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Complete Probabilistic Metric Spaces*

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1. Introduction

Menger [4] initiated the study of probabilistic metric spaces in 1942. A probabilistic metric space (briefly a PM space) is a space in which the "distance" between any two points is a probability distribution function. These spaces are assumed to satisfy axioms which are quite similar to the axioms satisfied in an ordinary metric space. The triangle inequality has been the subject of some controversy. Menger's triangle inequality was first challenged by Wald [16] who suggested an elegant form of the triangle inequality to replace Menger's. In [7] Schweizer and Sklar provided sufficient reason to indicate that Wald's inequality was restrictive and returned to Menger's triangle inequality which they modified slightly. The subject then began to grow rapidly due to the work of Schweizer, Sklar, Thorp and others. In [10, 11] Serstnev introduced yet another form of the triangle inequality. His formulation includes Wald's and Menger's formulation but for a slight exception which at present seems to be uninteresting. For this reason we shall state and prove our results in the setting of Serstnev whenever it is appropriate.

In Section 2 we shall do two things. First we shall briefly take note of the fact that the principal result of our paper on completions of PM spaces can be obtained under weaker hypotheses. Then we shall answer the following consistency questions: (1) Is a complete metric space obtained when a complete PM space is metrized? and (2) If a PM space and its completion are metrized, will the completions of the resulting metric spaces be isometric?

In Section 3 contraction maps on PM spaces will be investigated. A very natural definition of a contraction map was introduced by Sehgal [9]. In that paper he showed that every contraction map on a complete PM space satisfying the strongest form of Menger's triangle inequality has a unique fixed point. We shall give a strong plausibility argument which will indicate that this result is the exception rather than the rule for PM spaces. We shall prove that it is possible to construct complete PM spaces together with contraction maps which have no fixed points. In fact this will be done so that any of a very large class of triangle inequalities is satisfied. Finally we shall utilize the additional structure of an E-space to define a stronger contraction map which will have a fixed point whenever the space on which it is defined is complete.

In the last section the analogues for complete PM spaces of two other classical theorems for complete metric spaces will be proved.

A few definitions and conventions will be made here to fill in some background for the reader.

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We shall let I denote the closed unit interval $[0, 1]$. The letter Δ will denote the collection of nondecreasing, left-continuous functions F such that $F(0)=0$ and the range of F is a subset of I. The letter H will be reserved for the function defined via

$$H(x)=\begin{cases}0, & x\leqq 0\\ 1, & x>0.\end{cases}$$

The letter j will denote the identity function on the reals so that $j(x)=x$ for every real x.

A *triangular norm* (briefly, a *t-norm*) is a function T mapping $I\times I$ into I which is associative, commutative, non-decreasing in each place, and satisfies $T(a, 1)=a$ for each $a\in I$. A t-norm will be called an *l.c. t-norm* if it is left-continuous in each place. Some t-norms which will be of importance to us are the t-norms T_m, Prod and Min defined for each $a, b\in I$ via

$$T_m(a, b)=\text{Maximum}(a+b-1, 0),$$

$$\text{Prod}(a, b)=ab,$$

and

$$\text{Min}(a, b)=\text{Minimum}(a, b).$$

Let Δ be ordered via $F\leqq G$ if and only if $F(x)\leqq G(x)$ for all real x. Also $F<G$ if and only if $F\leqq G$ but $F\neq G$. A *triangle function* (briefly, a *t-function*) is a function τ mapping $\Delta\times\Delta$ into Δ which is associative, commutative, nondecreasing in each place and satisfies $\tau(F, H)=F$ for each $F\in\Delta$. In addition, throughout this paper we shall assume that all of our t-functions satisfy the following condition

$$\sup\{\tau(F, F)\colon F<H\}=H. \tag{1.1}$$

A t-function τ will be called an *l.c. t-function* if it is continuous in each place relative to nondecreasing sequences.

If T is an l.c. t-norm then the function τ defined via

$$\tau(F, G)(x)=\sup\{T(F(\alpha x), G(\beta x))\colon \alpha+\beta=1\} \tag{1.2}$$

for all real x and for all $F, G\in\Delta$ is an l.c. t-function [10].

A *probabilistic metric space* (briefly, a *PM space*) is an ordered pair $(S, \mathscr{F})$ where S is an abstract set and $\mathscr{F}$ is a mapping from $S\times S$ into Δ whose value $\mathscr{F}(p, q)$ at any pair $(p, q)\in S\times S$ is usually denoted by F_{pq} and assumed to satisfy

I. $\lim_{x\to+\infty} F_{pq}(x)=1$ for all $p, q\in S$,

II. For all $p, q\in S$, $F_{pq}=H$ if and only if $p=q$, and

III. $F_{pq}=F_{qp}$ for all $p, q\in S$, and either for some t-norm T

IVm. $F_{pr}(x+y)\geqq T(F_{pq}(x), F_{qr}(y))$ for all $p, q, r\in S$ and all $x, y>0$, or for some t-function τ

IVs. $F_{pr}\geqq\tau(F_{pq}, F_{qr})$ for all $p, q, r\in S$.

It should be noted that if T is an l.c. t-norm and τ is defined via (1.2), then $(S, \mathscr{F})$ satisfies IVm under T if and only if it satisfies IVs under τ. The inequality given in IVm was suggested by Menger, the one in IVs by Serstnev.

2. Completions of PM Spaces

In [5] Mushtari discusses the completion of PM spaces. However in his treatment he does not define complete PM space nor the completion of a PM space. He considers a certain uniform structure which arises quite naturally on the PM space and defines a probabilistic metric on the uniform completion which is an extension of the probabilistic metric on the original space. He further guarantees that the extended probabilistic metric defines a uniform structure which coincides with the extension of the original structure. He carries out this procedure for two uniformities which turn out to be identical if sufficient conditions are imposed on the triangle inequality.

Now whether a PM space is complete should be determined by the probabilistic metric alone and should not depend on which uniform structure one induces on that space. For this reason we have defined completeness of PM spaces in terms of the probabilistic metric. Then we prove under certain conditions that this notion of completeness is consistent with the previously existing notions in uniform spaces.

In [12] we defined the concepts mentioned above in terms of the PM space axioms. We shall continue to use these definitions. In that paper we showed that every PM space which satisfies IVm under a continuous t-norm has a completion which is unique up to an isometry. The next theorem includes this earlier result.

In [14] Sibley introduced a metric $\mathscr{L}$ on Δ which is a modified form of the well-known Lévy metric for distribution functions. He shows that $(\Delta, \mathscr{L})$ is a compact metric space in which convergence is equivalent to weak convergence of the functions in Δ. It is not difficult to prove that for the l.c. t-function τ and for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $\mathscr{L}(\tau(F, G), G) < \varepsilon$ for every $G \in \Delta$ whenever $F \in \Delta$ and $\mathscr{L}(F, H) < \delta$. Having noted this fact, the proof of the following theorem is almost identical to the proof of the earlier result given in [12]. For this reason we omit its proof.

Theorem 2.1. *Any PM space $(S, \mathscr{F})$ satisfying* IVs *under an l.c. t-function τ has a completion which is unique up to isometry.*

Since whenever T is an l.c. t-norm and τ is defined via (1.2) then τ is an l.c. t-function we obtain the following corollary.

Corollary. *Any PM space satisfying* IVm *under an l.c. t-norm has a completion which is unique up to isometry.*

Now we shall investigate the consistency of PM space completeness and metric completeness. Let $(S, \mathscr{F})$ be a PM space satisfying IVs under a t-function τ satisfying (1.1). For each $\varepsilon, \lambda > 0$, let

$$U(\varepsilon, \lambda) = \{(p, q) \colon p, q \in S \text{ and } F_{pq}(\varepsilon) > 1 - \lambda\}.$$

The collection

$$\mathscr{B} = \{U(\varepsilon, \lambda) \colon \varepsilon > 0 \text{ and } \lambda > 0\}$$

is a basis [6, 8] for a Hausdorff uniformity $\mathscr{U}$ on $S \times S$ which we call the uniformity on S generated by $\mathscr{F}$. This uniformity is metrizable since it is Hausdorff and has a countable base.

It is well-known (see e.g. [2]) that if (S, d) is a metric space, then the one-parameter family of subsets

$$\mathscr{C} = \{V(\delta): \delta > 0\}$$

where

$$V(\delta) = \{(p, q): p, q \in S \text{ and } d(p, q) < \delta\}$$

is a basis for a uniformity $\mathscr{V}$ which we call the uniformity on S generated by d.

Definition 2.1. Let $(S, \mathscr{F})$ be a PM space satisfying IVs under an l.c. t-function τ. The metric d for S *metrizes* $(S, \mathscr{F})$ if and only if the uniformity on S generated by $\mathscr{F}$ is the same as the uniformity on S generated by d.

In the interest of saving space, the proofs of the remaining theorems in this section are also omitted. These proofs are not difficult to reproduce.

Theorem 2.2. *Let $(S, \mathscr{F})$ be a complete PM space satisfying* IVs *under an l.c. t-function τ. Let $\mathscr{U}$ be the uniformity on S generated by $\mathscr{F}$. Then $(S, \mathscr{U})$ is a complete uniform space.*

Theorem 2.3. *Let $(S, \mathscr{F})$ be a complete PM space satisfying* IVs *under an l.c. t-function τ. Let d be a metric which metrizes $(S, \mathscr{F})$. Then (S, d) is a complete metric space.*

Theorem 2.4. *Let $(S, \mathscr{F})$ be a PM space satisfying* IVs *under l.c. t-function τ. Let $(S^*, \mathscr{F}^*, \tau)$ be the completion of $(S, \mathscr{F}, \tau)$; let $\mathscr{U}$ be the uniformity on S generated by $\mathscr{F}$; and let $\mathscr{U}^*$ be the uniformity on S^* generated by $\mathscr{F}^*$. Then $(S^*, \mathscr{U}^*)$ is the completion of $(S, \mathscr{U})$.*

Let $(S, \mathscr{F})$ be a PM space satisfying IVs under an l.c. t-function τ. Since $(S, \mathscr{F})$ is metrizable in the sense of Definition 2.1, let d be a metric on S which metrizes $(S, \mathscr{F})$. Now (S, d) has a completion (S', d'). Furthermore $(S, \mathscr{F})$ also has a completion $(S^*, \mathscr{F}^*)$ which is metrizable by a metric d^*. From Theorem 2.3 it follows that the metric space (S^*, d^*) is complete. However, since there are many metrics which will metrize a given metrizable uniform space, the spaces (S^*, d^*) and (S', d') need not be isometric. The following theorem provides the best result which can be expected along these lines.

Theorem 2.5. *Let $(S, \mathscr{F})$ be a PM space satisfying* IVs *under an l.c. t-function τ. Let $(S^*, \mathscr{F}^*)$ be the completion of $(S, \mathscr{F})$; let the metric d on S metrize $(S, \mathscr{F})$, and let (S', d') be the completion of (S, d). Then there is a metric d^* on S^* which metrizes $(S^*, \mathscr{F}^*)$ such that (S^*, d^*) and (S', d') are isometric.*

3. Contraction Maps on Complete PM Spaces

The following definition of a contraction map was suggested and studied by Sehgal in [9] where he also proved Theorem 3.1.

Definition 3.1. Let $(S, \mathscr{F})$ be a PM space. A mapping $M: S \to S$ is a *contraction map* on $(S, \mathscr{F})$ if and only if there is an $\alpha \in (0, 1)$ such that

$$F_{MpMq}(x) \geqq F_{pq}(x/\alpha)$$

for every $p, q \in S$.

Theorem 3.1. *A contraction map on a PM space has at most one fixed point.*

Definition 3.2. Let M be a contraction map on the PM space $(S, \mathscr{F})$. Let $p_0 \in S$.

(i) The *sequence of iterates* of p_0 under M is the sequence $\{p_n\}$ defined inductively via $p_n = M p_{n-1}$ for every positive integer n.

(ii) The function G_{p_0} is defined via

$$G_{p_0}(x) = \inf\{F_{p_0 p_m}(x) \colon m \text{ is a positive integer}\}$$

for each real x where $\{p_n\}$ is the sequence of iterates of p_0 under M.

Theorem 3.2. *Let $(S, \mathscr{F})$ be a complete PM space satisfying* IVs *under a triangle function τ satisfying* (1.1). *Let M be a contraction map on $(S, \mathscr{F})$. Then, either*

(i) *M has a unique fixed point*

or

(ii) *For every $p_0 \in S$, $\sup\{G_{p_0}(x)\colon x \text{ is real}\} < 1$.*

Proof. Suppose there exists $p_0 \in S$ such that $\sup\{G_{p_0}(x)\colon x \text{ is real}\} = 1$. Then if $\{p_n\}$ is the sequence of iterates of p_0 under M,

$$F_{p_n p_{n+m}}(x) \geqq F_{p_0 p_m}(x/\alpha^n) \geqq G_{p_0}(x/\alpha^n).$$

Thus, since G_{p_0} is non-decreasing,

$$\lim_{n\to\infty} F_{p_n p_{n+m}}(x) = 1$$

for all $x > 0$ independent of m, i.e., $\{p_n\}$ is a Cauchy sequence in $(S, \mathscr{F})$ which is complete. Consequently there is a point $p^* \in S$ such that $\{p_n\}$ converges to p^*. To see that $M p^* = p^*$ it suffices to notice that, for every positive integer n,

$$F_{M p^* p^*}(x) \geqq \tau(F_{M p^* p_n}, F_{p_n p^*})(x) \geqq \tau(F_{p^* p_{n-1}}(j/\alpha), F_{p_n p^*})(x).$$

Thus for all $x > 0$,

$$F_{M p^* p^*}(x) \geqq \lim_{n\to\infty} \tau(F_{p^* p_{n-1}}(j/\alpha), F_{p_n p^*})(x) \geqq H(x) = 1.$$

Hence p^* is the unique fixed point of M.

Corollary. *Every contraction map on a complete PM space $(S, \mathscr{F})$ satisfying* IVm *under* Min *has a unique fixed point.*

Proof. Let $\alpha \in (0, 1)$, M be a contraction map on $(S, \mathscr{F})$ relative to α. Let $p_0 \in S$ and let $\{p_n\}$ be the sequence of iterates of p_0 under M. Then for every positive integer m,

$$(1-\alpha)(\alpha + \alpha^2 + \cdots + \alpha^m) = \alpha - \alpha^{m+1} < 1;$$

whence

$$\begin{aligned} F_{p_0 p_m}(x) &\geqq F_{p_0 p_m}((1-\alpha)(1 + \alpha + \cdots + \alpha^{m-1})x) \\ &\geqq \operatorname{Min}(F_{p_0 p_1}((1-\alpha)x), \ldots, F_{p_{m-1} p_m}(\alpha^{m-1}(1-\alpha)x)) \\ &= F_{p_0 p_1}((1-\alpha)x). \end{aligned}$$

Thus $G_{p_0} \geqq F_{p_0 p_1}((1-\alpha)j)$ and the conclusion now follows from the theorem.

The result of this corollary was the principal result concerning contraction maps obtained by Sehgal in [9].

Conclusion (ii) of Theorem 3.2 is a statement concerning every point of the PM space. The following theorem points out that this condition is satisfied by some point if and only if it is satisfied by every point.

Theorem 3.3. *Let M be a contraction map on the PM space $(S, \mathscr{F})$ satisfying* IVs *under the triangle function τ satisfying* (1.1). *Then for every pair $p_0, q_0 \in S$,*

$$\sup\{G_{p_0}(x)\colon x \text{ is real}\} = 1$$

if and only if

$$\sup\{G_{q_0}(x)\colon \text{is real}\} = 1.$$

Proof. Let p_0 and q_0 belong to S and suppose

$$\sup\{G_{q_0}(x)\colon x \text{ is real}\} = 1.$$

Let G^* denote the left-continuous function which agrees with G_{q_0} at its points of continuity. Let $\{p_n\}$ and $\{q_n\}$ be the iterates of p_0 and q_0, respectively. Then for some α with $0<\alpha<1$, and for every positive integer m,

$$F_{q_m p_m} \geqq F_{q_0 p_0}(j/\alpha^m) \geqq F_{q_0 p_0}.$$

Hence for every positive integer m,

$$F_{p_0 p_m} \geqq \tau\big(F_{p_0 q_0}, \tau(F_{q_0 q_m}, F_{q_m p_m})\big) \geqq \tau\big(F_{p_0 q_0}, \tau(G^*, F_{q_0 p_0})\big)$$

from which it follows that

$$G_{p_0} \geqq \tau\big(F_{p_0 q_0}, \tau(G^*, F_{q_0 p_0})\big),$$

and the theorem is proved.

Theorem 3.4. *Let $(S, \mathscr{F})$ be a complete PM space satisfying* IVs *under an l.c. t-function τ. Suppose further that for every pair $p, q \in S$ there is a real number x such that $F_{pq}(x) = 1$. Let M be a contraction map on $(S, \mathscr{F})$. Then M has a unique fixed point.*

Proof. For each $p, q \in S$ let

$$d(p, q) = \inf\{x\colon F_{pq}(x) = 1\}.$$

It is easy to show that d is a metric on S and that M is a contraction map on the metric space (S, d). Let $p_0 \in S$ and let $\{p_n\}$ be the sequence of iterates of p_0 under M. The usual proof of the contraction mapping theorem for metric spaces shows that $\{p_n\}$ is a Cauchy sequence in (S, d). It is evident that $\{p_n\}$ is a Cauchy sequence in $(S, \mathscr{F})$, and since $(S, \mathscr{F})$ is complete there is a point $p \in S$ which is the limit of $\{p_n\}$. The proof is obviously concluded by showing that p is the unique fixed point of M.

In order to state and prove the next theorem we need to introduce some new notation. Let τ be a triangle function defined on $\Delta \times \Delta$ and let $\{F_n\}$ be a sequence in Δ. Since τ is associative the product, $\tau_{i=1}^{m} F_i$, is well-defined for all m; and since $\tau_{i=1}^{n} F_i \geqq \tau_{i=1}^{n+1} F_i$, the limit

$$\lim_{n\to\infty} \tau_{i=1}^{n} F_i$$

always exists. This limit will be denoted by $\tau_{i=1}^{\infty} F_i$. If T is a t-norm and $\{a_n\}$ is a sequence in I, we define $T_{i=1}^{n} a_i$ and $T_{i=1}^{\infty} a_i$ in a similar fashion.

Theorem 3.5. *Let T be an l.c. t-norm and let τ be the l.c. t-function defined via* (1.2). *There exists a complete PM space $(S, \mathscr{F})$ satisfying* IVs *under τ and a contraction map M on $(S, \mathscr{F})$ which has no fixed point if and only if there exists a $G \in \Delta$ with $\lim_{x\to+\infty} G(x) = 1$ and a number $\alpha \in (0, 1)$ such that*

$$\sup\{(\tau_{i=1}^{\infty} G(j/\alpha^{i-1}))(x)\colon x \text{ is real}\} < 1.$$

Proof. Suppose for every $G \in \Delta$ with $\lim_{x\to+\infty} G(x) = 1$ and for every $\alpha \in (0, 1)$ that

$$\sup\{(\tau_{i=1}^{\infty} G(j/\alpha^{i-1}))(x)\colon x \text{ is real}\} = 1$$

and suppose M is a contraction map on the complete PM space $(S, \mathscr{F})$ satisfying IVs under τ. We shall show that M has a fixed point. To this end, let $p_0 \in S$ and let $\{p_n\}$ be the sequence of iterates of p_0 under M. Then for some $\alpha \in (0, 1)$,

$$F_{p_0 p_m} \geqq \tau_{i=1}^{m} F_{p_{i-1} p_i} \geqq \tau_{i=1}^{m} F_{p_0 p_1}(j/\alpha^{i-1}) \geqq \tau_{i=1}^{\infty} F_{p_0 p_1}(j/\alpha^{i-1}).$$

Thus

$$G_{p_0} \geqq \tau_{i=1}^{\infty} F_{p_0 p_1}(j/\alpha^{i-1});$$

and it follows from Theorem 3.2 that M has a fixed point.

Turning to the converse, suppose $G \in \Delta$ and $\alpha \in (0, 1)$ are such that

$$\sup\{(\tau_{i=1}^{\infty} G(j/\alpha^{i-1}))(x)\colon x \text{ is real}\} < 1.$$

We shall define a probabilistic distance function $\mathscr{F}$ on the collection S of positive integers such that $(S, \mathscr{F})$ is a complete PM space satisfying IVs under τ and show that the mapping M taking n to $n+1$, which obviously has no fixed point, is a contraction mapping.

For any $m, n \in S$, define

$$F_{n+m,n} = F_{n,n+m} = \tau_{i=1}^{m} G(j/\alpha^{n+i-1}),$$

and let

$$F_{n,n} = H.$$

With this definition of $\mathscr{F}$, I, II and III clearly satisfied. Thus we have only to establish that $(S, \mathscr{F})$ satisfies IVs under τ. Case splitting shows that this will follow from the following three inequalities:

(1) $F_{n,n+m+k} \geqq \tau(F_{n,n+m}, F_{n+m,n+m+k})$,

(2) $F_{n,n+m} \geqq \tau(F_{n,n+m+k}, F_{n+m+k,n+m})$,

(3) $F_{n+m,n+m+k} \geqq \tau(F_{n+m,n}, F_{n,n+m+k})$.

Case (1). Since τ is associative,

$$\begin{aligned}F_{n,n+m+k} &= \tau_{i=1}^{m+k} G(j/\alpha^{n+i-1}) \\ &= \tau\left(\tau_{i=1}^{m} G(j/\alpha^{n+i-1}), \tau_{i=m+1}^{m+k} G(j/\alpha^{n+i-1})\right) \\ &= \tau\left(F_{n,n+m}, \tau_{i=1}^{k} G(j/\alpha^{n+m+i-1})\right) = \tau(F_{n,n+m}, F_{n+m,n+m+k}).\end{aligned}$$

Case (2). For any $F, G \in \Delta$, $F = \tau(F, H) \geqq \tau(F, G)$; also τ is associative. Thus

$$\begin{aligned}F_{n,n+m} &= \tau_{i=1}^{m} G(j/\alpha^{n+i-1}) \\ &\geqq \tau\left(\tau_{i=1}^{m} G(j/\alpha^{n+i-1}), \tau_{i=m+1}^{m+k} G(j/\alpha^{n+i-1})\right) \\ &= \tau_{i=1}^{m+k} G(j/\alpha^{n+i-1}) = F_{n,n+m+k} \geqq \tau(F_{n,n+m+k}, F_{n+m+k,n+m}).\end{aligned}$$

Case (3). The proof is similar to the proof given in Case (2) and is therefore omitted.

It will now be shown that $(S, \mathscr{F})$ is complete by showing that the only Cauchy sequences are those which are eventually constant. Suppose $\{p_n\}$ is a sequence in S which is not eventually constant. Then either (i) there is a positive integer N such that $p_n \leqq N$ for all n or (ii) for every positive integer k there exists a positive integer n_k such that $p_{n_k} > p_{n_{k-1}}$ where $n_0 = 1$.

Case (i). Since $\{p_n\}$ is not eventually constant it follows that for every positive integer K, there exist $m, n > K$ such that $p_m \neq p_n$; whence

$$F_{p_m p_n}(x) \leqq \operatorname{Max}\{F_{i,k}(x) \colon 0 \leqq i, j \leqq N, i \neq j\} < H(x)$$

for some $x > 0$. Thus $\{p_n\}$ is not a Cauchy sequence.

Case (ii). Suppose in addition that $\{p_n\}$ is a Cauchy sequence in $(S, \mathscr{F})$. It is easy to show by consideration of the isometric image of this sequence in the completion of $(S, \mathscr{F})$ that the limit function

$$\lim_{k\to\infty} F_{1,p_{n_k}}$$

is a distribution function and therefore has one as its supremum. However for any real x,

$$\begin{aligned}\lim_{k\to\infty} F_{1,p_{n_k}}(x) &= \lim_{k\to\infty} \left(\tau_{i=1}^{p_{n_k}-1} G(j/\alpha^{1+i-1})\right)(x) \\ &= \lim_{k\to\infty} \sup\left\{T_{i=1}^{p_{n_k}-1} G(\beta_i x/\alpha^i) \colon \sum_{i=1}^{p_{n_k}-1} \beta_i = 1, 0 \leqq \beta_i \leqq 1\right\} \\ &= \lim_{k\to\infty} \tau_{i=1}^{p_{n_k}-1} G(j/\alpha^{i-1})(x/\alpha) \\ &= \tau_{i=1}^{\infty} G(j/\alpha^{i-1})(x/\alpha).\end{aligned}$$

The supremum of this quantity for real x is strictly less than one, and we have arrived at a contradiction. Thus $(S, \mathscr{F})$ is complete.

Finally, to see that the mapping M: $S \to S$ defined via $M(n)=n+1$ for all $n \in S$ is a contraction map, it suffices to note that for $n>m$ positive integers and $x>0$,

$$F_{m+1,n+1}(x)=\left(\tau_{i=1}^{n-m} G(j/\alpha^{m+i})\right)(x)$$
$$=\sup\left\{T_{i=1}^{n-m} G(\beta_i x/\alpha^{m+i}) \colon \sum_{i=1}^{n-m} \beta_i=1, 0 \leqq \beta_i \leqq 1\right\}$$
$$=\left(\tau_{i=1}^{n-m} G(j/\alpha^{m+i-1})\right)(x/\alpha)=F_{m,n}(x/\alpha).$$

Corollary 1. *There is a complete PM space $(S, \mathscr{F})$ satisfying* IVm *under T_m and a contraction map on $(S, \mathscr{F})$ having no fixed point.*

Proof. Let α be any number with $0<\alpha<1$ and let G be defined via

$$G(x)=\begin{cases}0, & x \leqq 1/\alpha^2 \\ 1-(1/n), & (1/\alpha^n)<x \leqq 1/\alpha^{n+1}, \quad n>1.\end{cases}$$

It is easily verified that the conditions of Theorem 3.5 are satisfied.

Corollary 2. *There is a complete PM space $(S, \mathscr{F})$ satisfying* IVm *under Prod and a contraction map on $(S, \mathscr{F})$ having no fixed point.*

Proof. The same distribution G given in the proof of Corollary 1 suffices here as well.

The t-norms T_m and Prod are examples of Archimedean t-norms. An extension of Ling's results [3] yields a characterization of Archimedean t-norms which is useful for our purposes. A t-norm T is Archimedean if and only if there exists a function h which is defined, continuous and increasing on I, with $h(1)=1$, and such that, for every $(x, y) \in I \times I$,

$$T(x, y)=h^{[-1]}(h(x) \cdot h(y)),$$

where $h^{[-1]}$ is the function defined on I via

$$h^{[-1]}(x)=\begin{cases}0, & x \in [0, h(0)], \\ h^{-1}(x), & x \in [h(0), 1].\end{cases}$$

The function h is called a multiplicative generator for T.

Theorem 3.6. *If T is any Archimedean t-norm and τ is the l.c. t-function defined via* (1.2), *then there exists a complete PM space $(S, \mathscr{F})$ satisfying* IVs *under τ and a contraction map M on $(S, \mathscr{F})$ which has no fixed point.*

Proof. Let h be the multiplicative generator for the Archimedean t-norm T. We shall exhibit a $K \in \Delta$ with $\lim_{x \to +\infty} K(x)=1$ and a number $\alpha \in (0, 1)$ such that

$$\sup\left\{\left(\tau_{i=1}^{\infty} K(j/\alpha^{i-1})\right)(x) \colon x \text{ is real}\right\}<1.$$

Then by Theorem 3.5 the result follows. To this end let $\alpha \in (0, 1)$ and let G be the function defined in the proof of Corollary 1 to Theorem 3.5. Define K via $K(x)= h^{[-1]}(G(x))$ for every real number x. Clearly $K \in \Delta$ and $\lim_{x \to +\infty} K(x)=1$. Let x be any number such that $G(x)>h(0)$ and let n be the natural number such that $\frac{1}{\alpha^n}<x\leqq\frac{1}{\alpha^{n+1}}$. Then for $i\geqq 1$, $G(x/\alpha^{i-1})>h(0)$ so that

$$\begin{aligned}
(\tau_{i=1}^{\infty} K(j/\alpha^{i-1}))(x) &= \lim_{k\to\infty} (\tau_{i=1}^{k} K(j/\alpha^{i-1}))(x) \\
&= \lim_{k\to\infty} \sup\left\{T_{i=1}^{k} K(\beta_i x/\alpha^{i-1})\colon \sum_{i=1}^{k}\beta_i=1\right\} \\
&\leqq \lim_{k\to\infty} T_{i=1}^{k} K(x/\alpha^{i-1}) \\
&\leqq \lim_{k\to\infty} h^{[-1]}\left[\prod_{i-1}^{k} h K(x)/\alpha^{i-1})\right] \\
&= \lim_{k\to\infty} h^{[-1]}\left[\prod_{i=1}^{k} h\, h^{[-1]} G(x/\alpha^{i-1})\right] \\
&= h^{[-1]}\left[\lim_{k\to\infty}\prod_{i=1}^{k} G(x/\alpha^{i-1})\right] \\
&= h^{[-1]}\left[\lim_{k\to\infty}\prod_{i=1}^{k}\left(1-\frac{1}{n+i-1}\right)\right] \\
&= h^{[-1]}\left[\lim_{k\to\infty}\prod_{i=1}^{k}\frac{n+i-2}{n+i-1}\right] \\
&= h^{[-1]}\left[\lim_{k\to\infty}\frac{n-1}{n+k-1}\right] = h^{[-1]}[0]=0.
\end{aligned}$$

The desired conclusion follows.

Every E-space is a PM space satisfying IVm under T_m; moreover the results of Stevens [15] concerning metrically generated spaces together with our theorem [13] that every metrically generated space is isometric to an E-space clearly shows that this result is best possible in the sense that if T is a t-norm stronger than T_m then there is an E-space which fails to satisfy IVm under that t-norm T. Thus in particular the corollary of Theorem 3.2 does not apply. However E-spaces possess additional structure not shared by PM spaces in general and because of this it makes sense to ask whether this additional structure is sufficient to guarantee that every contraction map on a complete E-space has a unique fixed point. To date, this is an open question: we have neither a proof nor a counterexample. What we can do, however is use the special properties of E-spaces to refine the definition of a contraction map and show that a restricted class of contraction maps do have unique fixed points.

Definition 3.3. Let $(S,\mathscr{F})$ be an E-space over the metric space (M,d) and let $(\Omega,\mathscr{A},P)$ be the associated probability space. The mapping $A\colon S\to S$ is a *strict contraction* map on $(S,\mathscr{F})$ if and only if there is an $\alpha\in(0,1)$ such that for every $p,q\in S$ and for every real x,

$$\{t\in\Omega\colon d(pt,qt)<x/\alpha\}\subseteq\{t\in\Omega\colon d(Apt,Aqt)<x\}.$$

Notice that every strict contraction map is a contraction map.

Theorem 3.7. *Every strict contraction map on a complete E-space has a unique fixed point.*

Proof. Let $(S, \mathscr{F})$ be an E-space over the metric space (M, d) and let $(\Omega, \mathscr{A}, P)$ be the associated probability space. Let A be a strict contraction map on $(S, \mathscr{F})$, let $p_0 \in S$ and let $\{p_n\}$ be the sequence of iterates of p_0 under A. Let $x > 0$ be given and let m be a positive integer. Then

$$\begin{aligned}\{t \in \Omega: d(p_0 t, p_m t) < x\} &\supseteq \{t \in \Omega: d(p_0 t, p_1 t) + \cdots + d(p_{m-1} t, p_m t) < x\} \\ &\supseteq \{t \in \Omega: (1 + \alpha + \cdots + \alpha^{m-1})\, d(p_0 t, p_1 t) < x\} \\ &= \{t \in \Omega: (1 - \alpha^m)\, d(p_0 t, p_1 t) < (1 - \alpha)\, x\} \\ &\supseteq \{t \in \Omega: d(p_0 t, p_1 t) < (1 - \alpha)\, x\}.\end{aligned}$$

Thus $F_{p_0 p_m} \geqq F_{p_0 p_1}((1-\alpha)\, j)$; whence $G_{p_0} \geqq F_{p_0 p_1}((1-\alpha)\, j)$. The result now follows from Theorem 3.1 and 3.2.

This result is not surprising when the strict contraction map is viewed in the setting of pseudo-metrically generated PM spaces which are equivalent to E-spaces. In that setting, A is a strict contraction map if and only if A is a contraction map on each of the pseudo-metric spaces in the generating family. Thus in order for a map to be a strict contraction map it must satisfy a very stringent condition.

4. Analogues of Two Classical Theorems

This section is devoted to the analogues for complete PM spaces of two classical theorems concerning complete metric spaces. The first of these is the analogue of Cantor's theorem on nested sets which states that if a nested sequence of closed subsets of a complete metric space is such that the sequence of diameters of these subsets has limit zero then there is one and only one point in their intersection. In order to state this theorem the following definition, cf. [1], is needed.

Definition 4.1. Let $(S, \mathscr{F})$ be a PM space. Let A be a nonempty subset of S. The probabilistic diameter of A denoted by D_A is the function whose value at any real number x is given by

$$D_A(x) = \inf\{F_{pq}(x): p, q \in A\}.$$

Theorem 4.1. *Let $(S, \mathscr{F})$ be a complete PM space satisfying* IVs *under an l.c. t-function τ. Let $\{A_n\}$ be a nested sequence of nonempty subsets of S which are closed in the ε, λ-topology and such that $D_{A_n} \to H$ as $n \to \infty$. Then there is one and only one point $p_0 \in A_n$ for every n.*

Proof. Let $\{p_n\}$ be a sequence such that $p_n \in A_n$ for every positive integer n. Let $\varepsilon, \lambda > 0$ be given. Choose a positive integer N such that $D_{A_n}(\varepsilon) > 1 - \lambda$ whenever $n > N$. Let $m > n > N$. Then $p_m \in A_m \subseteq A_n$ and $p_n \in A_n$. Thus

$$F_{p_n p_m}(\varepsilon) \geqq \inf\{F_{pq}(\varepsilon): p, q \in A_n\} = D_{A_n}(\varepsilon) > 1 - \lambda.$$

Since ε, λ are arbitrary $\{p_n\}$ is a Cauchy sequence in the complete PM space $(S, \mathscr{F})$. Thus there is an element $p_0 \in S$ such that $\{p_n\} \to p_0$. Since $p_m \in A_n$ whenever $m > n$ and A_n is closed it follows that $p_0 \in A_n$ for every n. Finally, suppose there is another point q_0 belonging to each A_n. Then for each $x>0$,

$$F_{p_0 q_0}(x) \geqq \inf\{F_{pq}(x): p, q \in A_n\} \to 1$$

as $n \to \infty$, i.e., $p_0 = q_0$.

Baire's category theorem states that a complete metric space is of second category. This theorem has a topological version which states that if a topological space (X, σ) is pseudometrizable by a pseudo-metric d such that (X, d) is complete, then (X, σ) is of second category. This version can be carried over immediately to complete PM spaces.

Theorem 4.2. *A complete PM space* $(S, \mathscr{F})$ *satisfying* IVs *under an l.c. t-function* τ *is of second category.*

Proof. The ε, λ-topology [8] for $(S, \mathscr{F})$ is the uniform topology for the uniformity generated by $\mathscr{F}$. Thus it follows from Theorem 2.3 that the ε, λ-topology is metrizable by a metric d such that (S, d) is complete, and this completes the proof.

References

1. Egbert, R.J.: Products and quotients of probabilistic metric spaces. Pacific J. Math. **24**, 437–455 (1968).
2. Kelley, J.L.: General topology. Princeton: D. Van Nostrand Co., Inc. 1955.
3. Ling, C.H.: Representation of associative functions. Publ. math., Debrecen **12**, 189–212 (1965).
4. Menger, K.: Statistical metrics. Proc. nat. Acad. Sci., U.S.A. **28**, 535–537 (1942).
5. Mushtari, D.Kh.: On the completion of random metric spaces (Russian). Kazan. gosudarst. Univ., učenye Zapiski **127**, 109–119 (1967).
6. – Šerstnev, A.N.: On methods of introducing a topology into random metric spaces (Russian). Izvestiya vysš. učebn. Zaved., Mat. **6**, 99–106 (1966).
7. Schweizer, B., Sklar, A.: Statistical metric spaces. Pacific J. Math. **10**, 313–334 (1960).
8. – – Thorp, E.: The metrization of statistical metric spaces. Pacific J. Math. **10**, 673–675 (1960).
9. Sehgal, V.M.: Some fixed point theorems in functional analysis and probability. Ph. D. dissertation, Wayne State Univ., 1966.
10. Šerstnev, A.N.: On the probabilistic generalization of metric spaces. Kazan. gosudarst. Univ., učenye Zapiski **124**, 109–119 (1967).
11. – Triangle inequalities for random metric spaces. Kazan. gosudarst. Univ., učenye Zapiski **125**, 90–93 (1965).
12. Sherwood, H.: On the completion of probabilistic metric spaces. Z. Wahrscheinlichkeitstheorie verw. Geb. **6**, 62–64 (1966).
13. – On E-spaces and their relation to other classes of probabilistic metric spaces. J. London math. Soc. **44**, 441–448 (1969).
14. Sibley, D.A.: A metric for weak convergence of distribution functions. Notices Amer. math. Soc. **15**, 808 (1968).
15. Stevens, R.R.: Metrically generated probabilistic metric spaces. Fundamenta Math. **61**, 259–269 (1968).
16. Wald, A.: On a statistical generalization of metric spaces. Proc. nat. Acad. Sci., U.S.A. **29**, 196–197 (1943).

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