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On Convergence of Types and Processes in Euclidean Space

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Let $\mathscr{E}$ be an Euclidean space; Y_n, Z, U random vectors in $\mathscr{E}$; h_n, g_n affine transformations and let $\mathscr{H}$ be a subgroup of the group $\mathscr{G}$ of all the invertible affine transformations, closed relative to $\mathscr{G}$. Suppose that $g_n Y_n \xrightarrow{D} Z$ and $h_n Y_n \xrightarrow{D} U$ where Z is nonsingular. The behaviour of $\gamma_n = h_n g_n^{-1}$ as $n \to \infty$ is discussed first. The results are used then to prove that if $h_n Y_{[nt]} \xrightarrow{D} Z_t \in \mathscr{E}$ for all $t \in (0, \infty)$, where $h_n \in \mathscr{H}$ and Z_1 is nonsingular and nonsymmetric with respect to $\mathscr{H}$, then $\gamma_n(t) = h_n h_{[nt]}^{-1} \to \gamma(t) \in \mathscr{H}$, $Z_t \overset{D}{=} \gamma(t) Z_1$ for all $t \in (0, \infty)$ and γ is a continuous homomorphism of the multiplicative group of $(0, \infty)$ into $\mathscr{H}$. The explicit forms of the possible γ are shown.

1. Introduction

Let Y_n and Z be random vectors in a k-dimensional Euclidean space $\mathscr{E}$ and let h_n be an affine transformation. Whenever a limiting result of the form $h_n Y_n \xrightarrow{D} Z$ (as $n \to \infty$) is obtained, a natural question is then whether a functional limit theorem holds. Namely, does $h_n Y_{[nt]}$ converge in distribution? And if it does, what is the distribution of the limit Z_t, say, in terms of Z? The present paper, which is an attempt to answer these questions, generalizes the results of [6] obtained for $\mathscr{E} = R^1$.

Let $\mathscr{A}$ be the set of all affine transformations $h: \mathscr{E} \mapsto \mathscr{E}$. We write $h = \langle B; b \rangle$ if $hx = Bx + b$ $(x \in \mathscr{E})$, B a linear transformation and $b \in \mathscr{E}$. Let $\mathscr{G}$ be the group of all invertible (nonsingular) elements of $\mathscr{A}$ and let ε be the identity of $\mathscr{G}$. For h_n, $h \in \mathscr{A}$, $h_n \to h$ means $h_n x \to hx$ for each $x \in \mathscr{E}$. This definition gives to $\mathscr{A}$ a topology under which composition is continuous. The notation $h_n \sim g_n$ $(h_n, g_n \in \mathscr{G})$ means $h_n g_n^{-1} \to \varepsilon$.

A random vector $Z \in \mathscr{E}$ is *nonsingular* if (x, Z) is a nondegenerate random variable for each $x \neq 0$, $x \in \mathscr{E}$ (here $(\cdot, \cdot)$ is the inner product in $\mathscr{E}$).

In Section 2 we consider both relations

$$g_n Y_n \xrightarrow{D} Z, \qquad Z \text{ nonsingular} \tag{1.1}$$

and

$$h_n Y_n \xrightarrow{D} U \tag{1.2}$$

and draw some conclusions about $\gamma_n = h_n g_n^{-1}$ and U. In Section 3 we consider the situation under which

$$h_n Y_{[nt]} \xrightarrow{D} Z_t \qquad \forall t \in (0, \infty). \tag{1.3}$$

Our main result is that if Z_1 is nonsingular and the h_n belong to some subgroup $\mathscr{H} \subseteq \mathscr{G}$ with respect to which Z_1 is nonsymmetric then there exists a continuous homomorphism of the multiplicative group of $(0, \infty)$ γ: $(0, \infty) \mapsto \mathscr{H}$ such that $Z_t \overset{D}{=} \gamma(t) Z_1$. In Section 4 we show the explicit form of $\gamma(t)$ in matrix representation.

2. Convergence of Types in $\mathscr{E}$

For a random vector $Z \in \mathscr{E}$ we define $\mathscr{G}_Z \subseteq \mathscr{A}$ to be the *symmetry set* of Z i.e. $\eta Z \overset{D}{=} Z$ iff $\eta \in \mathscr{G}_Z$. The following two results of Billingsley [2] are essential.

(2.1) **Theorem.** *If Z is nonsingular then $\mathscr{G}_Z$ is a compact subgroup of $\mathscr{G}$.*

(2.2) **Theorem.** *Suppose* (1.1) *and* (1.2) *hold where both Z and U are nonsingular. Then for sufficiently large n, h_n and g_n are in $\mathscr{G}$ and there exist $\gamma \in \mathscr{G}$ and $\eta_n \in \mathscr{G}_Z$ such that*

$$\gamma_n = h_n g_n^{-1} \sim \gamma \eta_n \tag{2.1}$$

and

$$U \overset{D}{=} \gamma Z. \tag{2.2}$$

The fact that U and Z belong to the same type (namely (2.2) holds) was proved earlier by Fisz [4].

We shall need the following result when only Z is known to be nonsingular.

(2.3) **Theorem.** *If* (1.1) *and* (1.2) *hold then*

(a) *$\{\gamma_n\}$ is relatively compact in $\mathscr{A}$;*

(b) *if γ is a limit point of $\{\gamma_n\}$ then* (2.2) *holds;*

(c) *U is nonsingular iff $\{\gamma_n\}$ is relatively compact in $\mathscr{G}$ iff $\{\gamma_n\}$ has a limit point in $\mathscr{G}$.*

Proof. Since Z is nonsingular, γ_n is well defined for sufficiently large n. Suppose now that (a) is false. Then there exists a $\xi \in \mathscr{E}$ such that the sequence of real linear functions

$$f_{n\xi}(x) = (\xi, \gamma_n x) \qquad (x \in \mathscr{E}) \tag{2.3}$$

is unbounded. Let $\gamma_n = \langle D_n; d_n \rangle$ and get

$$f_{n\xi}(x) = (D'_n \xi, x) + (\xi, d_n) \equiv a_n(\zeta_n, x) + b_n, \tag{2.4}$$

where D'_n is the adjoint transformation of D_n, $a_n = \|D'_n \xi\|$, $\zeta_n = D'_n \xi / a_n$ and $b_n = (\xi, d_n)$. (Notice that $\|\zeta_n\| = 1$). Let $\xi \in \mathscr{E}$ and $\{m\}$ be a subsequence of $\{n\}$ such that $\{f_{m\xi}\}$

is unbounded and $\zeta_m \to \zeta \in \mathscr{E}$. By (1.1) we have

$$W_m = (\zeta_m, g_m Y_m) \xrightarrow{D} (\zeta, Z) \tag{2.5}$$

and (ζ, Z) is nondegenerate. But by (2.4), (2.3) and (1.2)

$$a_m W_m + b_m = f_{m\xi}(g_m Y_m) = (\xi, h_m Y_m) \xrightarrow{D} (\xi, U). \tag{2.6}$$

By the *convergence of types theorem* in R^1 (cf. [3], p. 246) (2.5) and (2.6) imply that $\{a_m\}$ and $\{b_m\}$ are convergent sequences and hence bounded. Thus $\{f_{m\xi}\}$ is bounded – a contradiction. This proves (a).

If $\gamma = \lim \gamma_m$ along some subsequence $\{m\}$ then (1.1) and (1.2) imply $h_m Y_m = \gamma_m g_m Y_m \xrightarrow{D} \gamma Z \overset{D}{=} U$ and (b) follows.

When U is nonsingular Theorem 2.2 applies and (2.1) follows. By Theorem 2.1 $\{\gamma_n\}$ is then relatively compact in $\mathscr{G}$ (i.e. all its limit points are in $\mathscr{G}$). Suppose now that $\{\gamma_n\}$ has a limit point $\gamma \in \mathscr{G}$. Then by (b) $U \overset{D}{=} \gamma Z$ and hence U is nonsingular. ▯

Let now $\mathscr{H} \subseteq \mathscr{G}$ be a subgroup, closed relative to $\mathscr{G}$. We say that Z is *nonsymmetric with respect to* $\mathscr{H}$ if $\mathscr{H} \cap \mathscr{G}_Z = \{\varepsilon\}$. An immediate consequence of the previous results is the following *convergence of types theorem* for $\mathscr{E}$.

(2.4) **Theorem.** *Let Z be nonsymmetric with respect to $\mathscr{H}$ and suppose* (1.1) *and* (1.2) *hold with Z, U nonsingular and $h_n, g_n \in \mathscr{H}$. Then there exists a $\gamma \in \mathscr{H}$ such that*

$$\gamma_n \to \gamma \tag{2.7}$$

and (2.2) *holds.*

Proof. We only have to prove that all the limit points of $\{\gamma_n\}$ are equal. Suppose γ and γ_* are both limit points. By Theorem 2.3 both belong to $\mathscr{G}$ and hence to $\mathscr{H}$ and $\gamma Z \overset{D}{=} \gamma_* Z$. This in turn implies $Z \overset{D}{=} \gamma^{-1}\gamma_* Z$ and hence $\gamma^{-1}\gamma_* \in \mathscr{G}_Z$. But since $\gamma^{-1}\gamma_* \in \mathscr{H}$ we must have $\gamma^{-1}\gamma_* = \varepsilon$ i.e. $\gamma = \gamma_*$. ▯

Notice that in the *classical convergence of types theorem* in R^1 ([3], p. 246) $\mathscr{H}$ is the group of all positive affine transformations.

3. Convergence to a Stochastic Process

In this section we shall assume throughout that

$$h_n Y_n \xrightarrow{D} Z_1, \qquad Z_1 \text{ is nonsingular}, \tag{3.1}$$

where $h_n \in \mathscr{H}$ and $\mathscr{H} \subseteq \mathscr{G}$ is a subgroup, closed relative to $\mathscr{G}$. For $t \in (0, \infty)$ we write $\gamma_n(t) = h_n h_{[nt]}^{-1}$ and thus

$$h_n Y_{[nt]} = \gamma_n(t)\, h_{[nt]} Y_{[nt]}. \tag{3.2}$$

If for some t (here and in the sequel, all t are in $(0, \infty)$)

$$\gamma_n(t) \to \gamma(t) \tag{3.3}$$

then we clearly have

$$h_n Y_{[nt]} \xrightarrow{D} \gamma(t) Z_1. \tag{3.4}$$

Let Λ be the set of all continuous homomorphisms of the multiplicative group of $(0, \infty)$ into $\mathscr{H}$.

(3.1) **Theorem.** *If* (3.3) *holds for all t with* $\gamma(t)\in\mathscr{H}$ *then* $\gamma\in\Lambda$ *and the convergence in* (3.3) *is uniform on compact subsets of* $(0, \infty)$.

Proof. Define $h(u)=h_{[u]}$ $(u>0)$ then

$$h(u)\, h^{-1}(ut)=\gamma_{[u]}(t)\, h_{[[u]\,t]} h^{-1}_{[ut]}. \tag{3.5}$$

Consider the product

$$\delta_n(t)=h_n h^{-1}_{[[nt]/t]}=\gamma_n(t)\, \gamma_{[nt]}(t^{-1}). \tag{3.6}$$

It is clear that for each $t, \delta_n(t)$ converges. In particular $\delta_n(1/2)$ converges. But $\delta_n(1/2)=\varepsilon$ if n is even and $\delta_n(1/2)=h_n h^{-1}_{n-1}$ if n is odd. Thus we have proved that

$$h_n h^{-1}_{n-1}\to\varepsilon. \tag{3.7}$$

Since $0\leqq[ut]-[[u]\,t]\leqq t+1$ for all $u>0$, (3.7) implies that $h_{[[u]\,t]} h^{-1}_{[ut]}\to\varepsilon$ as $u\to\infty$ and thus from (3.5) we have

$$\lim_{u\to\infty} h(u)\, h^{-1}(ut)=\gamma(t) \tag{3.8}$$

for all t. Thus $h(u)$ is a regularly varying function and hence (cf. [1] Section 9) $\gamma\in\Lambda$ and the convergence in (3.3) is uniform on compact subsets. □

Suppose now that

$$h_n Y_{[nt]}\xrightarrow{D} Z_t \qquad \forall t. \tag{3.9}$$

In general the Z_t need not be nonsingular nor need (3.3) hold. Let $T\subseteq(0, \infty)$ be the set of all t for which Z_t is nonsingular. The following theorem will help us decide when $t\in T$.

(3.2) **Theorem.** *Suppose* (3.1) *and* (3.9) *hold. Then*

(a) *for each t there exists a* $\gamma(t)\in\mathscr{A}$ *for which*

$$Z_t\overset{D}{=}\gamma(t)\, Z_1; \tag{3.10}$$

(b) *for a fixed t,* $\{\gamma_n(t)\}$ *has a limit point in* $\mathscr{H}$ *iff* $t\in T$;

(c) *if t is rational then* $t\in T$;

(d) (3.7) *implies* $T=(0, \infty)$;

(e) *the existence of* $t_0\neq 1$ *for which* $\gamma_n(t_0)\to\gamma(t_0)$ *and* $\gamma_n(t_0^{-1})\to\gamma^{-1}(t_0)$ *implies* (3.7);

(f) *if* Z_1 *is nonsymmetric with respect to* $\mathscr{H}$ *then* (3.7) *holds.*

Proof. The application of Theorem 2.3 immediately implies (a) and (b). Consider now $\delta_n(t)$ defined by (3.6), and suppose that ε is a limit point (t fixed). Since $\{\gamma_n(t)\}$ and $\{\gamma_{[nt]}(t^{-1})\}$ are both relatively compact in $\mathscr{A}$ (Theorem 2.3), there exists a subsequence $\{m\}$ along which $\delta_m(t)\to\varepsilon$ and $\gamma_m(t)$ and $\gamma_{[mt]}(t^{-1})$ both converge to $\gamma(t)$, $\gamma(t^{-1})$ in $\mathscr{A}$ (respectively). But by (3.6) we must have $\varepsilon=\gamma(t)\,\gamma(t^{-1})$ and hence $\gamma(t)$, $\gamma(t^{-1})\in\mathscr{H}$. Thus by (b) $t\in T$.

If t is rational then $\delta_n(t)=\varepsilon$ infinitely often (i.o.) (whenever nt is integral) and thus ε is a limit point and hence (c) follows.

Suppose $h_n h_{n-1}^{-1}\to\varepsilon$. For each t, $0\leqq n-[[nt]/t]\leqq t^{-1}+1$ is bounded. Hence $\delta_n(t)\to\varepsilon$ for each t and (d) follows.

For every $t\neq 1$ either $[[nt]/t]=n-1$ i.o. or $[[n/t]\,t]=n-1$ i.o. Let t_0 satisfy the condition of (e). Thus $\delta_n(t_0)\to\varepsilon$ and $\delta_n(t_0^{-1})\to\varepsilon$. Since at least one among $\{\delta_n(t_0), \delta_n(t_0^{-1})\}$ is equal to $h_n h_{n-1}^{-1}$ i.o. we must have $h_n h_{n-1}^{-1}\to\varepsilon$.

Suppose now that Z is nonsymmetric with respect to $\mathscr{H}$. Fix a rational (but not integral) t. We know that $t, t^{-1}\in T$ and hence by Theorem 2.4 $\gamma_n(t)$ and $\gamma_n(t^{-1})$ both converge in $\mathscr{H}$ and so does $\delta_n(t)$. But $\delta_n(t)=\varepsilon$ i.o. and $\delta_n(t)=h_n h_{n-1}^{-1}$ i.o., hence $\delta_n(t)\to\varepsilon$ and (f) follows. □

Notice that none of the conditions in (c)–(f) are necessary. But it is only under nonsymmetry that we can prove

(3.3) **Theorem.** *Suppose* (3.1) *holds with* Z_1 *nonsymmetric with respect to* $\mathscr{H}$. *Then* (3.9) *holds iff there exists a* $\gamma\in\Lambda$ *such that* (3.3) *and* (3.10) *hold for all* t.

Proof. If (3.9) holds then Theorem 3.2 (f), (d) imply that all the Z_t are nonsingular. In such a case, Theorem 2.4 applies and (3.3) and (3.10) follow; $\gamma\in\Lambda$ follows by Theorem 3.1 since (3.3) holds.

The converse follows easily by using the identity (3.2). □

4. The Possible Forms of γ

The fact that $\gamma_n(t)=h_n h_{[nt]}^{-1}\to\gamma(t)$ as a result of $h_n Y_{[nt]}\xrightarrow{D} Z_t$ does not restrict the possible γ.

Fact 1. *Every* $\gamma\in\Lambda$ *is obtained under* (3.9) *and* (3.10) *with every nonsingular* Z_1.

Proof. For given γ and Z_1 we define $Y_n\overset{D}{=}\gamma(n)\,Z_1$ and $h_n=\gamma(n^{-1})$. Then clearly $h_n Y_{[nt]}\xrightarrow{D}\gamma(t)\,Z_1$ and $\gamma_n(t)\to\gamma(t)$ for all t. □

Hence, the study of the possible forms of γ can be done without any reference to (3.1) or (3.9). We shall point out some facts which might be useful in applications. In this section, every γ is in Λ. We assume that a basis for $\mathscr{E}$ is specified and we identify linear transformations with their matrix representations. The notation $\gamma(t)=\langle D_t;\, d_t\rangle$ is now replaced by the matrix notation

$$\gamma(t)=\begin{pmatrix}1 & 0\\ d_t & D_t\end{pmatrix}\qquad (d_t, 0'\in R^k, D_t \text{ is } k\times k). \tag{4.1}$$

Let B be a square matrix. By $\exp\{B\}$ we mean $\sum_{i=0}^{\infty} B^i/i!$

Fact 2. *For each* $\gamma\in\Lambda$ *there exists a* $b\in R^k$ *and a* $k\times k$ *matrix* B *such that*

$$\gamma(t)=\exp\left\{\begin{pmatrix}0 & 0\\ b & B\end{pmatrix}\log t\right\},\qquad D_t=\exp\{B\log t\}. \tag{4.2}$$

Proof. Since both $\gamma(t)$ and D_t are continuous and satisfy Polya's functional equation ($\gamma(ts)=\gamma(t)\,\gamma(s)$, $\gamma(1)=I$) we must have $\gamma(t)=\exp\{C\log t\}$ and $D_t=\exp\{B\log t\}$

for some matrices C and B. The relation between C and B is determined by (4.1) and (4.2) is the result. □

Let ρ ($0 \leqq \rho \leqq k$) be the rank of B. Without loss of generality we can write

$$B = \begin{pmatrix} B_1 & 0 \\ B_2 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix},$$

where B_1 is a regular $\rho \times \rho$ matrix, B_2 is a $(k-\rho) \times \rho$ matrix, $b_1 \in R^\rho$ and $b_2 \in R^{k-\rho}$. By straightforward calculations we thus have

Fact 3.

$$\gamma(t) = \begin{pmatrix} 1 & 0 & 0 \\ c_1 & C_1 & 0 \\ c_2 & C_2 & 0 \end{pmatrix} \tag{4.3}$$

with

$$\begin{aligned} C_1 &= \exp\{B_1 \log t\} \\ C_2 &= B_2 (C_1 - I) B_1^{-1} \equiv B_2 U \\ c_1 &= (C_1 - I) B_1^{-1} b_1 = U b_1 \\ c_2 &= b_2 \log t + B_2 (U - I \log t) B_1^{-1} b_1. \end{aligned} \tag{4.4}$$

Fact 4. *If $\rho = k$ then $x_0 = -B^{-1} b$ is a common fixed point.*

Proof. When $\rho = k$ we have $B = B_1$, $D_t = C_1$ and $d_t = c_1$. Thus by (4.4)

$$\gamma(t)\, x = D_t x + d_t = \exp\{B \log t\}(x - x_0) + x_0 \qquad (x \in \mathscr{E}).\quad \square$$

Suppose that $B = P \Delta P^{-1}$ where Δ is of some desired canonical form (diagonal, Jordan, triangular, etc.). Then, since $B^n = P \Delta^n P^{-1}$ we have

$$\gamma(t) = \begin{pmatrix} 1 & 0 \\ 0 & P \end{pmatrix} \exp\left\{ \begin{pmatrix} 0 & 0 \\ a & \Delta \end{pmatrix} \log t \right\} \begin{pmatrix} 1 & 0 \\ 0 & P^{-1} \end{pmatrix}, \qquad a = P^{-1} b. \tag{4.5}$$

Let $\lambda_1, \ldots, \lambda_k$ be the characteristic roots of B and let $d_{t,i}$ and a_i be the i-th components of d_t and a respectively.

Fact 5. *If B can be diagonalized then (under a suitable coordinate system)*

$$D_t = \begin{pmatrix} t^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & t^{\lambda_k} \end{pmatrix} \qquad \begin{aligned} d_{t,i} &= (t^{\lambda_i} - 1)\, a_i / \lambda_i && \text{if } \lambda_i \neq 0 \\ &= a_i \log t && \text{if } \lambda_i = 0. \end{aligned} \tag{4.6}$$

Proof. If $\Delta = P^{-1} B P$ is diagonal then the λ_i are its diagonal elements (and can be arranged so that $\lambda_i \neq 0$, $i = 1, \ldots, \rho$ and $\lambda_i = 0$, $i = \rho + 1, \ldots, k$). To evaluate

$$\exp\left\{ \begin{pmatrix} 0 & 0 \\ a & \Delta \end{pmatrix} \log t \right\}$$

we apply (4.4) with diagonal B_1 and with $B_2 = 0$. □

Notice that diagonalization of B is possible e.g. when B is symmetric or when all the λ_i are distinct.

When $\mathscr{E}=R^1$, then $B=\lambda_1$ and (4.6) always holds. This case was treated in [6].

Lamperti [5] treats the case where $h_n=\langle\alpha_n I; b_n\rangle, \alpha_n\in R^1, \alpha_n\downarrow 0, b_n\in R^k, h_n Y_{[nt]}\to Z_t$ in the sense of convergence of all the finite dimensional laws and $\{Z_t\}$ is a continuous (in probability) process. In this case (4.6) automatically holds since D_t is of the form $t^\lambda I$.

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