Z. Wahrscheinlichkeitstheorie verw. Geb. 20, 134–142 (1971)

Approximation of Local Time at Regular Boundary Points of a Markov Chain

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The concept of local time, well-known in the theory of diffusions, was extended to arbitrary standard processes by Blumenthal and Getoor [1]. Recent work (e.g., [5]) has shown that Markov chains can be completed in such a way that the resulting processes are sufficiently regular to permit employment of the theory of additive functionals almost as in the standard case. Thus, for example, one may define local time at a "regular" boundary point of a (completed) Markov chain ([8, 9]); these local times can often be expressed in interesting ways, for example as weighted averages of local time at ordinary states or as limits of time spent in excursions.

The purpose of this paper is to exhibit a different characterization of the local time at a regular boundary point x of a Markov chain X_t. Local time at x measures, in some sense, the time spent by the process at x. If x is semipolar, we compute "local time" at x by simply counting up the visits to x; but if x is regular, $X_\bullet(\omega)$ visits x infinitely often in certain finite time intervals, so this simple technique cannot be employed. If one conceives of the regular boundary case as some type of limiting phenomenon of the semipolar case (cf. the remark in the middle of p. 158 of [4]), one is led to approximate local time at a regular boundary point x by "counting visits" to x on random subsets of the time axis. At the nth stage of the approximation, we fix a finite subset J_n of states (where $J_n \subset J_{n+1}$) and for each ω, we divide $[0, \infty)$ into countably many disjoint half-open intervals, numbered consecutively. $X_\bullet = x$ at the left-hand end-point of each of the even-numbered intervals, and $X_\bullet \notin J_n$ throughout; on the odd-numbered intervals, $X_\bullet \in J_n$ at the left-hand end-point and $X_\bullet \neq x$ throughout. For fixed t, we define $K_n(\omega)$ to be the largest integer m such that the Lebesgue measure of the first m odd-numbered intervals is $\leqq t$; $K_n(\omega) \to \infty$ as $n \uparrow \infty$. Multiplying by suitable constants C_n, where $C_n \to 0$ as $n \to \infty$, we find that $C_n \cdot K_n(\omega)$ converges a.s. to local time at x.

I. Assumptions and Notation

1. The Process

Let $\{X_t^*; t>0\}$ be a Markov chain on the integers I with transition matrix $p_{ij}(t)$ $(i, j \in I; t>0)$ satisfying $\lim_{t \to 0} p_{ii}(t) = 1$ for each $i \in I$. Doob [5] has shown how to compactify the state space I to yield a compact metric space K, such that I is dense in K. He also shows that there exists a standard modification X_t of X_t^*, defined for $t \geqq 0$, such that:

(a) With probability one, X_t takes its values and its left limits in a G_δ subset K_0 of K;

(b) X_t has right continuous paths a.s.;

(c) $P(X_t \in I) = 1$ for all $t > 0$.

2. *The Boundary*

The points in $K_0 - I$ will be called *boundary points*. We denote the set of boundary points B. If $x \in B$, x will be called a *regular* boundary point if $P^x(T_x = 0) = 1$, where $T_x = \inf\{t > 0\colon\ X_t = x \text{ or } X_{t-} = x\}$.

Points x for which $P^x(T_x = 0) < 1$ are called *branch points* by Doob; it follows easily from the fact that the set of branch points is negligible ([5], § 8) that $P^x(T_x = 0) = 0$ if x is a branch point.

Since quasi-left continuity holds at regular boundary points ([5], § 7), it follows that $X_{T_{x-}} = x$ implies $X_{T_x} = x$ if x is regular. For regular x we may therefore define

$$T_x = \inf\{t > 0\colon\ X_t = x\}.$$

If x is a regular boundary point, let $S^x(\omega) = \{t > 0\colon\ X_t(\omega) = x\}$. A regular boundary point is said to be *recurrent* if $P^x\{S^x(\omega)$ is an unbounded set$\} = 1$; otherwise it is called *transient*.

3. *Local Time at the Boundary*

A continuous additive functional $A_t(\omega)$ of the process X_t is a *local time* at x if the measure $dA_\bullet(\omega)$ has support on the closure of $S^x(\omega)$ with probability 1, i.e., $A_t(\omega)$ increases only when $X_t(\omega)$ is at x.

The existence of local time for a regular boundary point is shown in [8] and in [9][1]; it is unique up to a constant multiple.

We note two results about $A_t(\omega)$, the local time at x:

(A) If x is transient,

$$E^x(A_t) = P^x \quad [\text{last exit from } x \text{ occurs before time } t]. \tag{1.1}$$

In particular, $E^x(A_\infty) = 1$.

(B) If x is recurrent,

$$A_t(\omega) = \int_0^t e^s\, dB_s(\omega), \tag{1.2}$$

where B_t is an increasing process with $E^x(B_\infty) = 1$. If x is a non-regular boundary point, we define "local time" $K_t(\omega)$ at x to be the number of hits of x by $X_\bullet(\omega)$ on $(0, t]$. (Usually we will normalize $K_t(\omega)$ by a constant multiple for convenience.) $K_t(\omega)$ is then an additive functional, but plainly discontinuous.

[1] Both [8] and [9] employ completions of I differing somewhat from Doob's, but the existence of local time at x for the Doob boundary is clearly implied.

II. The Main Approximation Theorem

Let x be a regular boundary point, and $A_t(\omega)$ the local time at x. If J is a finite subset of I, let T_J denote the first hitting time of J. If $J_n \subset J_{n+1}$ for $n=1, 2, \ldots$, and $\bigcup_n J_n = I$, we shall write $J_n \uparrow I$.

It is convenient to consider separately the cases when x is recurrent and transient.

1. The Recurrent Case

Assume now that x is a regular recurrent boundary point.

Proposition 1. *$E^x(A_{T_J}) \to 0$ as $J \uparrow$ I.*

Proof. Let $\varphi_J(t, \omega) = \sum_{j \in J} \int_0^t 1_{[X_s = j]} ds$. Define $\psi_J(t, \omega) = A_t(\omega) + \varphi_J(t, \omega)$. Then $\psi_J(t, \omega)$ is a continuous additive functional, and if τ_t^J denotes its inverse, $X_{\tau_t^J}$ is a Markov chain with state space $J \cup \{x\}$. If $X_{\tau_t^J}$ is started at x, A_{T_J} is just the waiting time until the first jump. If $E^x(A_{T_J}) = \infty$, then x is an absorbing state for the process $X_{\tau_t^J}$, in which case $A_{T_J} = \infty$ a.s. But by remark B of I, § 2, $A_t(\omega) = \int_0^t e^s dB_s(\omega)$ where $B(\infty) < \infty$ a.s.; so that $T_J = \infty$ a.s. which is impossible for J large enough. Hence, $E^x(A_{T_J}) < \infty$ for large J. Since it is clear that $\lim_n T_{J_n} = 0$, and since $A_\bullet(\omega)$ is continuous, it follows that $E^x(A_{T_J}) \to 0$ as $J \uparrow I$.

Corollary. *A_{T_J} is exponentially distributed and independent of X_{T_J}.*

Proof. A_{T_J} is the waiting time until the first jump of $X_{\tau_t^J}$; it is therefore exponential and independent of the state jumped into, namely X_{T_J}.

For each finite subset J of I, define two sequences of stopping times as follows:

$$\begin{aligned}
&\tau_{J,0} = 0 && S_{J,0} = T_x \\
&\tau_{J,1} = T_J \circ \theta_{S_{J,0}} && S_{J,1} = T_x \circ \theta_{\tau_{J,1}} \\
&\quad\vdots && \quad\vdots \\
&\tau_{J,n} = T_J \circ \theta_{S_{J,n-1}} && S_{J,n} = T_x \circ \theta_{\tau_{J,n}}.
\end{aligned}$$

Definition 1. $K_J(t, \omega)$ (or simply $K_{J,t}$) $= \sup \left\{ n \colon \sum_{k=0}^{n-1} (S_{J,k} - \tau_{J,k}) \leqq t \right\}$.

Definition 2. $A_J(t, \omega) = E^x(A_{T_J})\, K_J(t, \omega)$.

(If we think of the process $X_t(\omega)$ restricted to the intervals $(\tau_{J,k}, S_{J,k}]$, $k = 0, 1, 2, \ldots$, then $K_J(t, \omega)$ is precisely the number of hits of x by this modified process in t units of time. In this sense $A_J(t, \omega)$ can be thought of as a (non-regular) "local time".)

Theorem 1. *Let $J_n \uparrow I$. Then there exists a set Ω_0, with $P^x(\Omega_0) = 1$, such that for $\omega \in \Omega_0$, $A_{J_n}(t, \omega) \to A_t(\omega)$ for each $t \geqq 0$.*

Proof. We first prove a useful lemma. (Henceforth, we assume $X(\bullet, \omega)$ to be started at x, so that we write T_J for $\tau_{J,1}$).

Lemma 1. *Adjoin an isolated point Δ to K and let*

$$Y_s^i = \begin{cases} X_{\tau_{J,i}+s} & \text{if } \tau_{J,i}+s<S_{J,i} \\ \Delta & \text{if } \tau_{J,i}+s \geqq S_{J,i}. \end{cases}$$

Let $\mathscr{H}$ denote the smallest σ-field generated by the random variables $\{Y_s^i\}_{i=0}^\infty$, s real. Denote by $\mathscr{I}$ the smallest σ-field generated by $\{A_{T_J} \circ \theta_{S_{J,k}}\}_{k=0}^\infty$. Then if X and Y are real-valued functions on Ω with $X \in \mathscr{H}$, $Y \in \mathscr{I}$,

$$E^x\{XY\} = E^x\{X\}\, E^x\{Y\}.$$

In particular, for any positive integer m and any $t>0$, $\sum_{k=1}^m A_{T_J} \circ \theta_{S_{J,k-1}}$ is independent of $K_{J,t}$.

Proof. Since X_t is right continuous and K a compact separable metric space the random variables $\{Y_s^i\}_{i=0}^\infty$, s rational, will generate $\mathscr{H}$. Let $\{k_i\}_{i=1}^\infty$ and $\{l_i\}_{i=1}^\infty$ be arbitrary sequences of integers, $\{s_j\}_{j=1}^\infty$ an arbitrary sequence of rationals. By a standard monotone class theorem ([2], p. 6) it will suffice to prove that

$$P^x\left\{\bigcap_{i=1}^m \{A_{T_J} \circ \theta_{S_{J,k_i}} \leqq x_i\} \cap \bigcap_{j=1}^p \bigcap_{i=1}^n \{Y_{s_j}^{l_i} \in B_{ij}\}\right\}$$
$$= P^x\left\{\bigcap_{i=1}^m \{A_{T_J} \circ \theta_{S_{J,k_i}} \leqq x_i\}\right\} P^x\left\{\bigcap_{j=1}^p \bigcap_{i=1}^n \{Y_{s_j}^{l_i} \in B_{ij}\}\right\},$$

where B_{ij} is a Borel set in $K \cup \Delta$. Since the case when one or more of the B_{ij} contains Δ presents no new difficulties, let us assume that all $B_{ij} \subset K$, so that the left-hand side above becomes

$$P^x\left\{\bigcap_{i=1}^m \{A_{T_J} \circ \theta_{S_{J,k_i}} \leqq x_i\} \bigcap_{i=1}^p \bigcap_{i=1}^n \{X_{\tau_{J,l_i}+s_j} \in B_{ij};\ \tau_{J,l_i}+s_j < S_{J,l_i}\}\right\}.$$

By repeated application of the strong Markov Property at the times $S_{J,j}$, it will suffice to prove that

$$\begin{aligned} &P^x\{A_{T_J} \circ \theta_{S_{J,k}} \leqq y;\ X_{\tau_{J,k+1}+s_j} \in B_{k+1,j};\ T_x \circ \theta_{\tau_{J,k+1}} > s_j\} \\ &\quad = P^x\{A_{T_J} \circ \theta_{S_{J,k}} \leqq y\}\, P^x\{X_{\tau_{J,k+1}+s_j} \in B_{k+1,j};\ T_x \circ \theta_{\tau_{J,k+1}} > s_j\}. \end{aligned} \tag{$*$}$$

Conditioning the left-hand side above by $\mathscr{F}_{S_{J,k}}$ yields

$$P^x\{A_{T_J} \leqq y;\ X_{T_J+s_j} \in B_{k+1,j};\ T_x \circ \theta_{T_J} > s_j\}.$$

Conditioning now on $\mathscr{F}_{T_J}$ and using the corollary to Proposition 1, this becomes

$$\begin{aligned} &\sum_{j \in J} P^x\{A_{T_J} \leqq y;\ X_{T_J}=j;\ P^j[X_{s_j} \in B_{k+1,j};\ s_j < T_x]\} \\ &\quad = \sum_{j \in J} P^x[A_{T_J} \leqq y]\, P^x[X_{T_J}=j;\ P^j\{X_{s_j} \in B_{k+1,j};\ s_j < T_x\}] \\ &\quad = P^x[A_{T_J} \leqq y]\, P^x[X_{T_J+s_j} \in B_{k+1,j};\ T_x \circ \theta_{T_J} > s_j], \end{aligned}$$

which equals the right-hand side of ($*$), by another application of the strong Markov property.

Definition 3. $\sigma_J(t,\omega)=$ Lebesgue measure $\left\{[0,t]\cap\bigcup_{k=0}^{\infty}[\tau_{J,k},S_{J,k})\right\}$

$$\Delta_J(t,\omega)=t-\sigma_J(t,\omega).$$

Lemma 2. *$\Delta_J(t,\omega)\to 0$ a.s. as $J\uparrow I$.*

Proof. $\Delta_J(t,\omega)+\sigma_J(t,\omega)=t$. Since $\sigma_J(t,\omega)\geqq\int_0^t 1_{[X_s\in J]}\,ds$ and the latter expression increases to t as $J\uparrow I$, it follows that $\Delta_J(t,\omega)\to 0$ a.s. as $J\uparrow I$.

Definition 4. $\beta_J(t,\omega)$ (or $\beta_{J,t}$) $=\inf\{s>0:\ \sigma_J(s,\omega)>t\}$.

By construction of $\beta_{J,t}$, it follows that

$$\sum_{k=1}^{K_{J,t}} A_{T_J}\circ\theta_{S_{J,k-1}}=A(\beta_{J,t}) \tag{1.3}$$

(where we have used the fact, established in [8], that $A_{T+R}=A_T+A_R\circ\theta_T$ for T optional, R measurable). Since $\sigma_{J,t}\uparrow t$ as $J\uparrow I$ by Lemma 2, we have $\beta_{J,t}\downarrow t$ as $J\uparrow I$. Hence by the continuity of $A_\bullet(\omega)$, it follows from (1.3) that

$$\sum_{k=1}^{K_{J,t}} A_{T_J}\circ\theta_{S_{J,k-1}}\xrightarrow{J\uparrow I} A_t(\omega)\quad\text{a.s.} \tag{1.4}$$

Hence if we can show that

$$\sum_{k=1}^{K_{J,t}} A_{T_J}\circ\theta_{S_{J,k-1}}-A_J(t,\omega)\xrightarrow{J\uparrow I} 0\quad\text{a.s.},$$

the proof will be complete.

We first show that

$$\sum_{k=1}^{K_{J,t}} A_{T_J}\circ\theta_{S_{J,k-1}}-A_{J,t}\xrightarrow{J\uparrow I} 0\qquad\text{in } L^2(P^x).$$

We have

$$E^x\left[\sum_{1}^{K_{J,t}} A_{T_J}\circ\theta_{S_{J,k-1}}-K_J(t,\omega)\,E^x(A_{T_J})\right]^2=E^x\left[\sum_{1}^{K_{J,t}}(A_{T_J}\circ\theta_{S_{J,k-1}}-E^x(A_{T_J}))\right]^2.$$

By an obvious use of the strong Markov property, the $A_{T_J}\circ\theta_{S_{J,k}}$ are independent, identically distributed random variables. Hence, if Y_k denotes $A_{T_J}\circ\theta_{S_{J,k-1}}-E^x(A_{T_J})$, we have

$$\begin{aligned}E^x\left(\sum_{1}^{K_{J,t}} Y_k\right)^2&=\sum_{n=1}^{\infty}E^x\left(\left(\sum_{1}^{n} Y_k\right)^2;\ K_J(t,\omega)=n\right)\\&=\sum_{n=1}^{\infty}E^x\left(\sum_{1}^{n} Y_k\right)^2 P^x\{K_{J,t}=n\}\qquad\text{(by Lemma 1)}\\&=\sum_{n=1}^{\infty}n\,E^x(Y_k)^2\,P^x\{K_{J,t}=n\}=\text{Variance}^x(A_{T_J})\cdot E^x(K_{J,t}).\end{aligned} \tag{1.5}$$

Since A_{T_J} is exponentially distributed, $\mathrm{Var}^x(A_{T_J}) = [E^x(A_{T_J})]^2$ so that

$$E^x\left[\sum_1^{K_{J,t}} A_{T_J}\circ\theta_{S_{J,k-1}} - A_{J,t}\right]^2 = E^x(A_{J,t})\cdot E^x(A_{T_J}). \tag{1.6}$$

But

$$E^x(A_{J,t}) = E^x(A_{T_J})\, E^x(K_{J,t}) = E^x\left(\sum_1^{K_{J,t}} A_{T_J}\circ\theta_{S_{J,k-1}}\right)$$

by Wald's equation (if $E^x(K_{J,t})<\infty$) and the latter expression is just $E^x(A_{\beta_{J,t}})$. Since $A_{\beta_{J,t}}$ decreases as $J\uparrow I$, it will follow from (1.6) and Proposition 1 that

$$E^x\left\{\sum_1^{K_{J,t}} A_{T_J}\circ\theta_{S_{J,k-1}} - A_{J,t}\right\}^2 \to 0$$

as $J\uparrow I$, provided $E^x(A_{J,t})<\infty$ for large enough J, i.e., provided $E^x(K_{J,t})<\infty$. It is easily seen using the strong Markov property that $P^x\{K_{J,t}\geqq n\}\leqq(1-p_t)^{n-1}$, where $p_t = P^x\{T_x\circ\theta_{T_J}>t\}$; we show $p_t>0$.

We can find a neighborhood N_j of each $j\in J$, and a neighborhood N of x, such that $N_j\cap N=\emptyset$ for all $j\in J$. Setting $H=N\cap I$, $p_t=0$ would imply ${}_Hp_{ji}(t+\varepsilon)=0$ for all $i\in I$, $\varepsilon>0$, and some $j\in J$, where ${}_Hp_{ji}$ is the transition probability with H taboo ([3], p. 188). By Theorem 4, p. 126 of [3], this implies ${}_Hp_{jj}(t)\equiv 0$; this cannot be, since $N_j\cap N=\emptyset$ and $\lim_{t\to 0} p_{jj}(t)=1$. Hence $E^x(K_{J,t})<\infty$, and it follows that

$$\sum_{k=1}^{K_{J,t}} A_{T_J}\circ\theta_{S_{J,k-1}} - A_{J,t} \xrightarrow{J\uparrow I} 0 \quad \text{in } L^2(P^x).$$

In particular, if $J_n\uparrow I$, there is a subsequence $\{n_k\}_{k=1}^\infty$ on which $A_{J_{n_k}} \xrightarrow{k\uparrow\infty} A_t$ a.s.

From the L^2 convergence we will now deduce a.s. convergence. By Lemma 2, we have $\beta_{J,t}\downarrow t$ as $J\uparrow I$; it then follows by the continuity of $A_\bullet(\omega)$ that

$$\sum_1^{K_{n,t}} A_{T_n}\circ\theta_{S_{n,k-1}} = A(\beta_{n,t}(\omega),\omega) \xrightarrow{n\uparrow\infty} A_t(\omega) \quad \text{a.s.} \tag{1.7}$$

(where we have written $K_{n,t}$, $\beta_{n,t}$, A_{T_n} for $K_{J_n,t}$, $\beta_{J_n,t}$, $A_{T_{J_n}}$ resp.).

Let G_n denote the Borel field generated by the random variable $K_{n,t}$; let $\bigvee_{i=1}^n G_i$ be the smallest Borel field containing $G_1, G_2, \ldots, G_n$.

Lemma 3. $E\left(A(\beta_{n,t}(\omega),\omega)\,\middle|\,\bigvee_{i=1}^n G_i\right) = A_{J_n}(t,\omega)$.

Proof. The σ-field $\bigvee_{i=1}^n G_i$ is generated by the countable partition

$$\Lambda_{k_1,k_2,\ldots,k_n} \quad \text{where} \quad \Lambda_{k_1,k_2,\ldots,k_n} = \{\omega\colon K_{1,t}=k_2;\ K_{2,t}=k_2;\ \ldots;\ K_{n,t}=k_n\}.$$

By definition,

$$E\left\{A(\beta_{n,t}(\omega),\omega)\,\middle|\,\bigvee_{i=1}^n G_i\right\}(\omega) = \sum \frac{1}{P(\Lambda_{k_1,k_2,\ldots,k_n})}\left(\int_{\Lambda_{k_1,k_2,\ldots,k_n}}\left[\sum_1^{K_{n,t}} A_{T_n}\circ\theta_{S_{n,k-1}}\right]dP^x\right) 1_{\Lambda_{k_1,k_2,\ldots,k_n}}(\omega)$$

(where we have written $S_{n,k-1}$ for $S_{J_{n,k-1}}$), the sum being taken over all n-tuples $(k_1, k_2, \ldots, k_n)$ of positive integers. Consider

$$E^x\left\{\sum_1^{k_n} A_{T_n}\circ\theta_{S_{n,k-1}};\ K_{1,t}=k_1;\ K_{2,t}=k_2;\ \ldots;\ K_{n,t}=k_n\right\}.$$

Lemma 1 assures us that any function measurable in the field $\mathscr{H}$ there constructed is P^x-independent of $\sum_1^{k_n} A_T\circ\theta_{S_{n,k-1}}$. If the sets $\{S_{J_i,k}-\tau_{J_i,k}\leqq y\}$ are $\mathscr{H}$-measurable for $i=1,2,\ldots,n-1$, k arbitrary, then it will follow that the sets $\{K_{i,t}=k_i\}$ are $\mathscr{H}$-measurable for $i=1,2,\ldots,n-1$. This can easily be shown by an argument which is as straightforward as it is tedious.

Therefore,

$$E^x\sum_1^{K_{n,t}}\{A_{T_n}\circ\theta_{S_{n,k-1}};\ \Lambda_{k_1,k_2,\ldots,k_n}\}=k_n E^x(A_{T_n})\,P^x\{\Lambda_{k_1,k_2,\ldots,k_n}\}$$

so that

$$E\left\{A(\beta_{n,t}(\omega),\omega)\,\middle|\,\bigvee_{i=1}^{n} G_i\right\}(\omega)=\sum k_n E^x(A_{T_n})\,1_{\Lambda_{k_1,k_n,\,-k_2}}(\omega) \tag{1.8}$$

where again the last sum is over all n-tuples $(k_1, k_2, \ldots, k_n)$ of positive integers.

But

$$\sum k_n E^x(A_{T_n})\,1_{\Lambda_{k_1,k_2,\,-k_n}}(\omega)=\sum_{k=1}^{\infty} k E^x(A_{T_n})\,1_{[K_{n,t}=k]}(\omega)=A_{J_n}(t,\omega),$$

proving the lemma.

Now by Lemma 3 and a lemma of Hunt ([6], p. 47) it follows that as $n\uparrow\infty$,

$$A_{J_n}(t,\omega)=E\left[A(\beta_{n,t}(\omega),\omega)\,\middle|\,\bigvee_{i=1}^{n} G_i\right]\to E\left[A_t(\omega)\,\middle|\,\bigvee_{i=1}^{\infty} G_i\right]\quad \text{a.s.}$$

But we have already proved that $A_{J_{n_k}}(t,\omega)\to A_t(\omega)$ a.s. for some subsequence $\{n_k\}_{k=1}^{\infty}$, so it follows that $A_t(\omega)$ is measurable in the field $\bigvee_{i=1}^{\infty} G_i$. We therefore conclude that $A_{J_n}(t,\omega)\to A_t(\omega)$ a.s. Denoting by Ω_t the set of probability one on which $A_{J_n}(t,\omega)\to A_t(\omega)$, let $\Omega_0=\bigcap_{t\,\text{rat}}\Omega_t$. For $\omega\in\Omega_0$, $A_{J_n}(r,\omega)\to A_r(\omega)$ for all rational r. Since $A_\bullet(\omega)$ is continuous and $A_J(\bullet,\omega)$ non-decreasing, the property extends by standard arguments to all $t\geqq 0$. Hence for $\omega\in\Omega_0$, $A_{J_n}(t,\omega)\to A_t(\omega)$ for all $t\geqq 0$, and Theorem 1 is proved.

2. *The Transient Case*

The transient case presents no new difficulties; indeed, it is simpler in most respects. If x is transient, then by (1.3) of Part I, $E^x(A_{T_J})$ has a convenient interpretation as $P^x\{X_\bullet\circ\theta_{T_J}$ does not hit $x\}$. If x is transient, however, one of the $S_{J,k}$ may be infinite with positive probability, since x need not be hit from J. In this case $A_t(\omega)$ will simply be "flat" from $\tau_{J,k}(\omega)$ on.

3. *Approximation in* $\mathcal{F}_t$

It may be objected that the above approximation of $A_t(\omega)$ makes use of information not contained in the field $\mathcal{F}_t$, and that ideally $A_t(\omega)$ should be approximated within $\mathcal{F}_t$, since $A_t \in \mathcal{F}_t$. We now indicate how this may be done.

Definition 5. $N_J(t,\omega) = \sup\{n: \tau_{J,n}(\omega) \leqq t\}$.

Lemma 4. $K_J(t,\omega) - N_J(t,\omega) \to 0$ *a.s.* P^x *as* $J \uparrow I$.

Proof. $P^x\{K_{J_n}(t,\omega) - N_{J_n}(t,\omega) > 0 \text{ i.o.}\} \leqq P^x\{X(\bullet,\omega) \text{ hits } x \text{ in } (t, \beta_{J_n,t}] \text{ i.o.}\}$. As $J_n \uparrow I$, by Lemma 2 $\sigma_{J_n}(t,\omega) \uparrow t$ a.s. so $\beta_{J_n}(t,\omega) \downarrow t$ a.s. Hence, if $X(\bullet,\omega)$ hits x in $(t, \beta_{J_n}(t,\omega))$ infinitely often, we must have $X_t(\omega) = x$. Therefore, $P^x\{K_{J_n}(t,\omega) - N_{J_n}(t,\omega) > 0 \text{ i.o.}\} \leqq P^x\{X_t = x\} = 0$, so that $K_{J_n}(t,\omega) - N_{J_n}(t,\omega)$ decreases to 0 a.s. From Theorem 1 we conclude that

$$N_J(t,\omega) E^x(A_{T_J}) \xrightarrow{J\uparrow I} A_t(\omega) \quad \text{a.s.},$$

and clearly $N_J(t,\omega) \in \mathcal{F}_t$.

III. Inverse Local Time as a Limit of Compound Poissons

Let us denote by α_t the inverse local time of A, i.e., $\alpha_t(\omega) = \inf\{s > 0: A_s(\omega) > t\}$. α_t is known to be infinitely divisible ([2], p. 218). We can now exhibit a sequence of compound Poisson random variables which converge a.s. to α_t on $\{\alpha_t < \infty\}$.

Fix $t > 0$.

Definition 1. $W_J(t,\omega) = W_{J,t}(\omega) = \sup\left\{n: \sum_{k=1}^{n} A_{T_J} \circ \theta_{S_{J,k-1}} \leqq t\right\}$. Since A_{T_J} is exponential, $W_J(t,\omega)$ is Poisson with mean $t/E^x(A_{T_J})$.

Theorem 2. $\sum_{k=1}^{W_{J,t}} T_x \circ \theta_{\tau_{J,k}} \xrightarrow{J\uparrow I} \alpha_t(\omega)$ *a.s. on* $\{\alpha_t < \infty\}$.

Proof.

$$\alpha_t - \left(\sum_{k=1}^{W_{J,t}} T_x \circ \theta_{\tau_{J,k}} + \sum_{k=1}^{W_{J,t}} T_J \circ \theta_{S_{J,k-1}}\right) \leqq T_J \circ \theta_{S_{J,m}}$$

where $m = W_J(t,\omega)$. But

$$T_J \circ \theta_{S_{J,m}} \xrightarrow{J\uparrow I} 0 \text{ a.s.} \quad \text{on } \{\alpha_t < \infty\}.$$

We show that

$$\sum_{k=1}^{W_{J,t}} T_J \circ \theta_{S_{J,k-1}} \xrightarrow{J\uparrow I} 0 \text{ a.s.} \quad \text{on } \{\alpha_t < \infty\}.$$

$$\begin{aligned}\sum_{k=1}^{W_{J,t}} T_J \circ \theta_{S_{J,k-1}} &\leqq \sum_{N=0}^{\infty} \left(\sum_{k=1}^{W_{J,t}} T_J \circ \theta_{S_{J,k-1}}\right) 1_{[N \leqq \alpha_t < N+1]} \\ &\leqq \sum_{N=0}^{\infty} \Delta_J(N+1,\omega) 1_{[N \leqq \alpha_t < N+1]} \xrightarrow{J\uparrow I} 0 \text{ a.s.} \quad \text{on } \{\alpha_t < \infty\},\end{aligned}$$

by Lemma 2 and the fact that $\Delta_{J_n}(N+1,\omega)$ decreases as $n\to\infty$, for all $N\geqq 0$. Hence

$$\sum_{1}^{W_{J,t}} T_x\circ\theta_{\tau_{J,k}} \xrightarrow{J\uparrow I} \alpha_t \quad \text{a.s. on } \{\alpha_t<\infty\}.$$

Now by Lemma 1 it follows that each $T_x\circ\theta_{\tau_{J,k}}$ is independent of each $A_{T_J}\circ\theta_{S_{J,i-1}}$ and that $W_{J,t}(\omega)$ is independent of the $T_x\circ\theta_{\tau_{J,k}}$; and clearly the $T_x\circ\theta_{\tau_{J,k}}$ are independent and identically distributed. Hence $\sum_{1}^{W_{J,t}} T_x\circ\theta_{\tau_{J,k}}$ is a compound Poisson random variable and converges to $\alpha_t(\omega)$ a.s. on $\{\alpha_t<\infty\}$ as $J\uparrow I$.

Acknowledgement. The results of this paper form part of the author's doctoral dissertation, written at Stanford University under the direction of K. L. Chung. The author expresses his thanks to Professor Chung for his advice and encouragement, and to Professor John Walsh for many helpful discussions.

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(Received August 15, 1970)