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Limit Theorems for Infinite Particle Systems*

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§ 1. Introduction

Let I be a countable set and suppose that at time zero a certain (possibly random) number $A_0(x)$ of particles are placed at each $x \in I$ and then they move independently according to some transition law. If B is a nonempty subset of I then some quantities of interest are $A_n(B)$ – the number of particles in B at time n; $S_n(B)$ – the total occupation time of B by time n; $L_n(B)$ – the number of distinct particles in B by time n; and $J_n(B)$ – the number of particles which are in B for a last time at time n.

In [2], Derman investigated a system of this type where he proved the following: Suppose $A_0(x)$, $x \in I$ are independent Poisson variables with means $\mu(x)$ and the particles move independently according to the transition function $P(x, y)$ of a Markov chain. Also suppose that μ is invariant for P; that is, $\sum \mu(x) P(x, y) = \mu(y)$ for all y. Then the system is in statistical equilibrium in the sense that at any time n, $A_n(x)$, $x \in I$ are independent Poisson variables with means $\mu(x)$. Port further examined this system in [4] under the hypothesis that $P(x, y)$ is the transition function of a transient chain. Here he proved several limit theorems involving the aforementioned quantities. For example, he showed that $S_n(B)/n \to \mu(B)$ a.s. and that $S_n(B)$ is asymptotically normally distributed.

Now, if the Poisson assumption on the initial distribution of particles is dropped several problems immediately arise. In the Poisson case, the random variables $\{A_n(B)\}_n$ form a strictly stationary sequence and the pointwise ergodic theorem applies. But if the Poisson assumption is dropped, the sequence $\{A_n(B)\}$ is in general not stationary. Thus different techniques must be developed and applied to get the strong law of large numbers for the $S_n(B)$. Another difficulty is met when attempting to determine the asymptotic behavior of the variance of $S_n(B)$; an additional term is encountered in the non-Poisson case and the discovery of its asymptotic behavior is not only crucial for the proof of the central limit theorem for $S_n(B)$, but is also vital in proving the strong law for this quantity. Finally, in the Poisson case, the proof of the central limit theorem for $S_n(B)$ rests on the infinite divisibility of the Poisson distribution. Consequently, in the non-Poisson case some modifications must be made.

The purpose of this paper is to establish the appropriate limit theorems for the particle system without the Poisson hypothesis while assuming $P(x, y)$ is the transition function of a transient aperiodic random walk. As indicated above, some major modifications are necessary in dealing with the non-Poisson case.

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We assume that $A_0(x)$, $x\in Z$ are independent nonnegative integer-valued random variables with finite fourth moments and that there are constants $\lambda>0$, ν, and M such that

$$\begin{aligned} &\text{(i)}\ \mu_1(x)\to\lambda \quad \text{as} \quad |x|\to\infty\\ &\text{(ii)}\ \mu_2(x)\to\nu \quad \text{as} \quad |x|\to\infty\\ &\text{(iii)}\ \mu_j(x)\leqq M \qquad 1\leqq j\leqq 4,\ x\in Z \end{aligned} \tag{1.1}$$

where $\mu_j(x)=E(A_0(x)(A_0(x)-1)\dots(A_0(x)-j+1))$.

Remark. In particular, if $A_0(x)$, $x\in Z$ are independent and identically distributed random variables with finite fourth moments, then (i)–(iii) will hold.

§ 2. Preliminaries and Notation

Suppose that X_n, $n\geqq 0$ are independent integer-valued random variables and for $n\geqq 1$ the variables are identically distributed. Then the process $\{Y_n;\ n\geqq 0\}$ defined by $Y_n=X_0+X_1+\cdots+X_n$ is called a random walk. This process is a Markov chain with n step transition function $P_n(x,y)$ given by $P_n(x,y)=P(X_1+\cdots+X_n=y-x)$. $P(x,y)\equiv P_1(x,y)$ is called the transition function of the random walk. Let $F_n(x,y)=P(Y_n=y;\ Y_\nu\neq y,\ 1\leqq\nu\leqq n-1\,|\,Y_0=x)$. Then the random walk is said to be recurrent if $\sum F_n(0,0)=1$ and transient otherwise. The random walk is called aperiodic if the group generated by the set $\Theta=\{x\colon P(0,x)>0\}$ is the group of all integers. The following renewal theorem will be used frequently in the sequel and can be found in [5].

Theorem. *Suppose $P_n(x,y)$ is the n step transition function of a transient aperiodic random walk. Let $G(x,y)=\sum_{n=1}^{\infty}P_n(x,y)$. If $\sum|x|\,P(0,x)=\infty$, then*

$$\lim_{|x|\to\infty} G(x,y)=0.$$

On the other hand, if $\sum|x|\,P(0,x)<\infty$ and $\sum x\,P(0,x)=m$ then

$$\lim_{x\to\infty} G(x,y)=0$$
$$\lim_{x\to-\infty} G(x,y)=m^{-1}$$

if $m>0$ and if $m<0$

$$\lim_{x\to\infty} G(x,y)=(-m)^{-1}$$
$$\lim_{x\to-\infty} G(x,y)=0.$$

Let $\{Y_n;\ n\geqq 0\}$ be a transient random walk with n step transition function $P_n(x,y)$. Also let B be a finite nonempty subset of the integers. The following notation will be employed

$$P_n(x,B)=\sum_{y\in B}P_n(x,y);\qquad P_n(x,\{y\})=P_n(x,y)$$

$$G_n(x,B)=\sum_{k=1}^{n}P_k(x,B);\qquad G(x,B)=\sum_{k=1}^{\infty}P_k(x,B)$$

$$V_B=\inf\{n\geqq 1;\ Y_n\in B\}\qquad (=\infty\ \text{if}\ Y_n\notin B\ \forall\, n\geqq 1).$$

On $\{V_B<\infty\}$, we define

$$T_B=\sup\{n\geqq 1: Y_n\in B\}$$

and T_B is undefined otherwise. $1_B(x)=1$ if $x\in B$ and equals 0 otherwise. $Z=$ integers, $\mathcal{N}=$ positive integers, $|B|=$ cardinality of B.

§ 3. Statement of Results

Suppose at time zero we place $A_0(x)$ particles at $x\in Z$ where $A_0(x)$ are independent random variables satisfying (i)–(iii) of § 1. The particles are then assumed to move independently according to the transition function $P(x,y)$ of a transient aperiodic random walk $\{Y_n\}$. More precisely, we assume that the random variables $X_{nx}^{(k)}$, for $n\in\mathcal{N}$, $k\in\mathcal{N}$, $x\in Z$ are independent and identically distributed with $P(X_{nx}^{(k)}=y)=P(0,y)$. It is also assumed that the variables $A_0(x)$ and $X_{nx}^{(k)}$ are independent. Then the process $\{Y_{nz}^{(k)}\}_{n=0}^{\infty}$ defined by $Y_{0z}^{(k)}=z$ and $Y_{nz}^{(k)}=z+X_{1z}^{(k)}+\cdots+X_{nz}^{(k)}$ for $n\geqq 1$ is a random walk with transition function $P(x,y)$ and represents the position of the k-th particle starting at z at time n.

Throughout B will denote a finite nonempty subset of Z. Our first result gives the asymptotic behavior of the variance of $S_n(B)$. Results in [4] along with some theorems in [6] lead to the conjecture that $\operatorname{Var} S_n(B)\sim n[\lambda|B|+2\lambda\sum_{y\in B} G(y,B)]$. It turns out, suprisingly enough, that although this is correct in the case $\sum|x|\,P(0,x)=\infty$, it is not correct when $\sum|x|\,P(0,x)<\infty$.

Theorem 1. *Let $S_n(B)$ denote the total occupation time of B by time n. If $\sum|x|\,P(0,x)=\infty$ then*

$$\lim_{n\to\infty}\frac{\operatorname{Var} S_n(B)}{n}=\lambda|B|+2\lambda\sum_{y\in B}G(y,B). \tag{3.1}$$

On the other hand if $\sum|x|\,P(0,x)<\infty$ and $\sum x\,P(0,x)=m$ then

$$\lim_{n\to\infty}\frac{\operatorname{Var} S_n(B)}{n}=\lambda|B|+2\lambda\sum_{y\in B}G(y,B)+\frac{\nu-\lambda^2}{|m|}|B|^2. \tag{3.2}$$

Using the fact that the variance of $S_n(B)$ grows like n along with a fourth moment argument, we get the strong law of large numbers for the $S_n(B)$.

Theorem 2. *Let the notation be as above. Then with probability one*

$$\lim_{n\to\infty} S_n(B)/n=\lambda|B|. \tag{3.3}$$

Notice that (3.3) *shows that the number of particles per unit time in B equals* $\lambda|B|$.

The next theorem shows that $S_n(B)$ is asymptotically normally distributed.

Theorem 3. *Let*

$$\sigma^2(B)=\begin{cases}\lambda|B|+2\lambda\sum_{y\in B}G(y,B) & \text{if } \sum|x|\,P(0,x)=\infty\\ \lambda|B|+2\lambda\sum_{y\in B}G(y,B)+\frac{\nu-\lambda^2}{|m|}|B|^2 & \text{if } \sum x\,P(0,x)=m.\end{cases} \tag{3.4}$$

Then for all $u \in R$

$$\lim_{n\to\infty} P\left(\frac{S_n(B)-ES_n(B)}{[n\sigma^2(B)]^{\frac{1}{2}}} \leqq u\right) = \Phi(u) \tag{3.5}$$

where Φ *is the standard normal distribution.*

The investigation of the system continues with the study of the number of distinct particles which enter B by time n which we denote by $L_n(B)$. The first result for $L_n(B)$ gives the behavior of the variance of $L_n(B)$ for large n.

Theorem 4. *Let* $C(B) = \sum_{y\in B} P_y(V_B = \infty)$. *If* $\sum |x|\, P(0, x) = \infty$, *then*

$$\lim_{n\to\infty} \frac{\operatorname{Var} L_n(B)}{n} = \lambda\, C(B). \tag{3.6}$$

On the other hand, if $\sum |x|\, P(0, x) < \infty$ *and* $\sum x\, P(0, x) = m$ *then*

$$\lim_{n\to\infty} \frac{\operatorname{Var} L_n(B)}{n} = C(B)\left[\lambda + \frac{v-\lambda^2}{|m|}\, C(B)\right]. \tag{3.7}$$

With the aid of Theorem 4 we show that the number of new particles in B per unit time is $\lambda\, C(B)$ and that $L_n(B)$ is asymptotically normal.

Theorem 5. *Let*

$$\tau^2(B) = \begin{cases} \lambda\, C(B) & \text{if } \sum |x|\, P(0, x) = \infty \\ C(B)\left[\lambda + (v-\lambda^2)\, |m|^{-1}\, C(B)\right] & \text{if } \sum x\, P(0, x) = m. \end{cases} \tag{3.8}$$

If $\tau^2(B) > 0$, *then for all* $u \in R$

$$\lim_{n\to\infty} P\left(\frac{L_n(B)-EL_n(B)}{[n\tau^2(B)]^{\frac{1}{2}}} \leqq u\right) = \Phi(u) \tag{3.9}$$

and with probability one

$$\lim_{n\to\infty} L_n(B)/n = \lambda\, C(B). \tag{3.10}$$

Similar theorems to the one above are proved in [4]. But the proofs there depend heavily on the Poisson nature of the system. So, as for the $S_n(B)$, we are forced to develop new techniques for dealing with $L_n(B)$ in the non-Poisson case. The study of the system terminates with the investigation of the quantity $J_n(B)$ — the number of particles which are in B for a last time at time n.

Theorem 6. *Let* $\tau^2(B)$ *be given by* (3.8) *and* $C(B)$ *be as above. Then, if* $D_n(B) = J_1(B) + \cdots + J_n(B)$ *we have*

$$\lim_{n\to\infty} \operatorname{Var} D_n(B)/n = \tau^2(B) \tag{3.11}$$

and with probability one

$$\lim_{n\to\infty} D_n(B)/n = \lambda\, C(B). \tag{3.12}$$

Moreover, if $\tau^2(B) > 0$, *then for all* $u \in R$

$$\lim_{n\to\infty} P\left(\frac{D_n(B)-ED_n(B)}{[n\tau^2(B)]^{\frac{1}{2}}} \leqq u\right) = \Phi(u). \tag{3.13}$$

As a consequence of (3.10) and (3.12), we get the fact that the number of new particles in B per unit time equals the number of particles per unit time which leave B never to return. Hence, although the system in not in general in statistical equilibrium, as in the Poisson case, it does have some basic properties characteristic of such a system. The paper concludes with several illustrative examples.

§ 4. Proof of Theorem 1

In order to prove Theorem 1, we must first establish several lemmas. Let $\{Y_{nx}\}_{n=0}^{\infty}$ be a random walk with $Y_{0x}=x$ and the transition function $P(y,z)$ which regulates the movement of the particles. For a finite nonempty set B let $N_{nx}(B)$ denote the number of times the process $\{Y_{nx}\}$ is in B by time n. Then

$$N_{nx}(B)=\sum_{j=1}^{n} 1_B(Y_{jx}). \tag{4.1}$$

For each $n\in\mathcal{N}$, $k\in\mathcal{N}$, $x\in Z$ let $Z_{nx}^{(k)}(B)$ be the occupation time of B by time n of the k-th particle starting at x. Then $Z_{nx}^{(k)}(B)=\sum_{j=1}^{n} 1_B(Y_{jx}^{(k)})$ and for each $n\in\mathcal{N}$ the random variables $\{Z_{nx}^{(k)}(B)\}_{k,x}$ are independent. Moreover $Z_{nx}^{(k)}(B)$ is distributed as $N_{nx}(B)$ for all k. Then

$$S_{nx}(B)=\sum_{k=1}^{A_0(x)} Z_{nx}^{(k)}(B) \quad (=0 \text{ if } A_0(x)=0) \tag{4.2}$$

gives the total occupation time of B by time n of the particles starting at x. Note that for each n the random variables $\{S_{nx}(B)\}_x$ are independent and finally that

$$S_n(B)=\sum_x S_{nx}(B). \tag{4.3}$$

We will now exhibit the first four moments of $S_{nx}(B)$. From the assumptions made, it is clear that $E\big(S_{nx}(B)|A_0(x)=m\big)=m\,EN_{nx}(B)$ and consequently

$$ES_{nx}(B)=\mu_1(x)\,EN_{nx}(B). \tag{4.4}$$

Next we have that $E\big(S_{nx}(B)^2|A_0(x)=m\big)=E\left(\sum_{k=1}^{m} Z_{nx}^{(k)}(B)\right)^2$. But

$$\left(\sum_{k=1}^{m} Z_{nx}^{(k)}(B)\right)^2=\sum_{k=1}^{m}(Z_{nx}^{(k)}(B))^2+\sum_{i\neq j} Z_{nx}^{(i)}(B)\,Z_{nx}^{(j)}(B)$$

and so using the independence of the $\{Z_{nx}^{(k)}(B)\}_k$ we get

$$ES_{nx}(B)^2=\mu_1(x)\,EN_{nx}(B)^2+\mu_2(x)[EN_{nx}(B)]^2. \tag{4.5}$$

Similar arguments give

$$ES_{nx}(B)^3=\mu_1(x)\,EN_{nx}(B)^3+3\mu_2(x)\,EN_{nx}(B)^2\,EN_{nx}(B)+\mu_3(x)[EN_{nx}(B)]^3, \tag{4.6}$$

$$\begin{aligned} ES_{nx}(B)^4=&\mu_1(x)\,EN_{nx}(B)^4+3\mu_2(x)[EN_{nx}(B)^2]^2\\ &+4\mu_2(x)\,EN_{nx}(B)^3\,EN_{nx}(B)+6\mu_3(x)\,EN_{nx}(B)^2[EN_{nx}(B)]^2\\ &+\mu_4(x)[EN_{nx}(B)]^4. \end{aligned} \tag{4.7}$$

Using (4.3)–(4.5) and the independence of $\{S_{nx}(B)\}_x$ we obtain for each $n \in \mathcal{N}$

$$\operatorname{Var} S_n(B) = \sum_x \mu_1(x)\, EN_{nx}(B)^2 + \sum_x (\mu_2(x) - \mu_1(x)^2)\,[EN_{nx}(B)]^2. \tag{4.8}$$

The second term on the right in (4.8) is the additional term alluded to in the introduction. In the Poisson case $\mu_2(x) = \mu_1(x)^2$ and so it does not arise. It turns out that the second term in (4.8) is much more difficult to handle than the first and our first lemmas deal with obtaining its asymptotic behavior. The following fact will be used.

Lemma 1. *Suppose for each x that $\{a_n(x)\}_n$ is a nonnegative sequence of real numbers such that $\sum_x a_n(x) \to a$ as $n \to \infty$ and for each x, $a_n(x) \to 0$ as $n \to \infty$. Also suppose that $\{q(x)\}_x$ satisfies $q(x) \to b$ as $|x| \to \infty$. Then*

$$\lim_{n \to \infty} \sum_x q(x)\, a_n(x) = b\, a.$$

Now, by (4.1) $EN_{nx}(B) = \sum_{j=1}^{n} P_j(x, B) = G_n(x, B)$, and for all n, $\sum_x EN_{nx}(B) = n\,|B|$. With this in mind, we prove

Lemma 2. *Let the notation be as above and suppose that $\sum |x|\, P(0, x) = \infty$. Then*

$$\lim_{n \to \infty} \frac{1}{n} \sum_x [EN_{nx}(B)]^2 = 0. \tag{4.9}$$

Proof. Let $a_n(x) = G_n(x, B)/n\,|B|$ and $q(x) = G(x, B)$. Since the random walk is transient $G_n(x, B) \leqq G(x, B) < \infty$ and hence $a_n(x) \to 0$ as $n \to \infty$ for all $x \in Z$. Also, by the renewal theorem $q(x) \to 0$ as $|x| \to \infty$ since $\sum |x|\, P(0, x) = \infty$. Since $\sum_x a_n(x) = 1$ for all n we can apply Lemma 1 to conclude $\sum_x G(x, B)\, G_n(x, B)/n\,|B| = o(1)$ and (4.9) follows since $EN_{nx}(B) = G_n(x, B) \leqq G(x, B)$.

The next lemma gives the behavior of $\sum [EN_{nx}(B)]^2$ in case $\sum |x|\, P(0, x) < \infty$. The proof in this case is much more subtle as a result of the different limiting values of $G(x, B)$ when $x \to \infty$ and $x \to -\infty$.

Lemma 3. *Suppose $\sum |x|\, P(0, x) < \infty$ and $m = \sum x\, P(0, x)$. Then*

$$\lim_{n \to \infty} \frac{1}{n} \sum_x [EN_{nx}(B)]^2 = \frac{|B|^2}{|m|}. \tag{4.10}$$

Proof. Without loss of generality assume $m > 0$. Let $z \in B$. The weak law of large numbers implies that $\sum_{x \geqq 0} P_n(x, z) \to 0$ as $n \to \infty$ and since $|B| < \infty$ it follows that

$$\lim_{n \to \infty} n^{-1} \sum_{x \geqq 0} G_n(x, B) = 0. \tag{4.11}$$

The next thing that must be done is to establish a lower bound for $n^{-1} \sum_x [G_n(x, B)]^2$. To do this we estimate the difference $G(x, B) - G_n(x, B)$ for large n. The Markov property and spatial homogeneity of random walks imply

$$G(-x, B) - G_n(-x, B) = \sum_{z \in B} \sum_y P_n(0, y)\, G(y, x + z). \tag{4.12}$$

Let $K=1+G(0,0)$ and $z_0=\max\{z\colon z\in B\}$. The weak law of large numbers implies that for given $\varepsilon>0$, $\sum_{y<n(m-\varepsilon)} P_n(0,y)<\varepsilon/2K|B|$ for large n and the renewal theorem implies $G(0,w)<\varepsilon/2|B|$ for $-w$ large. Using these facts, it follows that for large n and $x\leqq n(m-2\varepsilon)-z_0$

$$\sum_y P_n(0,y)\,G(y,x+z)\leqq\varepsilon/|B|,\qquad z\in B. \tag{4.13}$$

Using (4.12) and (4.13) we get that for large n and $x\leqq n(m-2\varepsilon)-z_0$

$$G(-x,B)-G_n(-x,B)\leqq\varepsilon. \tag{4.14}$$

By the renewal theorem $G(0,x+B)\to m^{-1}|B|$, as $x\to\infty$. Hence using (4.14) we can choose M so that for large n and $M\leqq x\leqq n(m-2\varepsilon)-z_0$, $G_n(0,x+B)\geqq m^{-1}|B|-2\varepsilon$. Let $A_n=\{x\in Z\colon M\leqq x\leqq n(m-2\varepsilon)-z_0\}$. Then the above facts imply $\sum_x [G_n(x,B)]^2\geqq(m^{-1}|B|-2\varepsilon)^2|A_n|$ and consequently

$$\liminf_{n\to\infty} n^{-1}\sum_x [G_n(x,B)]^2\geqq\left(\frac{|B|}{m}-2\varepsilon\right)^2(m-2\varepsilon).$$

Letting $\varepsilon\downarrow 0$ we get the desired lower bound:

$$\liminf_{n\to\infty} n^{-1}\sum_x [G_n(x,B)]^2\geqq\frac{|B|^2}{m}. \tag{4.15}$$

To finish the proof of the lemma, note that (4.11), the renewal theorem, Lemma 1, and $G_n(x,B)\leqq G(x,B)$ imply

$$\limsup_{n\to\infty} n^{-1}\sum_{x<0} [G_n(x,B)]^2\leqq m^{-1}|B|^2 \tag{4.16}$$

and this along with (4.15) gives (4.10).

We are now in a position to prove Theorem 1:

Simple calculations show that

$$\sum EN_{nx}(B)^2=n|B|+2\sum_{y\in B}\sum_{i=1}^{n-1} G_i(y,B).$$

Consequently,

$$\lim_{n\to\infty} n^{-1}\sum EN_{nx}(B)^2=|B|+2\sum_{y\in B} G(y,B). \tag{4.17}$$

It now follows from (4.17), Lemma 1, and (1.1) that

$$\lim_{n\to\infty} n^{-1}\sum_x \mu_1(x)\,EN_{nx}(B)^2=\lambda\big[|B|+2\sum_{y\in B} G(y,B)\big]. \tag{4.18}$$

Lemmas 1–3 along with (1.1) imply that

$$\lim_{n\to\infty} n^{-1}\sum_x (\mu_2(x)-\mu_1(x)^2)[EN_{nx}(B)]^2$$
$$=\begin{cases}0 & \text{if } \sum|x|\,P(0,x)=\infty\\[4pt] \dfrac{\nu-\lambda^2}{|m|}|B|^2 & \text{if } \sum x\,P(0,x)=m.\end{cases} \tag{4.19}$$

Theorem 1 now follows from (4.8), (4.18) and (4.19).

§ 5. Proof of Theorem 2

As pointed out in the introduction, in the non-Poisson case the sequence $\{A_n(B)\}$ is not in general stationary. Hence, by necessity, an entirely different method than in [4] must be used if we are to prove the strong law of large numbers for $S_n(B)$ in the non-Poisson case. It turns out that a fourth moment argument works. We begin with

Lemma 4. *Let $S_{nx}(B)$ denote the total occupation time of B by time n of the particles starting at x* (see § 4). *Then*

$$\sum_x E[S_{nx}(B)-ES_{nx}(B)]^4=o(n^2). \tag{5.1}$$

Proof. First of all note that (4.17), Lemma 2, and Lemma 3 imply that $\sum EN_{nx}(B)^2=O(n)$ and $\sum [EN_{nx}(B)]^2=O(n)$. Using these facts, the fact that $EN_{nx}(B)$ and $EN_{nx}(B)^2$ are uniformly bounded in $x\in Z$, $n\in\mathcal{N}$, and $G_n(x,y)\uparrow G(x,y)<\infty$ some rather lengthy calculations show that the following hold:

$$\sum_x EN_{nx}(B)^2\, EN_{nx}(B)=O(n), \tag{5.2}$$

$$\sum_x [EN_{nx}(B)]^3=O(n), \tag{5.3}$$

$$\sum_x EN_{nx}(B)^3=O(n), \tag{5.4}$$

$$\sum_x EN_{nx}(B)^4=O(n). \tag{5.5}$$

Using the above results along with (1.1) and (4.4)–(4.7) it is not difficult to see that

$$\sum_x E[S_{nx}(B)-ES_{nx}(B)]^4=O(n) \tag{5.6}$$

and so in particular (5.1) holds.

We are now in a position to prove Theorem 2. First of all we show that

$$ES_n(B)\sim n\lambda|B|. \tag{5.7}$$

It follows from (4.3) and (4.4) that $ES_n(B)=\sum \mu_1(x)\,EN_{nx}(B)$. But $\sum \mu_1(x)\,EN_{nx}(B)=\sum_{y\in B}\sum_{k=1}^{n}(\sum_x \mu_1(x)\,P_k(x,y))$ and (1.1) and Lemma 1 imply $\sum_x \mu_1(x)\,P_n(x,y)\to\lambda$ for all y and these facts imply that (5.7) holds.

For brevity let $A_{nx}=S_{nx}(B)-ES_{nx}(B)$. Then the random variables $\{A_{nx}\}_x$ are independent for each n and $EA_{nx}=0$ while $EA_{nx}^2=\operatorname{Var} S_{nx}(B)$. Writing

$$\begin{aligned}(\sum_x A_{nx})^4=&\sum_x A_{nx}^4+4\sum_{x\neq y}A_{nx}^3A_{ny}+3\sum_{x\neq y}A_{nx}^2A_{ny}^2\\ &+\sum_{x\neq y\neq z}A_{nx}^2A_{ny}A_{nz}+\sum_{x\neq y\neq z\neq w}A_{nx}A_{ny}A_{nz}A_{nw}\end{aligned}$$

and using the above facts we conclude that

$$\begin{aligned}E[S_n(B)-ES_n(B)]^4=&\sum_x E[S_{nx}(B)-ES_{nx}(B)]^4\\ &+3\sum_{x\neq y}\operatorname{Var} S_{nx}(B)\operatorname{Var} S_{ny}(B).\end{aligned} \tag{5.8}$$

Now, $\sum_{x \neq y} \operatorname{Var} S_{nx}(B) \operatorname{Var} S_{ny}(B) \leqq [\sum_x \operatorname{Var} S_{nx}(B)]^2 = [\operatorname{Var} S_n(B)]^2 \sim n^2 [\sigma^2(B)]^2$ by Theorem 1. Here $\sigma^2(B)$ is given by (3.4). Using this fact, along with Lemma 4 and (5.8), we see that $E[S_n(B) - ES_n(B)]^4 = O(n^2)$. Thus for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \frac{E[S_n(B) - ES_n(B)]^4}{n^4 \varepsilon^4} < \infty. \tag{5.9}$$

By Chebychev's inequality

$$P\left(\left|\frac{S_n(B) - ES_n(B)}{n}\right| > \varepsilon\right) \leqq \frac{E[S_n(B) - ES_n(B)]^4}{n^4 \varepsilon^4}.$$

Using this along with (5.9) and the Borel-Cantelli lemma, we conclude that for any $\varepsilon > 0$

$$P\left(\left|\frac{S_n(B) - ES_n(B)}{n}\right| > \varepsilon \text{ i.o.}\right) = 0. \tag{5.10}$$

From (5.10) it follows easily that with probability one $[S_n(B) - ES_n(B)] = o(n)$ and consequently (5.7) shows that (3.3) holds. This completes the proof of Theorem 2.

§ 6. Proof of Theorem 3

In this section we prove that $S_n(B)$, suitably normalized, converges in distribution to the standard normal distribution. The method of characteristic functions is used. In the course of the proof several error terms arise and the following lemmas are necessary in order to deal with these terms.

Lemma 5. *With the notation as above we have*

$$\sup_x \operatorname{Var} S_{nx}(B) = o(n) \tag{6.1}$$

and

$$\sup_x E|S_{nx}(B) - ES_{nx}(B)|^3 = o(n^{\frac{3}{2}}). \tag{6.2}$$

Proof. By the renewal theorem $\sup_x G(x, B) < \infty$. From this it follows that $\operatorname{Var} N_{nx}(B)$, $EN_{nx}(B)$, $EN_{nx}(B)^2$, and $EN_{nx}(B)^3$ are uniformly bounded in n and x. Using (4.4)–(4.8) along with (1.1) and the above facts, some lengthy calculations show that the quantities in (6.1) and (6.2) are in fact $O(1)$. This completes the proof of the lemma.

Lemma 6. *Let the notation be as above. Then*

$$\sum_x E|S_{nx}(B) - ES_{nx}(B)|^3 = o(n^{\frac{3}{2}}). \tag{6.3}$$

Proof. With the aid of (4.4)–(4.6), (1.1), and (5.2)–(5.4) and some calculations it is not too difficult to show that $\sum_x E|S_{nx}(B) - ES_{nx}(B)|^3 = O(n)$ and consequently (6.3) is valid.

To prove Theorem 3 first let $\phi_n(\theta)$ and $\psi_{nx}(\theta)$ be the characteristic functions of $[S_n(B) - ES_n(B)]\, n^{-\frac{1}{2}}$ and $[S_{nx}(B) - ES_{nx}(B)]\, n^{-\frac{1}{2}}$, respectively. Since $S_n(B) = \sum_x S_{nx}(B)$ and the random variables $\{S_{nx}(B)\}_x$ are independent for each n we have

$$\phi_n(\theta) = \prod_x \psi_{nx}(\theta). \tag{6.4}$$

If X is a random variable with finite third moment and $f(\theta)$ is the characteristic function of X then (see Feller [3], p. 487) $f(\theta)=1+i\theta EX-\theta^2 EX^2/2+\varepsilon(\theta)$ where $|\varepsilon(\theta)|\leqq|\theta|^3 E|X|^3/3!$. Applying this to $\psi_{nx}(\theta)$ we get

$$\psi_{nx}(\theta)=1-\theta^2 \operatorname{Var} S_{nx}(B)/2n+R_{nx}(\theta) \tag{6.5}$$

where

$$|R_{nx}(\theta)|\leqq|\theta|^3 E|S_{nx}(B)-ES_{nx}(B)|^3/3!\, n^{\frac{3}{2}}. \tag{6.6}$$

For convenience let $A_{nx}(\theta)=-\theta^2 \operatorname{Var} S_{nx}(B)/2n+R_{nx}(\theta)$. Then Lemma 5 and (6.6) show that $\sup_x |A_{nx}(\theta)|\to 0$ as $n\to\infty$. For $|z|\leqq\frac{1}{2}$, $\log(1+z)=z+\varepsilon(z)|z|^2$ where $|\varepsilon(z)|\leqq 1$. Hence for large n, $\log(1+A_{nx}(\theta))=A_{nx}(\theta)+\Lambda_{nx}(\theta)|A_{nx}(\theta)|^2$ for all $x\in Z$, where $|\Lambda_{nx}(\theta)|\leqq 1$. Recall that $\operatorname{Var} S_n(B)=\sum_x \operatorname{Var} S_{nx}(B)$. Then Theorem 1, (6.2), and (6.6) imply that

$$\lim_{n\to\infty}\sum_x A_{nx}(\theta)=-\theta^2\sigma^2(B)/2 \tag{6.7}$$

and

$$\limsup_{n\to\infty}\sum_x |A_{nx}(\theta)|\leqq\theta^2\sigma^2(B)/2 \tag{6.8}$$

Since

$$\left|\sum_x \Lambda_{nx}(\theta)|A_{nx}(\theta)|^2\right|\leqq\left[\sup_x |A_{nx}(\theta)|\right]\sum_x |A_{nx}(\theta)|$$

we can use (6.8) and $\sup_x |A_{nx}(\theta)|\to 0$ to get

$$\lim_{n\to\infty}\sum_x \Lambda_{nx}(\theta)|A_{nx}(\theta)|^2=0. \tag{6.9}$$

Finally (6.4) can be written as $\phi_n(\theta)=\exp\left[\sum_x \log\psi_{nx}(\theta)\right]$ and (6.7) and (6.9) imply $\sum_x \log\psi_{nx}(\theta)\to-\theta^2\sigma^2(B)/2$ as $n\to\infty$. Theorem 3 now follows by the continuity theorem.

§ 7. Proof of Theorem 4

For each x let $\{Y_{nx}\}$ be a random walk with $Y_{0x}=x$ and transition function $P(y,z)$. Let $M_{nx}(B)$ be 1 or 0 according as $\{Y_{nx}\}$ does or does not visit B by time n. That is, $M_{nx}(B)=1\left[\bigcup_{k=1}^{n}\{Y_{kx}\in B\}\right]$. For each $n\in\mathcal{N}$, $k\in\mathcal{N}$, $x\in Z$ let $U_{nx}^{(k)}(B)$ be 1 if the k-th particle starting at x visits B by time n and 0 otherwise. Then for each n the random variables $\{U_{nx}^{(k)}(B)\}_{k,x}$ are independent. Moreover, $U_{nx}^{(k)}(B)$ is distributed as $M_{nx}(B)$ for all k. It is clear that if $L_{nx}(B)$ is the number of distinct particles starting from x which hit B by time n, then for each n the random variables $\{L_{nx}(B)\}_x$ are independent and

$$L_{nx}(B)=\sum_{k=1}^{A_0(x)} U_{nx}^{(k)}(B) \quad (=0 \text{ if } A_0(x)=0), \tag{7.1}$$

$$L_n(B)=\sum_x L_{nx}(B). \tag{7.2}$$

Note that $EM_{nx}(B)=P_x(V_B \leqq n)$. With this in mind arguments as in §4 give the following:

$$EL_{nx}(B)=\mu_1(x)\,P_x(V_B \leqq n), \tag{7.3}$$

$$EL_{nx}(B)^2=\mu_1(x)\,P_x(V_B \leqq n)+\mu_2(x)[P_x(V_B \leqq n)]^2, \tag{7.4}$$

$$EL_{nx}(B)^3=\mu_1(x)\,P_x(V_B \leqq n)+3\mu_2(x)[P_x(V_B \leqq n)]^2+\mu_3(x)[P_x(V_B \leqq n)]^3, \tag{7.5}$$

$$\begin{aligned} EL_{nx}(B)^4 &= \mu_1(x)\,P_x(V_B \leqq n)+7\mu_2(x)[P_x(V_B \leqq n)]^2 \\ &\quad +6\mu_3(x)[P_x(V_B \leqq n)]^3+\mu_4(x)[P_x(V_B \leqq n)]^4 \end{aligned} \tag{7.6}$$

and it follows immediately from (7.2), (7.3) and (7.4) that

$$\operatorname{Var} L_n(B)=\sum_x \mu_1(x)\,P_x(V_B \leqq n)+\sum_x (\mu_2(x)-\mu_1(x)^2)[P_x(V_B \leqq n)]^2. \tag{7.7}$$

In the Poisson case $\mu_2(x)=\mu_1(x)^2$ and consequently the second term on the right of (7.7) does not have to be contended with. As in the case of $S_n(B)$, it turns out that this second term is more difficult to handle than the first. As before the renewal theorem does the trick.

Lemma 7. *If the random walk is such that* $\sum |x|\,P(0,x)=\infty$, *then*

$$\lim_{|x|\to\infty} P_x(V_B<\infty)=0. \tag{7.8}$$

On the other hand if $\sum |x|\,P(0,x)<\infty$ *and* $\sum x\,P(0,x)=m$, *then*

$$\begin{aligned} &\lim_{x\to\infty} P_x(V_B<\infty)=0 \\ &\lim_{x\to-\infty} P_x(V_B<\infty)=m^{-1}\,C(B) \end{aligned} \tag{7.9}$$

if $m>0$ *and if* $m<0$

$$\begin{aligned} &\lim_{x\to\infty} P_x(V_B<\infty)=-m^{-1}\,C(B) \\ &\lim_{x\to-\infty} P_x(V_B<\infty)=0. \end{aligned} \tag{7.10}$$

Here $C(B)=\sum_{y\in B} P_y(V_B=\infty)$.

Proof. Since the chain is assumed to be transient $P_x(T_B=\infty)=0$ and thus

$$\begin{aligned} P_x(V_B<\infty) &= \sum_{k=1}^{\infty} P_x(T_B=k)=\sum_{k=1}^{\infty}\sum_{y\in B} P_k(x,y)\,P_y(V_B=\infty) \\ &= \sum_{y\in B} G(x,y)\,P_y(V_B=\infty). \end{aligned}$$

Consequently, since $|B|<\infty$ the result follows from the renewal theorem.

In order to get the asymptotic variance of $L_n(B)$ it is first necessary to determine the behavior of $\sum_x [P_x(V_B \leqq n)]^2$ for large n. In case $\sum |x|\,P(0,x)=\infty$ it is not difficult to show that Lemma 1 and (7.8) imply $\sum_x [P_x(V_B \leqq n)]^2=o(n)$. However, if $\sum_x x\,P(0,x)=m$ this is not the case and a bit more care must be taken. Note also that since the chain is transient and $|B|<\infty$, $P_y(V_B=\infty)>0$ for some $y\in B$ and thus $C(B)>0$. Before getting the asymptotic behavior of $\sum[P_x(V_B\leqq n)]^2$ in the finite mean case we first determine the behavior of $EL_n(B)$.

Lemma 8. *Let the notation be as above. Then*

$$EL_n(B)\sim n\,\lambda\,C(B). \tag{7.11}$$

Proof. We have $EL_n(B)=\sum_x \mu_1(x)\,P_x(V_B\leqq n)$ and a last entrance decomposition gives

$$\sum_x \mu_1(x)\,P_x(V_B\leqq n)=\sum_{y\in B}\sum_{k=1}^{n}\Big(\sum_x \mu_1(x)\,P_k(x,y)\Big)\,P_y(V_B>n-k).$$

It follows from Lemma 1 that $\sum_x \mu_1(x)\,P_k(x,y)\to\lambda$ as $k\to\infty$ and clearly $P_y(V_B>n)\downarrow P_y(V_B=\infty)$. A summability argument now yields $EL_n(B)\sim n\sum_{y\in B}\lambda\,P_y(V_B=\infty)$ and thus (7.11) holds.

Lemma 9. *Suppose* $\sum|x|\,P(0,x)<\infty$ *and* $\sum x\,P(0,x)=m$. *Then*

$$\sum_x[P_x(V_B\leqq n)]^2\sim n\,|m|^{-1}[C(B)]^2. \tag{7.12}$$

Proof. Without loss of generality assume $m>0$. Since $P_x(V_B\leqq n)\leqq G_n(x,B)$, (4.11) implies that $\sum_{x\geqq 0}P_x(V_B\leqq n)=o(n)$. An argument similar to the one used in the proof of Lemma 8 gives $\sum_x P_x(V_B\leqq n)\sim n\,C(B)$. Consequently,

$$\sum_{x<0}P_x(V_B\leqq n)\sim n\,C(B). \tag{7.13}$$

For convenience, let $h_n(x)=P_{-x}(V_B\leqq n)$ and $h(x)=P_{-x}(V_B<\infty)$. It is easy to see that $h(x)-h_n(x)\leqq G(-x,B)-G_n(-x,B)$ and hence (4.14) applies to give that for large n and $x\leqq n(m-2\varepsilon)-z_0$ we have $h(x)-h_n(x)\leqq\varepsilon$. Let $\alpha=m^{-1}C(B)$. Then (7.9) implies $h(x)\to\alpha$ as $x\to\infty$. Hence for large n and M (independent of n) $h_n(x)\geqq\alpha-2\varepsilon$ uniformly for $M\leqq x\leqq n(m-2\varepsilon)-z_0$. Similar arguments as in Lemma 3 now yield

$$\liminf_{n\to\infty} n^{-1}\sum_x[P_x(V_B\leqq n)]^2\geqq m^{-1}[C(B)]^2. \tag{7.14}$$

Using (7.13), $P_x(V_B\leqq n)\leqq P_x(V_B<\infty)$, Lemma 7, and Lemma 1 it is not hard to see that

$$\limsup_{n\to\infty} n^{-1}\sum_{x<0}[P_x(V_B\leqq n)]^2\leqq m^{-1}[C(B)]^2. \tag{7.15}$$

This completes the proof of Lemma 9.

To prove Theorem 4 first note that (7.7) and (7.3) imply $\operatorname{Var}L_n(B)=EL_n(B)+\sum_x(\mu_2(x)-\mu_1(x)^2)[P_x(V_B\leqq n)]^2$. Lemma 1 applied to (1.1) and the above results give

$$\sum_x(\mu_2(x)-\mu_1(x)^2)[P_x(V_B\leqq n)]^2 =\begin{cases} o(n) & \text{if } \sum|x|\,P(0,x)=\infty\\ n(\nu-\lambda^2)\,|m|^{-1}[C(B)]^2+o(n) & \text{if } \sum_x P(0,x)=m.\end{cases} \tag{7.16}$$

Theorem 4 now follows from Lemma 8 and (7.16).

§ 8. Proof of Theorem 5

To get (3.9) first note that $M_{nx}(B)^j = M_{nx}(B)$ for all $j \in \mathcal{N}$ and $EM_{nx}(B) = P_x(V_B \leq n) \leq 1$. Using these facts and (7.3)–(7.5) some calculations show that

$$\sup_x \operatorname{Var} L_{nx}(B) = o(n), \tag{8.1}$$

$$\sup_x E\,|L_{nx}(B) - EL_{nx}(B)|^3 = O(1). \tag{8.2}$$

Also, it is not too difficult to show that $E\,|L_{nx}(B) - EL_{nx}(B)|^3 = O(EL_{nx}(B))$ and consequently Lemma 8 implies

$$\sum_x E\,|L_{nx}(B) - EL_{nx}(B)|^3 = O(EL_n(B)) = o(n^{\frac{3}{2}}). \tag{8.3}$$

Using (8.1)–(8.3) and arguments based on characteristic functions similar to those which appear in the proof of Theorem 3 we are able to prove (3.9).

In order to prove the strong law of large numbers for the quantity $L_n(B)$ we again turn to a fourth moment argument similar to the one used in Theorem 2. The essential facts needed to complete the argument are Lemma 8 and $E[L_{nx}(B) - EL_{nx}(B)]^4 = O(EL_{nx}(B))$. The last fact along with Lemma 8 implies that

$$\sum_x E[L_{nx}(B) - EL_{nx}(B)]^4 = o(n^2). \tag{8.4}$$

Armed with (8.4) and Lemma 8, arguments as in Theorem 2 yield (3.10).

§ 9. Proof of Theorem 6

The proof of Theorem 6 can be accomplished by arguments similar to those used in Sections 7 and 8. Consequently, we will only formulate the problem here and make a few comments regarding the quantity $D_n(B)$. Once these comments are made it should not be too difficult to see how the proof of Theorem 6 should proceed.

Let $\{Y_{nx}\}$ be as in § 7. Let $R_{nx}(B)$ equal 1 if the process $\{Y_{nx}\}_{n=1}^{\infty}$ leaves B, never to return, by time n; and let it equal 0 otherwise. Then

$$R_{nx}(B) = 1\left[\bigcup_{k=1}^{n} \{Y_{kx} \in B;\ Y_{jk} \notin B, j > k\}\right].$$

For each n, $k \in \mathcal{N}$, $x \in Z$ let $W_{nx}^{(k)}(B)$ be 1 if the k-th particle starting at x leaves B by time n and never returns; and let it equal 0 otherwise. Then $W_{nx}^{(k)}(B)$ is distributed as $R_{nx}(B)$ for all k, the random variables $\{W_{nx}^{(k)}(B)\}_{k,x}$ are independent for each n, and

$$D_{nx}(B) = \sum_{k=1}^{A_0(x)} W_{nx}^{(k)}(B) \qquad (=0 \text{ if } A_0(x) = 0) \tag{9.1}$$

represents the number of particles starting at x which by time n leave B and never return. Moreover, for each n, the variables $\{D_{nx}(B)\}_x$ are independent and

$$D_n(B) = \sum_x D_{nx}(B). \tag{9.2}$$

Formulas similar to those given in (7.3)-(7.7) hold for $D_{nx}(B)$, with $P_x(V_B \leqq n)$ replaced by $P_x(T_B \leqq n)$ since the latter quantity is the mean of $R_{nx}(B)$. Since $P_x(T_B < \infty) = P_x(V_B < \infty)$ Lemma 7 hold with $P_x(T_B < \infty)$ replacing $P_x(V_B < \infty)$. Now,

$$ED_n(B) = \sum \mu_1(x)\, P_x(T_B \leqq n) = \sum_{y \in B} \sum_{k=1}^{n} \left(\sum_x \mu_1(x)\, P_k(x, y)\right) P_y(V_B = \infty)$$

and so arguments as before show that

$$ED_n(B) \sim n\, \lambda\, C(B). \tag{9.3}$$

Finally it is clear that $P_x(T_B < \infty) - P_x(T_B \leqq n) \leqq G(x, B) - G_n(x, B)$ and this permits us to get the asymptotic behavior of the quantity $\sum_x [P_x(T_B \leqq n)]^2$. The rest of the details regarding Theorem 6 are left to the reader.

§ 10. Examples

First we give some examples where (i)-(iii) hold.

Example 1. As pointed out previously, if the random variables $\{A_0(x)\}$ are independent and identically distributed with finite fourth moments then (i)-(iii) will hold.

Example 2. Let $\{\mu_1(x)\}$ be a sequence of nonnegative real numbers such that $\mu_1(x) \to \lambda > 0$ as $|x| \to \infty$. Choose L so that $\mu_1(x) < L$ for all x and let the random variables $A_0(x)$, $x \in Z$ be independent and binomially distributed with parameters L and $\mu_1(x)/L$. Then (i) holds by definition and it is easy to show that (iii) holds with $M = L^4$. Finally $\mu_2(x) = (L-1)\, \mu_1(x)^2/L$ and so (ii) holds with $\nu = \lambda^2(L-1)/L$.

Next we give an example where some of the quantities introduced previously are calculated.

Example 3. Bernoulli random walk. Consider the random walk with transition function given by $P(0, 1) = p$, $P(0-1) = 1-p$ where $0 < p < 1$. We assume $p \neq \frac{1}{2}$ so that the random walk is transient. It is well known that $\sum_{n=0}^{\infty} P_n(0, 0) = |1-2p|^{-1}$ (see Chung [1], p. 23). Consequently, $G(0, 0) = |1-2p|^{-1} - 1$. Now for any transient state of any Markov chain we have $P_x(V_{\{x\}} = \infty) = [1 + G(x, x)]^{-1}$. In the present case we have $C(\{0\}) = P_0(V_{\{0\}} = \infty) = [1 + G(0, 0)]^{-1} = |1-2p|$. Also, $m = \sum x\, P(0, x) = 2p - 1$. Using these facts we get for $B = \{0\}$,

$$\sigma^2(B) = \lambda + 2\lambda(|1-2p|^{-1} - 1) + (\nu - \lambda^2)\, |1-2p|^{-1}$$
$$\tau^2(B) = |1-2p|\, (\lambda + \nu - \lambda^2).$$

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