$
$$+P\left[(pq)^{-1/2}\left|\sum_{j=n\lambda_1+1}^{n\lambda_2} Z^{(n)}_j\right| \geqq 8^{-1} n^{3/4} v_n^{-1/2}\theta(pq)^{1/2}\right]$$
$$+P\left[\{1+\tfrac{1}{2}(p/q)^{1/2}\}\left|\sum_{j=1}^{n} Z^{(n)}_j\right| \geqq 8^{-1} n^{3/4} v_n^{-1/2}\theta(pq)^{1/2}\right]$$
$$\leqq C n^{-1/2} v_n.$$

The region $R'_{33}\cap T_1$ is treated in the same way. For $(r_1,r_2)\in R'_{12}\cap T_1$, R.H.S. of (23)

$$= n v_n^{-1}\theta\{d(1-d)pq\}^{1/2} - n v_n^{-1}\theta pq$$
$$+ v_n^{-1}\theta\{(n\lambda_1-r_1)p+(n\lambda_2-r_2)q\} = n v_n^{-1}\theta\{d(1-d)pq\}^{1/2}$$
$$+ v_n^{-1}\theta(n\lambda_1-r_1) - n v_n^{-1}\theta q d \geqq 2^{-1} n v_n^{-1}\theta(dpq)^{1/2} \geqq 2^{-1} n^{3/4} v_n^{-1/2}\theta(pq)^{1/2} \tag{25}$$

for sufficiently small ε and sufficiently large n, since $(n\lambda_1-r_1)\geqq 0$ for $d\leqq\varepsilon$, $\{d(1-d)\}^{1/2} \simeq d^{1/2}$ dominates d. Similarly, for $(r_1,r_2)\in R'_{13}\cap T_1$, R.H.S. of (23)

$$= n v_n^{-1}\theta\{d(1-d)pq\}^{1/2} - n v_n^{-1}\theta pq + v_n^{-1}\theta\{(n\lambda_1-r_1)p+(r_2-n\lambda_2)p\}$$
$$= n v_n^{-1}\theta\{d(1-d)pq\}^{1/2} - n v_n^{-1}\theta p(1-d)$$
$$\geqq 2^{-1} n v_n^{-1}\theta\{(1-d)pq\}^{1/2} \geqq 2^{-1} n^{3/4} v_n^{-1/2}\theta(pq)^{1/2}. \tag{26}$$

Proceeding as in (24) and using (25) and (26), $P[\bigcup_{R'_{12}\cap T_1} E'_n(r_1,r_2)]$ and $P[\bigcup_{R'_{13}\cap T_1} E'_n(r_1,r_2)]$ are seen to be bounded by $Cn^{-1/2}v_n$. The regions $R'_{22}\cap T_1$ and $R'_{23}\cap T_1$ are treated in the same way. Now consider the regions $R'_{i_1 i_2}\cap T_2$ which are the same as $R_{i_1 i_2}\cap T_2$. First note that the L.H.S. of (23) differs from the L.H.S. of (22) by the term

$$[\{d(1-d)/(pq)\}^{1/2}-1]\cdot\left\{p\sum_{j=1}^{n} Z^{(n)}_j - \sum_{j=n\lambda_1+1}^{n\lambda_2} Z^{(n)}_j\right\},$$

the maximum absolute value of which (with respect to d) is $O_p(\sqrt{n})$. This term can, therefore, be neglected for our purpose. For (r_1,r_2) in $(R_{11}\cup R_{33})\cap T_2$, R.H.S. of (23) $\geqq n v_n^{-1}\theta p\varepsilon$, from which it follows in the same way as in the proof of part (a) that $P[\bigcup_{R_{11}\cap T_2} E'_n(r_1,r_2)]$ and $P[\bigcup_{R_{33}\cap T_2} E'_n(r_1,r_2)]$ are bounded by $Cn^{-1}v_n^2$. We shall now show that the same bound also holds for R_{12}, R_{13}, R_{22}, R_{23}. *Let us* write R.H.S. of (23)

$$= n v_n^{-1}\theta pq[\{(d(1-d)/(pq)\}^{1/2}-1] + v_n^{-1}\theta\{h_n(r_1,r_2)+npq\}$$
$$= \gamma_n(r_1,r_2) + \alpha\cdot \text{R.H.S. of (22)}$$

where

$$\gamma_n(r_1,r_2) = n v_n^{-1}\theta pq[\{d(1-d)/(pq)\}^{1/2}-1] + (1-\alpha) v_n^{-1}\theta\{h_n(r_1,r_2)+npq\}.$$

It is enough to show that for some $0<\alpha<1$,

$$\min_{R_{i_1 i_2}\cap S\cap T_2} \gamma_n(r_1, r_2) \geqq o(\sqrt{n}) \tag{27}$$

for the regions R_{12}, R_{13}, R_{22} and R_{23}. The rest will follow from part (a) of the lemma. First consider $R_{12}\cap S_1$ and write

$$u_i = n^{-1}(r_i - n\lambda_i), \qquad v_i = |u_i|, \; i=1,2. \tag{28}$$

Then for $(r_1, r_2)\in R_{12}\cap S_1$, $0\leqq v_1 = -u_1 \leqq \lambda_1$ and $0\leqq v_2 = -u_2 \leqq Kn^{-1} v_n^2$, so

$$\begin{aligned}
&\gamma_n(r_1, r_2)\\
&= n v_n^{-1}\theta p q[\{(1-(u_1-u_2)/p)(1+(u_1-u_2)/q)\}^{1/2}-1]\\
&+ n v_n^{-1}\theta(1-\alpha)(p|u_1|+q|u_2|)\\
&= n v_n^{-1}\theta p q[\{(1+(v_1-v_2)/p)(1-(v_1-v_2)/q\}^{1/2}-1] + n v_n^{-1}\theta(1-\alpha)(p v_1 + q v_2)\\
&= n v_n^{-1}\theta p v_1[(1-\alpha)+f(v_1; p)] + o(\sqrt{n})
\end{aligned}$$

where

$$f(v; p) = (q/v)[\{(1+v/p)(1-v/q)\}^{1/2}-1]. \tag{29}$$

The derivative of $f(v; p)$ with respect to v is

$$f'(v, p) = q v^{-2}[1 - \text{a.m.}\{v+p^{-1}, v-q^{-1}\}/\text{g.m.}\{v+p^{-1}, v-q^{-1}\}], \tag{30}$$

$0\leqq v\leqq\lambda_1$, where a.m. and g.m. denote the arithmetic and the geometric means respectively. Hence $f'(v, p)\leqq 0$ and

$$\min_{0\leqq v_1\leqq\lambda_1} [(1-\alpha)+f(v_1; p)] = (1-\alpha)+f(\lambda_1; p),$$

which is positive for $0<\alpha<1+f(\lambda_1; p)$ and that concludes the proof for this case.

Note that the proof for this case did not require the restriction to T_2. Now consider $R_{12}\cap S_2\cap T_2$ and let u_1, u_2, v_1, v_2 be as in (28). Here $0\leqq v_1 = -u_1 \leqq Kn^{-1} v_n^2$ and $0\leqq v_2 = -u_2 \leqq p$. Furthermore, since $(r_1, r_2)\in T_2$, $v_2 - v_1 \leqq p-\varepsilon$. Hence $v_2 \leqq p-\varepsilon+Kn^{-1}v_n^2 \leqq p-\varepsilon/2$ for large n.

$$\gamma_n(r_1, r_2) = n v_n^{-1}\theta q v_2[(1-\alpha)+f(v_2; q)] + o(\sqrt{n})$$

where $f(v; q)$ is obtained by interchanging p and q in (29), Again, $f'(v_2; q)\leqq 0$ as before, so

$$\min_{0\leqq v_2\leqq p-\varepsilon/2} [(1-\alpha)+f(v_2; q)] = (1-\alpha)+f(p-\varepsilon/2; q),$$

which is positive for $0<\alpha<1+f(p-\varepsilon/2; q)$. That concludes the proof for this case. Now suppose $(r_1, r_2)\in R_{12}\cap S_3\cap T_2$. Here $0\leqq v_1\leqq\lambda_1$, $0\leqq v_2\leqq p$ and $v_1 - v_2 \geqq -p+\varepsilon$. Let $w_1 = v_1 - v_2$ and $w_2 = p v_1 + q v_2$. Then either $0\leqq w_1\leqq\lambda_1$ and $w_2\geqq p w_1$, or $-p+\varepsilon\leqq w_1\leqq 0$ and $w_2\geqq -q w_1$. Now

$$\gamma_n(r_1, r_2) = n v_n^{-1}\theta g(w_1, w_2),$$

where

$$g(w_1, w_2) = p q[\{(1+w_1/p)(1-w_1/q)\}^{1/2}-1] + (1-\alpha) w_2.$$

For $w_1 \geqq 0$,

$$\min_{w_2 \geqq p w_1} g(w_1, w_2) = g(w_1, p w_1) = g_1(w_1), \quad \text{say.}$$

It is easy to see that $g_1''(w_1) < 0$, so

$$\min_{0 \leqq w_1 \leqq \lambda_1,\, w_2 \geqq p w_1} g(w_1, w_2) = \min_{0 \leqq w_1 \leqq \lambda_1} g_1(w_1) = \min\{g_1(0), g_1(\lambda_1)\}.$$

Now $g_1(0) = 0$ and

$$(p\lambda_1)^{-1} g_1(\lambda_1) = (q/\lambda_1)\,[\{(1+\lambda_1/p)(1-\lambda_1/q)\}^{1/2} - 1] + (1-\alpha)$$
$$= f(\lambda_1; p) + (1-\alpha),$$

which has already been shown to be positive in the case of $R_{12} \cap S_1$ for $0 < \alpha < 1 + f(\lambda_1; p)$. Thus

$$\min_{0 \leqq w_1 \leqq \lambda_1,\, w_2 \geqq p w_1} g(w_1, w_2) \geqq 0$$

for the above choice of α. Again, for $w_1 < 0$

$$\min_{w_2 \geqq -q w_1} g(w_1, w_2) = g(w_1, -q w_1) = g_2(w_1), \quad \text{say.}$$

As before, $g_2''(w_1) < 0$, so

$$\min_{-p+\varepsilon \leqq w_1 \leqq 0,\, w_2 \geqq -q w_1} g(w_1, w_2) = \min_{-p+\varepsilon \leqq w_1 \leqq 0} g_2(w_1) = \min\{g_2(0), g_2(-p+\varepsilon)\}.$$

Again $g_2(0) = 0$ and

$$\{q(p-\varepsilon)\}^{-1} g_2(-p+\varepsilon) = f(p-\varepsilon; q) + (1-\alpha),$$

which is positive as in the case of $R_{12} \cap S_2 \cap T_2$ for $0 < \alpha < 1 + f(p-\varepsilon, q)$. Thus

$$\min_{-p+\varepsilon \leqq w_1 \leqq 0,\ w_2 \geqq -q w_1} g(w_1, w_2) \geqq 0$$

for the above choice of α and that concludes the proof for the case of $R_{12} \cap S_3 \cap T_2$. The treatments of the regions R_{13}, R_{22} and R_{23} vary only in details but the essential arguments are the same, so they are omitted. The proof of the lemma is now complete.

We need one more lemma before proving the main result of this section. Let $\mathscr{F}_K$ denote the set of all real-valued functions on $[-K, K]^2$ which attain their minimum values and give $\mathscr{F}_K$ the sup norm topology. Recall that $S(f)$ is the set of points at which the minimum of f is attained and $\psi(f)$ is an unambiguously defined point in $S(f)$.

Lemma 4. *ψ is continuous at all $f \in \mathscr{F}_K$ which have unique minima.*

Proof. Suppose $f \in \mathscr{F}_K$ has unique minimum at (x_0, y_0) and let $f_n \to f$. For each $\varepsilon > 0$ let

$$\delta(\varepsilon) = \min_{\|(x-y)-(x_0, y_0)\| \geqq \varepsilon} f(x,y) - f(x_0, y_0) > 0.$$

Choose n large enough such that $\|f_n - f\| < \delta(\varepsilon)/2$. Then

$$\|(x, y) - (x_0, y_0)\| > \varepsilon \quad \text{implies} \quad f_n(x, y) > f_n(x_0, y_0).$$

Consequently, $S(f_n)$ is entirely contained in a circle of radius ε centered at $(x_0, y_0)=\psi(f)$. Hence $\|\psi(f_n)-\psi(f)\|<\varepsilon$ and the lemma is proved.

Theorem 3. *$n v_n^{-2}\{(\hat{\lambda}_{1n}, \hat{\lambda}_{2n})-(\lambda_1, \lambda_2)\} \to_{\mathscr{L}} \psi(\xi)=(T_1, T_2)$ and $n v_n^{-2}\{(\lambda_{1n}, \lambda_{2n}) -(\lambda_1, \lambda_2)\} \to_{\mathscr{L}} \psi(\eta)=(T_1, T_2)$ where ξ and η are given by* (20) *and* (21).

Proof. By (9) and (10), it suffices to show that $\psi(\xi_n)\to\psi(\xi)$ and $\psi(\eta_n)\to_{\mathscr{L}}\psi(\eta)$. We shall prove only the former, the proof of the latter being exactly analogous. Let $\xi_{n,K}$ and ξ_K denote respectively the restrictions of ξ_n to $[-K, K]^2$ and of ξ to $[-K, K]^2$. Let $F_{n,K}$, F_n, F_K^* and F^* denote the distribution functions of $\psi(\xi_{n,K})$, $\psi(\xi_n)$, $\psi(\xi_K)$ and $\psi(\xi)$ respectively. It will be shown in Remark 5.3 that for each K, almost all sample paths of ξ_K have unique minima. It therefore follows from Theorem 2 and Lemma 4 that for each K, $\psi(\xi_{n,K})$ converges in law to $\psi(\xi_K)$ and by Remark 5.2, see below, $\psi(\xi_K)$ has a continuous distribution. Also, $\psi(\xi_K)$ converges to $\psi(\xi)$ almost surely and so in law. Thus for all (t_1, t_2),

$$\lim_{n\to\infty} F_{n,K}(t_1, t_2)=F_K^*(t_1, t_2) \qquad \text{for each } K \tag{31}$$

and

$$\lim_{K\to\infty} F_K^*(t_1, t_2)=F^*(t_1, t_2). \tag{32}$$

Finally, since $\psi(\xi_{n,K})=\psi(\xi_n)$ on

$$A_{n,K}\subset\{\omega\,|\,S(\xi_n(\cdot, \omega))\subset[-K, K]^2\}$$

and since

$$A_{n,K}^c\subset\{\omega\,|\,\min_{R\cap S} U_n(r_1, r_2)\leqq U_n([n\lambda_1], [n\lambda_2])\},$$

it follows from Lemma 3 that

$$\begin{aligned} |F_{n,K}(t_1, t_2)-F_n(t_1, t_2)| &= |P[\psi(\xi_{n,K})\in(-\infty, t_1]\times(-\infty, t_2];\, A_{n,K}^c] \\ &\quad -P[\psi(\xi_n)\in(-\infty, t_1]\times(-\infty, t_2];\, A_{n,K}^c]|\leqq P[A_{n,K}^c] \\ &\leqq P[\min_{R\cap S} U_n(r_1, r_2)\leqq U_n([n\lambda_1], [n\lambda_2])]\leqq C(K^{-1}+n^{-1}v_n^2). \end{aligned} \tag{33}$$

The theorem now follows from (31), (32) and (33) by first choosing K sufficiently large and then allowing $n\to\infty$.

Remark 5.1. By (7) and (8), $v_n^{-2}\{(\hat{\tau}_{1n}, \hat{\tau}_{2n})-([n\lambda_1], [n\lambda_2])\} \to_{\mathscr{L}} (\hat{T}_1, \hat{T}_2)=\psi(\xi)$ and $v_n^{-2}\{(\tau_{1n}, \tau_{2n})-([n\lambda_1], [n\lambda_2])\} \to_{\mathscr{L}} (T_1, T_2)=\psi(\eta)$.

We now derive the distributions of $\psi(\xi)$ and $\psi(\eta)$ from Corollary 2. Before that we shall establish certain facts about ξ_K, the restriction of ξ on $[-K, K]^2$, that have been used in the proof of Theorem 3.

Remark 5.2. Let T_K denote the point of time at which a B.M. with positive drift attains its minimum on $[0, K]$. The law of iterated logarithm implies $P(T_K=0)=0$. By the Markov property $P(T_K=s)\leqq P(T_{K-s}=0)=0$ for all $0<s<K$; reversal of the process shows that $P(T_K=K)=0$. Thus T_K is a continuous random variable. In proving Theorem 3 we have used the fact that $\psi(\xi_K)=(\hat{T}_{1K}, \hat{T}_{2K})$ has a continuous distribution. Because of the independence of $\hat{T}_{1K}$ and $\hat{T}_{2K}$, it is enough

to show that each of these random variables is continuous. This follows easily from what we have shown in the beginning of this paragraph.

Remark 5.3. In the proof of Theorem 3 we have also used the fact that almost all paths of ξ_K have unique minima. Let $X(t)=B_i(t)+\theta g_i(t)$. In view of Remark 1.2, it is enough to show that

$$P[\min_{-K\leq t\leq 0} X(t)=\min_{0\leq t\leq K} X(t)]=0, \qquad i=1,2. \tag{34}$$

For each rational $-K\leq r<0<s\leq K$, let $Z_{rs}=\min_{-K\leq t\leq r} X(t)-\min_{s\leq t\leq K} X(t)$. Since by the first part of Remark 5.2, $X(t)$ can attain neither its minimum on $[-K,0]$ nor its minimum on $[0,K]$ at 0, (34) will be proved by showing that $P[Z_{rs}=0]=0$. This is proved as in Remark 1.2.

To find the distribution of $\psi(\xi)$, first consider $\{X(t), -\infty<t<\infty\}$ where $X(t)=X_1(-t)$, $t\leq 0$ and $X(t)=X_2(t)$, $t\geq 0$, $\{X_i(t), t\geq 0\}$, $i=1,2$ being independent standard B.M.'s with positive drifts θ_1 and θ_2 respectively. Let M_i denote the minimum of $\{X_i(t)\}$ and T_i the time at which this minimum is attained. Define

$$T=\begin{cases} -T_1 & \text{if } M_1\leq M_2 \\ T_2 & \text{if } M_1>M_2. \end{cases}$$

By virtue of Corollary 2, $P[M_1=M_2]=0$. The random variable T is therefore, the time at which $\{X(t), -\infty<t<\infty\}$ attains its (a.s.) unique minimum. The density function of T is obtained from Corollary 2.

Corollary 4. *The density of T is,*

$$f_T(t)=\begin{cases} f(-t;\,\theta_1,\theta_2), & t\leq 0 \\ f(t;\,\theta_2,\theta_1), & t>0 \end{cases} \tag{35}$$

where

$$\begin{aligned} f(t;\,\theta_1,\theta_2)=&2\theta_1[t^{-1/2}\phi(\theta_1 t^{1/2})-\theta_1\{1-\Phi(\theta_1 t^{1/2})\}] \\ &-2\theta_1 \exp[2\theta_2(\theta_1+\theta_2)t] \\ &\cdot[t^{-1/2}\phi((\theta_1+2\theta_2)t^{1/2})-(\theta_1+2\theta_2)\{1-\Phi((\theta_1+2\theta_2)t^{1/2})\}]. \end{aligned}$$

Proof. Let $U_i=-M_i$. The marginal distribution of U_i is easily seen to be negative exponential with mean $(2\theta_i)^{-1}$. Since (T_1,U_1) and (T_2,U_2) are independent, we have for $t>0$,

$$\begin{aligned} f_T(t)=&\int_0^\infty P[U_1<u]\,f_{T_2,U_2}(t,u)\,du=\int_0^\infty \{1-\exp[-2\theta_1 u]\} \\ &\cdot 2\theta_2 u t^{-3/2}\phi(u t^{-1/2}+\theta_2 t^{1/2})\,du=f(t;\,\theta_2,\theta_1) \end{aligned}$$

after some simplification. The case of $t\leq 0$ is proved similarly.

It is now one easy step to write down density functions of $\psi(\xi)=(\hat{T}_1,\hat{T}_2)$ and $\psi(\eta)=(T_1,T_2)$. To do this, we only have to note that $\hat{T}_1$ and $\hat{T}_2$ are independent, $\hat{T}_1$ has density function $f_{\hat{T}_1}(t)$ of the form (35) with $\theta_1=p\theta$, $\theta_2=q\theta$ and $\hat{T}_2$ has density function $f_{\hat{T}_2}(t)$ of the form (35) with $\theta_1=q\theta$ and $\theta_2=p\theta$. We summarize this as

Corollary 5. (a) *The density function of* $\psi(\xi)=(\hat{T}_1, \hat{T}_2)$ *is*

$$f_{\hat{T}_1, \hat{T}_2}(t_1, t_2)=f_{\hat{T}_1}(t_1) f_{\hat{T}_2}(t_2)$$

where $f_{\hat{T}_1}(t)=f_{\hat{T}_2}(-t)$ *is given by* (35) *with* $\theta_1=p\theta$ *and* $\theta_2=q\theta$.

(b) *The density function of* $\psi(\eta)=(T_1, T_2)$ *is obtained as a special case of* (a) *for* $p=q=\frac{1}{2}$.

6. Miscellaneous Remarks

6.1. A Comparison between the Estimators $(\lambda_{1n}, \lambda_{2n})$ *and* $(\hat{\lambda}_{1n}, \hat{\lambda}_{2n})$

The distribution of $\psi(\xi)$ depends on λ_1 and λ_2 only through $p=\lambda_2-\lambda_1$ (i.e., it depends on the extent of the signal but not an its actual location in the observed series), whereas the distribution of $\psi(\eta)$ is altogether independent of λ_1 and λ_2. For the random variable T treated in Corollary 4, a little calculation shows that $P[T<0]=\theta_2/(\theta_1+\theta_2)$. Examining Corollary 5 in the light of this fact we see that for $p>\frac{1}{2}$, $\hat{\lambda}_{1n}$ is more likely to overestimate λ_1 and $\hat{\lambda}_{2n}$ is more likely to underestimate λ_2 and for $p<\frac{1}{2}$, the tendency is in the opposite direction. This means that longer signals will tend to be shortened and vice versa. The estimator $(\lambda_{1n}, \lambda_{2n})$, however, does not suffer from this kind of bias.

6.2. The Degenerate Case

Suppose in the signal given by (1), λ_2 is known to be 1 and only λ_1 is to be estimated. Let $U_n'(r)=\sum_{j=1}^{r} Y_j^{(n)}-rn^{-1}\sum_{j=1}^{n} Y_j^{(n)}$, $c_n'(r)=n\{r(n-r)\}^{-1/2}$ and $V_n'(r)=c_n'(r)\,U_n'(r)$, $1\leqq r \leqq n-1$. Then analogous estimators of the two estimators discussed in this paper are defined as n^{-1} times the values of r which minimize $U_n'(r)$ and $V_n'(r)$ respectively. $U_n'(r)$ is minimized over the entire range, whereas $V_n'(r)$ is minimized over $\{r \mid \min(r, n-r) \geqq n^{1/2} v_n\}$. Call these estimators $\hat{\lambda}_{1n}$ and λ_{1n} respectively. The following properties hold for $\{U_n'(r)\}$ and $\{V_n'(r)\}$ with $p=1-\lambda_1$ and $q=\lambda_1$.

(i) By an almost sure invariance principle on compact sets similar to Theorem 2,

$$\xi_n(t)=(v_n\sigma)^{-1}\{U_n'([n\lambda_1+v_n^2 t])-U_n'([n\lambda_1])\} \to B(t)+\theta g_1(t)$$

and

$$\eta_n(t)=(pq)^{1/2}(v_n\sigma)^{-1}\{V_n'([n\lambda_1+v_n^2 t])-V_n'([n\lambda_1])\} \to B(t)+\tfrac{1}{2}\theta|t|,$$

where $B(t)$ is a two-sided standard B.M. and $g_1(t)$ is as in (12).

(ii) $$P\Big[\min_{|r-n\lambda_1|\geqq K v_n^2} U_n'(r)\leqq U_n'([n\lambda_1])\Big]\leqq C(K^{-1}+n^{-1}v_n^2)$$

and

$$P\Big[\min_{|r-n\lambda_1|\geqq K v_n^2} V_n'(r)\leqq V_n'([n\lambda_1])\Big]\leqq C(K^{-1}+n^{-1/2}v_n).$$

6.3. An Asymptotic Test for H_0

Consider the stochastic process

$$\zeta_n(t)=n^{-1/2}\hat{\sigma}_n^{-1}\left[\sum_{j=1}^{[nt]} Y_j^{(n)}-n^{-1}[nt]\sum_{j=1}^{n} Y_j^{(n)}\right], \qquad 0<t<1$$

where $\hat{\sigma}_n^2=\{2(n-1)\}^{-1}\sum_{j=1}^{n-1}(Y_{j+1}^{(n)}-Y_j^{(n)})^2$. Since $\hat{\sigma}_n^2$ is a consistent estimator of σ^2, it can easily be seen that

(i) under H_0, $\{\zeta_n(t)\}$ converges weakly to $\{B^*(t)\}$, and

(ii) under H_1, $\{\zeta_n(t)-\theta n^{1/2}v_n^{-1}g(t)\}$ converges weakly to $\{B^*(t)\}$

where $\{B^*(t)\}$ is a Brownian bridge on (0,1) and

$$g(t)=\begin{cases}-pt, & 0<t\leqq\lambda_1\\ -p\lambda_1+q(t-\lambda_1), & \lambda_1<t\leqq\lambda_2\\ p(1-\lambda_2)-p(t-\lambda_2), & \lambda_2<t<1.\end{cases}$$

It then follows that if $\{k_n\}$ is chosen such that

$$\lim_{n\to\infty} k_n=\infty \qquad \text{but} \quad \lim_{n\to\infty} k_n n^{-1/2}v_n=0,$$

then a test with rejection region $\sup_{0<t<1}|\zeta_n(t)|\geqq k_n$ will have probabilities of both types of error tending to 0 as $n\to\infty$.

6.4. Estimation of the Intensity of a Signal

Suppose a signal is detected by the test described in the last paragraph and $([n\lambda_1],[n\lambda_2])$ is estimated by (τ_{1n},τ_{2n}). We now consider the problem of estimating δ, the intensity of the signal. The mle of δ for normally distributed noise is given by

$$\delta_n=v_n\left[(\tau_{2n}-\tau_{1n})^{-1}\sum_{j=\tau_{1n}+1}^{\tau_{2n}} Y_j^{(n)}-(n-\tau_{2n}+\tau_{1n})^{-1}\left\{\sum_{j=1}^{\tau_{1n}} Y_j^{(n)}+\sum_{j=\tau_{2n}+1}^{n} Y_j^{(n)}\right\}\right].$$

If λ_1 and λ_2 were known and if τ_{1n} and τ_{2n} were replaced by $[n\lambda_1]$ and $[n\lambda_2]$ respectively in the above formula, then $n^{1/2}v_n^{-1}(\delta_n-\delta)$ would be asymptotically normal with mean 0 and variance $\sigma^2/(pq)$. It can be verified by straightforward analysis that even though δ_n uses the estimates τ_{1n} and τ_{2n} instead of $[n\lambda_1]$ and $[n\lambda_2]$, it still has the same asymptotic distribution. The key thing is to check that $\sum_{j=1}^{\tau_{in}} Y_j^{(n)}-\sum_{j=1}^{[n\lambda_i]} Y_j^{(n)}=O_p(v_n^2)$, $i=1,2$. In fact it makes no difference whether the estimates τ_{1n} and τ_{2n} are the mle's of $[n\lambda_1]$ and $[n\lambda_2]$ or not, so long as they satisfy $\tau_{in}-[n\lambda_i]=O_p(v_n^2)$, $i=1,2$. A test for a specified value δ_0 of δ can also be constructed on the basis of the statistic $(n\hat{p}_n\hat{q}_n)^{1/2}(\delta_n-\delta_0)/(v_n\hat{\sigma}_n)$, where $\hat{p}_n=\lambda_{2n}-\lambda_{1n}$, $\hat{q}_n=1-\hat{p}_n$ and $\hat{\sigma}_n$ is as in 6.3.

6.5. *Application to Image Processing*

Consider a discretized picture in black and white in which the darkness Y_{ij} at the (i,j)-th grid, $1 \leqq i \leqq j \leqq n$, is the sum of a noise X_{ij} and a possible signal A_{ij}. The simplest kind of noise consists of iid random variables X_{ij} with find second moment and the simplest kind of signal is a shadow of uniform darkness, i.e., A_{ij} has a constant positive value on some set S and is 0 elsewhere. Suppose n is large and $A_{ij} = \delta v_n^{-1}$ for $(i,j) \in S$ where v_n satisfies the assumption of Section 2. If further, each row-section of S, i.e., the intersection of S with $\{(i,j) \mid j = 1, \dots, n\}$ for each fixed i, is either empty or a set of consecutive integers $\{[n\lambda_{i1}]+1, \dots, [n\lambda_{i2}]\}$ with $0 < \lambda_{i1} < \lambda_{i2} < 1$, then the picture can be analyzed by its row-sections by means of the methods discussed in this paper. By means of the test described in the Remark 6.3, each row-section can be first tested to determine whether it contains any part of the shadow. Since such a test will have both error probabilities quite small, this part of the problem presents little difficulty. Furthermore, most shadows will have non-empty intersections with a number of consecutive rows and this will help correcting occasional mistakes in the tests. We can then estimate $(\lambda_{i1}, \lambda_{i2})$ by $(\lambda_{i1n}, \lambda_{i2n})$ for those i where the test rejects H_0. Putting these row-by-row estimators together, the estimated shadow is obtained. Successful recognition of the shape and size of the shadow will depend on the errors of estimation in the different rows. Suppose a shadow becomes unrecognizable if either λ_{i1} or λ_{i2} is estimated with an error exceeding ε in absolute value in a proportion α or more of rows having a non-empty section. Then a criterion for successful recognition can be computed from our results. As we have mentioned above, most shadows will have non-empty sections in a number of consecutive rows. Also, the values of $(\lambda_{i1}, \lambda_{i2})$ in nearby rows will frequently be close to each other. We can take advantage of such a situation by applying the above procedure on the averages of several rows in a row-wise moving average scheme. Faint shadows are obtained in electron microscopy, hard X-ray photography through bones, etc. However, our assumption that each non-empty row-section of the shadow should be a set of consecutive integers, is too restrictive. To adapt the method of this paper to situations not satisfying this condition, we note that if a shadow has several segments in a row-section, then with a high probability the stochastic process $\{\zeta_n(t)\}$ for that row, given in 6.3, will have a local minumum near the beginning and a local maximum near the end of each segment. Thus the number of segments in a row-section and their approximate locations can be determined by a preliminary examination by analyzing $\{\zeta_n(t)\}$ for that row. On the basis of these approximate findings, the row can then be broken into several parts, each containing a single segment, and then the segments are more precisely estimated in the way discussed earlier.

7. The Minimum Content of a Dam in $[0, t]$

Of prime interest in storage theory is the distribution of the time taken for a dam to become empty for the first time. However by using the results of Section 1 we can derive the much more informative joint distribution of the minimum content

v_t in $[0, t]$ and time τ_t at which this minimum value first occurs. We shall use the classical model of an infinite dam (see Moran (1959), Prabhu (1965)) for which the input in $[0, t]$ is $Z(t)$ where $\{Z(t), t \geqq 0\}$ is an additive process with non-decreasing sample-paths, and for which the release rate as a function of the content x is $r(x) = c I_{(0,\infty)}(x)$, $c > 0$. The joint distribution can be expressed quite explicitly if $\{Z(t)\}$ is stable with index 1/2 or if we consider (as in Bather (1968)) the somewhat different model in which $\{Z(t) - ct\}$ is a Brownian motion with drift.

Suppose that at time zero the dam has content $a > 0$. If we define

$$X(t) = Z(t) - ct, \qquad t \geqq 0,$$

then $\{X(t)\}$ is a process of the class considered in Section 1, and if $F(-v)$, T_t and M_t are defined in terms of $\{X(t)\}$ as before, then

$$P[\tau_t \leqq u, v_t \leqq v] = P[F(-a) \leqq \min(u, t)] + P[T_t \leqq u, -a < M_t \leqq v - a], \qquad v \geqq 0.$$

If $\{Z(t)\}$ is stable with index 1/2 the right hand side can be written down quite explicitly using the results of Theorem 1, Corollary 3. If $\{X(t)\}$ is a Brownian motion with $EX(t) = \mu t$ and $\operatorname{Var} X(t) = t$, the joint distribution of (τ_t, v_t) can again be written explicitly, this time using the results of Theorem 1, Corollary 2. In the latter case we see in particular that the distribution of v_t consists of a mass

$$P[v_t = 0] = \int_0^t a y^{-3/2} \phi(a y^{-1/2} + \mu y^{1/2}) \, dy,$$

and a density,

$$f_{v_t}(v) = I_{(0,a)}(v) \int_t^\infty y^{-5/2} [y(1 - a\mu + v\mu) - (a - v)^2] \, \phi((a - v) y^{-1/2} + \mu y^{1/2}) \, dy + I_{(0,a)}(v) \, \beta(0+) \exp(-\beta(0+)(a - v)),$$

where $\beta(0+) = I_{(0,\infty)}(\mu) \, 2\mu$.

8. The Minimum of the Integral of a Markov Chain and Its Location

Let $\{Y(t), t \geqq 0\}$ be an irreducible Markov chain with state-space $\{\mu_1, \ldots, \mu_m\}$ and infinitesimal generator $Q = [q_{ij}]$. Suppose that $\mu_i < 0$, $i = 1, \ldots, n$, $\mu_i > 0$, $i > n$, and define a diagonal matrix, $D = \operatorname{diag}\{\mu_i\}$. To the version of $\{Y(t)\}$ whose sample-paths are right-continuous step-functions there corresponds a process

$$X(t) = \int_0^t Y(u) \, du,$$

whose sample-paths are continuous. We shall conclude this paper by sketching the analogues for $\{X(t)\}$ of the results obtained in Section 1 for additive processes with no negative jumps.

For $v \geqq 0$ define

$$F(-v)=\begin{cases}+\infty & \text{if } X(t)>-v \quad \text{for all } t>0,\\ \inf\{t>0\colon X(t)\leqq -v\} & \text{otherwise,}\end{cases}$$

and

$$F(-v,j)=F(-v)\,I_{\{Y(F(-v))=\mu_j\}}+\infty\,I_{\{Y(F(-v))\neq\mu_j\}},$$

where I_A denotes the indicator function of the event A. We shall use the notation $P_i(\cdot)$ and $E_i(\cdot)$ to denote probabilities and expectations conditional on $Y(0)=\mu_i$, $i=1,\ldots,m$, and $\underset{\sim}{P}(\cdot)$, $\underset{\sim}{E}(\cdot)$ for the column vectors with components $P_i(\cdot)$, $E_i(\cdot)$ respectively. The random variables M_t and T_t are defined in terms of the process $\{x(t)\}$ as in Section 1.

Let $P_{F(-v,j)}(\cdot\,|\,i)$ denote the (possibly defective) distribution of $F(-v,j)$ conditional on $Y(0)=\mu_i$. Then from the results of Brockwell (1973) it is easily deduced that the matrix $S^{(-v)}(\xi)$ of Laplace transforms $E_i(\exp-\xi F(-v,j))$, $i,j=1,\ldots,m$, is given by

$$S^{(-v)}(\xi)=\exp(-vA(\xi))\,S(\xi), \qquad v\geqq 0,\ \xi>0, \tag{36}$$

where

$$A(\xi)=D^{-1}(Q-\xi I),$$

and $S(\xi)=[S_{ij}(\xi)]$ is uniquely determined by the conditions

$$\text{(i)}\quad S_{ij}(\xi)=\begin{cases}0, & j>n\\ \delta_{ij}, & i,j=1,\ldots,n,\end{cases}$$

and

(ii) each column of $S(\xi)$ is orthogonal to the $(m-n)$-dimensional subspace of $\mathbb{R}^m$ associated with the $(m-n)$ eigenvalues of $A(\xi)$ with negative real parts.

Using the result (36) together with the equations,

$$P_i[T_t\leqq u, M_t\leqq -v] = \sum_{j=1}^{m}\int_{[0,u]} P_j[T_{t-s}\leqq u-s]\,P_{F(-v,j)}(ds\,|\,i), \qquad 0\leqq u\leqq t,\ v\geqq 0,$$

and arguing as in the proof of Theorem 1, we obtain the following result.

Theorem 4. $$\int_0^\infty \exp(-\xi t)\,\underset{\sim}{P}[T_t\leqq u, |M_t|\leqq v]\,dt=\xi^{-1}[I-S(\xi)]\,\underset{\sim}{1} + \xi^{-1}\int_{(0,v]} dy \int_{(0,u]} \exp(-\xi x)\,P_F^{(-y)}(dx)\,A(\xi)\,S(\xi)\,\underset{\sim}{1} \qquad u,v\geqq 0,\ \xi>0,$$

where $P_F^{(-y)}(\cdot)$ is the matrix of measures $[P_{F(-y,j)}(\cdot\,|\,i)]$ whose Laplace transform is specified by Equation (36).

Corollary 6. *Let $\underset{\sim}{y}'$ be the stationary distribution of the chain $\{Y(t)\}$, i.e., $\underset{\sim}{y}'\,Q=0$ and $\underset{\sim}{y}'\underset{\sim}{1}=1$. Then*

(i) *if* $\underset{\sim}{y}' D \underset{\sim}{1} > 0$, $(T_t, M_t) \xrightarrow[\text{a.s.}]{} (T, M)$ *as* $t \to \infty$, *where*

$$\underset{\sim}{P}[T \leqq u, |M| \leqq v] = [I - S(0)] \underset{\sim}{1} + \int_{(0,v]} dy \int_{(0,u]} P_F^{(-y)}(dx) D^{-1} Q S(0) \underset{\sim}{1},$$

(ii) *if* $\underset{\sim}{y}' D \underset{\sim}{1} < 0$, $\lim_{t \to \infty} \underset{\sim}{P}[T_t \leqq u, |M_t| \leqq v] = \underset{\sim}{0}$.

Remark 8.1. Explicit inversion of the Laplace transform in Theorem 4 (in terms of modified Bessel functions) is possible when $\{Y(t)\}$ has only two states. We omit the details.

Remark 8.2. Theorem 4 may be used to derive results analogous to those of Section 7 for the dam whose net input rate at time t (prior to emptiness) is $Y(t)$. (Net input rate here means input rate minus release rate.) Defining v_t and τ_t as in Section 7 we obtain in particular,

$$\begin{aligned} \xi \int_0^\infty \exp(-\xi t) \underset{\sim}{E} \exp[-\rho \tau_t - \theta v_t] \, dt = {} & \exp(-a\theta) [I - S(\xi)] \underset{\sim}{1} \\ & + [\exp(-aA(\xi + \rho))] S(\xi + \rho) \underset{\sim}{1} \\ & + \exp(-a\theta) \int_0^a \exp[v(\theta - A(\xi + \rho))] S(\xi + \rho) A(\xi) S(\xi) \underset{\sim}{1} \, dv, \\ & \rho, \theta \geqq 0, \ \xi > 0. \end{aligned}$$

Explicit inversion is again possible if $\{Y(t)\}$ has only two possible states (such a dam was considered by McNeil (1972)).

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