Z. Wahrscheinlichkeitstheorie verw. Gebiete 29, 241 – 252 (1974)

Positivity of the K-Entropy on Non-Abelian K-Flows

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A non-commutative extension of certain aspects of classical probability theory is presented in such a manner that the notion of Kolmogorov entropy can be extended to a large class of non-classical dynamical systems. In particular, the generalized K-entropy so defined is shown to be strictly positive on the class of non-abelian K-flows.

I. Introduction

A *classical flow* $\{\Omega, \mu, T\}$ is usually [1, 14] defined as a Lebesgue space (Ω, μ) equipped with a one-parameter group $\{T(t) | t \in \mathbb{R}\}$ of automorphisms (mod 0) of (Ω, μ), with $T(t)$ depending measurably of t. As is well-known [1, 9], one can associate to every classical flow a triple $\{\mathfrak{N}, \Phi, \alpha\}$ consisting of: (a) the maximal abelian von Neumann algebra $\mathfrak{N} = \mathscr{L}^\infty(\Omega, \mu)$, the elements of which are the "Stochastic variables" of the theory, acting by multiplication on the separable Hilbert space $\mathfrak{H} = \mathscr{L}^2(\Omega, \mu)$; (b) the vector Φ in $\mathfrak{H}$, defined by $\Phi(\omega) = 1$ for all ω in Ω; we notice that Φ is cyclic in $\mathfrak{H}$ with respect to $\mathfrak{N}$, and is separating for $\mathfrak{N}$; moreover $\langle\phi; N\rangle = (N\Phi, \Phi) = \int_\Omega N(\omega)\, \mu(d\omega)$, $\forall N \in \mathfrak{N}$ defines a positive linear functional ϕ on $\mathfrak{N}$, which is normalized to 1, countably additive and faithful on $\mathfrak{N}$; in particular, for every μ-measurable subset Δ of Ω, $\mu(\Delta) = \langle\phi; \chi_\Delta\rangle$ where χ_Δ is the indicator of Δ; finally (c): the homomorphism $\alpha: \mathbb{R} \to \mathrm{Aut}(\mathfrak{N})$ defined by $\alpha(t)[N](\omega) = N(T(t)[\omega])$, which is continuous in the weak-operator topology, and leaves ϕ invariant, i.e. $\langle\phi; \alpha(t)[N]\rangle = \langle\phi; N\rangle$ for every N in $\mathfrak{N}$ and every t in $\mathbb{R}$.

Motivated to a large extend by the need to provide *classical* statistical mechanics with a mathematical foundation [7], the ergodic theory of classical flows has been successfully developped in the last forty years (see for instance [1]). The extension of this theory to the situations encountered in *quantum* statistical mechanics now requires [10, 15] (for a review see for instance [4]) substituting a non-abelian algebra of "non-commutative observables" to the abelian algebra $\mathfrak{N}$ of the "stochastic variables" of the classical theory. With such an extension in view, we concentrate our attention in this paper on the following generalization of a classical flow. A *dynamical system* is defined as a triple $\{\mathfrak{N}, \Phi, \alpha\}$ consisting of: (a) a von Neumann algebra $\mathfrak{N}$ acting on a separable Hilbert space $\mathfrak{H}$; (b) a vector Φ in $\mathfrak{H}$, normalized to 1, cyclic in $\mathfrak{H}$ with respect to $\mathfrak{N}$ and separating for $\mathfrak{N}$; we denote by ϕ the faithfull normal state defined on $\mathfrak{N}$ by $\langle\phi; N\rangle = (N\Phi, \Phi)$ for all N in $\mathfrak{N}$; and (c) an homomorphism $\alpha: \mathbb{R} \to \mathrm{Aut}(\mathfrak{N})$, continuous for the weak-operator topology and leaving ϕ invariant, i.e. $\langle\phi; \alpha(t)[N]\rangle = \langle\phi; N\rangle$ for all N in $\mathfrak{N}$, and all t in $\mathbb{R}$.

The aim of the present paper is to obtain for such dynamical systems an object $\hat{H}_\phi(\alpha)$ which generalizes the Kolmogorov-Sinai entropy [8, 16] of a classical flow [1, 12].

In analogy with the classical case our first step, in obtaining $\hat{H}_\phi(\alpha)$, will be to define a *conditional entropy*, i.e. we will first have to prescribe a class $\hat{\mathfrak{F}}$ of partitions of the identity in $\mathfrak{N}$, a class $\hat{\mathfrak{C}}$ of von Neumann subalgebras of $\mathfrak{N}$, and a mapping $\hat{H}_\phi: \hat{\mathfrak{F}} \times \hat{\mathfrak{C}} \to \mathbb{R}^+$ such that:

(i) $\hat{H}_\phi(\mathfrak{F}|\mathfrak{C})=0$ if and only if $\mathfrak{F} \subseteq \mathfrak{C}$;

(ii) $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$ implies $\hat{H}_\phi(\mathfrak{F}_1|\mathfrak{C}) \leqq \hat{H}_\phi(\mathfrak{F}_2|\mathfrak{C})$;

(iii) $\mathfrak{C}_1 \subseteq \mathfrak{C}_2$ implies $\hat{H}_\phi(\mathfrak{F}|\mathfrak{C}_1) \geqq \hat{H}_\phi(\mathfrak{F}|\mathfrak{C}_2)$.

A fourth property of this conditional entropy will naturally be required, namely that if $\mathfrak{N}$ is abelian, $\hat{H}_\phi$ should coincide with the classical conditional entropy. Intuitively, $\hat{H}_\phi(\mathfrak{F}|\mathfrak{C})$ should be a measure of the information gained in measuring $\mathfrak{F}$ when the expectations relative to $\mathfrak{C}$ are known.

The main tool for the present investigation will be Takesaki's theory [17] of Tomita algebras. We recall in this connection that if ϕ is a faithful normal state on a von Neumann algebra $\mathfrak{N}$, there is a unique homomorphism $\alpha^\beta: \mathbb{R} \to \mathrm{Aut}(\mathfrak{N})$ such that for every N and M in $\mathfrak{N}$ there exists a function $\phi^\beta_{N,M}(z)$ holomorphic in, and continuous on, the strip $0 \leqq \mathrm{Im}(z) \leqq \beta$ ($\beta > 0$) with boundary values

$$\phi^\beta_{N,M}(t) = \langle \phi; \alpha^\beta(t)[M]\, N \rangle,$$
$$\phi^\beta_{N,M}(t+i\beta) = \langle \phi; N\, \alpha^\beta(t)[M] \rangle.$$

To distinguish $\alpha(\mathbb{R})$ from $\alpha^\beta(\mathbb{R})$ we will refer to the former as the "true evolution", and to the latter as the "free evolution". Accordingly we will refer to the von Neumann algebra $\mathfrak{N}_0$ of the fixed points of $\mathfrak{N}$ under $\alpha^\beta(\mathbb{R})$ as the algebra of the "constants of the motion under the free evolution".

It is well-known, and easy to check (see for instance [17], or [4]) that ϕ is a fixed point of the dual action $\{\alpha^\beta(t)^* \mid t \in \mathbb{R}\}$ on $\mathfrak{N}^*$, and that ϕ is a trace on $\mathfrak{N}$ if and only if $\alpha^\beta(t)$ is the identity automorphism of $\mathfrak{N}$ for all t in $\mathbb{R}$, i.e. if and only if $\mathfrak{N} = \mathfrak{N}_0$. This situation is in particular encountered when $\mathfrak{N}$ is abelian, so that the occurrence of a non-trivial $\alpha^\beta(\mathbb{R})$ is linked to the generalization considered here, namely that $\mathfrak{N}$ is allowed to be non-abelian. We shall implicitly assume throughout this investigation that $\mathfrak{N}_0$ is "large enough" in $\mathfrak{N}$, and in particular that $\alpha^\beta(\mathbb{R})$ does not act in an ergodic manner on $\mathfrak{N}$ (i.e. that $\mathfrak{N}_0 \neq \mathbb{C} I$). The latter assumption will in particular be satisfied in the case of the "non-abelian K-flows" considered in the last two sections of this paper: for these particular dynamical systems every maximal abelian von Neumann subalgebra of $\mathfrak{N}_0$ is also maximal abelian in $\mathfrak{N}$. In Section V we extend the classical results of Kolmogorov and Sinai, and show that the K-entropy $\hat{H}_\phi(\alpha)$ defined for the general dynamical systems of Sections I–III is strictly positive on the class of the non-abelian K-flows.

II. The Classes $\hat{\mathfrak{F}}$ and $\hat{\mathfrak{C}}$

The aim of this section is to describe the classes $\hat{\mathfrak{F}}$ and $\hat{\mathfrak{C}}$ on which the conditional entropy $\hat{H}_\phi: \hat{\mathfrak{F}} \times \hat{\mathfrak{C}} \to \mathbb{R}^+$ will be defined in the next section.

Throughout, $(\mathfrak{N}, \phi)$ denotes a pair consisting of a von Neumann algebra $\mathfrak{N}$ acting on a separable Hilbert space $\mathfrak{H}$; and the faithful normal state ϕ on $\mathfrak{N}$ corresponding to a vector Φ in $\mathfrak{H}$, normalized to 1, cyclic in $\mathfrak{H}$ with respect to $\mathfrak{N}$, and separating for $\mathfrak{N}$.

We first recall that a partition $\mathfrak{F}$ of the identity in $\mathfrak{N}$ is, by definition, a collection $\{F_i\}$ of mutually orthogonal projectors belonging to $\mathfrak{N}$ and adding up to the identity I on $\mathfrak{H}$. We will simply denote by $\mathfrak{F}''$ the abelian von Neumann subalgebra of $\mathfrak{N}$ generated by $\mathfrak{F}$.

We next recall that a von Neumann subalgebra $\mathfrak{C}$ of $\mathfrak{N}$ is said to admit a conditional expectation if there exists a σ-weakly continuous, faithful projection $\mathscr{E}_\phi(\cdot|\mathfrak{C})$ of norm one from $\mathfrak{N}$ onto $\mathfrak{C}$ such that

$$\langle\phi; N\rangle = \langle\phi; \mathscr{E}_\phi(N|\mathfrak{C})\rangle, \qquad \forall N \in \mathfrak{N}$$

(see for instance [21]; for the meaning of this definition in the framework of classical probability theory, see [11]). By Tomiyama's result [20], the *$\mathfrak{C}$-conditional expectation $\mathscr{E}_\phi(\cdot|\mathfrak{C})$ with respect to ϕ* has the following properties (for all N in $\mathfrak{N}$ and all C_1, C_2 in $\mathfrak{C}$):

(i) $0 \leqq \mathscr{E}_\phi(N|\mathfrak{C})^* \mathscr{E}_\phi(N|\mathfrak{C}) \leqq \mathscr{E}_\phi(N^* N|\mathfrak{C})$;

(ii) $\mathscr{E}_\phi(C_1 N C_2|\mathfrak{C}) = C_1 \mathscr{E}_\phi(N|\mathfrak{C}) C_2$

so that in particular:

(iii) $N_1 \leqq N_2$ implies $\mathscr{E}_\phi(N_1|\mathfrak{C}) \leqq \mathscr{E}_\phi(N_2|\mathfrak{C})$;

(iv) $\langle\phi; C_1 \mathscr{E}_\phi(N|\mathfrak{C}) C_2\rangle = \langle\phi; C_1 N C_2\rangle$.

Two important results should be mentioned here. First, Takesaki [18] proved that $\mathscr{E}_\phi(\cdot|\mathfrak{C})$ exists if and only if $\mathfrak{C}$ is *stable* under the group $\alpha^\beta(\mathbb{R})$ canonically associated to ϕ (see Section I). Second, Theorem 3.6 in [13], when applied to the situation studied here, says that the fixed point algebra $\mathfrak{N}_0$ of $\mathfrak{N}$ under $\alpha^\beta(\mathbb{R})$ is the "centralizer" of $\mathfrak{N}$ with respect to ϕ, i.e.

$$\mathfrak{N}_0 = \{X \in \mathfrak{N} \,|\, \langle\phi; XN\rangle = \langle\phi; NX\rangle \;\forall N \in \mathfrak{N}\}.$$

We should however notice that the existence of $\mathscr{E}_\phi(\cdot|\mathfrak{C})$ does *not* imply in general the existence of $\mathscr{E}_\phi(\cdot|\mathfrak{D})$ for an arbitrary von Neumann subalgebra $\mathfrak{D}$ of $\mathfrak{C}$; this situation makes difficult (both conceptually and technically) a straightforward generalization of classical probability theory to the situation we want to study. *Our forthcoming introduction of the classes $\hat{\mathfrak{F}}$ and $\hat{\mathfrak{C}}$ is precisely devised so as to bypass these difficulties* (see Corollary II.1 below).

Theorem II.1. *For any abelian von Neumann subalgebra $\mathfrak{B}$ of $\mathfrak{N}$, the following conditions are equivalent:*

(i) *for every partition $\mathfrak{F} = \{F_i\}$ of the identity in $\mathfrak{B}$, $\Sigma_i \lambda_i \phi_i$ (with $\lambda_i = \langle\phi; F_i\rangle$, and $\langle\phi_i; N\rangle = \langle\phi; F_i\rangle^{-1} \langle\phi; F_i N F_i\rangle$ for every N in $\mathfrak{N}$) is a convex decomposition of ϕ into states ϕ_i on $\mathfrak{N}$ with mutually orthogonal supports;*

(ii) *for every B in $\mathfrak{B}$ and every N in $\mathfrak{N}$ $\langle\phi; BN\rangle = \langle\phi; NB\rangle$;*

(iii) $\mathfrak{B} \subseteq \mathfrak{N}_0$;

(iv) *for every von Neumann subalgebra* $\mathfrak{C} \subseteq \mathfrak{B}$, *the* $\mathfrak{C}$*-conditional expectation with respect to* ϕ *exists;*

(v) *the* $\mathfrak{B}$*-conditional expectation with respect to* ϕ *exists.*

Proof. (i) implies (ii). For every projector F in $\mathfrak{B}$, let $\mathfrak{F} = \{F, (I-F)\}$; we have thus for every N in $\mathfrak{N}$: $\langle\phi; N\rangle = \langle\phi; FNF\rangle + \langle\phi; (I-F)N(I-F)\rangle$; in particular we get, upon replacing successively N by NF and by FN: $\langle\phi; NF\rangle = \langle\phi; FNF\rangle = \langle\phi; FN\rangle$; for every finite sum $\Sigma_k a_k F_k$ with $a_k \in \mathbb{C}$ and F_k projector in $\mathfrak{B}$, we have thus: $\langle\phi; N(\Sigma_k a_k F_k)\rangle = \langle\phi; (\Sigma_k a_k F_k) N\rangle$ since finally every B in $\mathfrak{B}$ is the weak-operator limit of such sums, and since ϕ is a vector-state, we obtain: $\langle\phi; NB\rangle = \langle\phi; BN\rangle$, which is (ii). (ii) implies (i). For any partition $\mathfrak{F} = \{F_i\}$ of the identity in $\mathfrak{B}$, we have $\langle\phi; N\rangle = \langle\phi; (\Sigma_i F_i) N\rangle = \Sigma_i \langle\phi; F_i N\rangle = \Sigma_i \langle\phi; F_i N F_i\rangle = \langle\Sigma_i \lambda_i \phi_i; N\rangle$; hence $\Sigma_i \lambda_i \phi_i$ is indeed a convex decomposition of ϕ; now ϕ faithfull implies that the support of ϕ_i is F_i, thus proving that the ϕ_i's have mutually orthogonal supports. (ii) and (iii) are equivalent by the result of [13] quoted above. (iii) implies (iv). $\mathfrak{C} \subseteq \mathfrak{B} \subseteq \mathfrak{N}_0$ clearly implies that $\mathfrak{C}$ is stable under $\alpha^\beta(\mathbb{R})$, so that the $\mathfrak{C}$-conditional expectation with respect to ϕ exists by [18]. (iv) implies (v) as a particular case. (v) implies (iii). By [18] (v) implies that $\mathfrak{B}$ is stable with respect to $\alpha^\beta(\mathbb{R})$; let thus $\alpha_0^\beta(\mathbb{R})$ and ϕ_0 denote respectively the restrictions of $\alpha^\beta(\mathbb{R})$ and of ϕ to $\mathfrak{B}$; since $\mathfrak{B}$ is abelian, ϕ_0 is a trace on $\mathfrak{B}$ and thus $\alpha_0^\beta(\mathbb{R}) = \mathrm{id}$, i.e. $\mathfrak{B} \subseteq \mathfrak{N}_0$, q.e.d.

We now define the classes $\hat{\mathfrak{F}}$ and $\hat{\mathfrak{C}}$ as follows. $\hat{\mathfrak{F}}$ is the collection of all finite partitions $\mathfrak{F}$ of the identity in $\mathfrak{N}$ such that $\mathfrak{F}''$ satisfies any (and thus all) of the five equivalent conditions of Theorem II.1. $\hat{\mathfrak{C}}$ is the collection of all von Neumann subalgebras $\mathfrak{C}$ of $\mathfrak{N}$ such that every abelian von Neumann subalgebra $\mathfrak{B} \subseteq \mathfrak{C}$ satisfies any (and thus all) of the five equivalent conditions of Theorem II.1. When $\mathfrak{F}$ (resp. $\mathfrak{C}$) belongs to $\hat{\mathfrak{F}}$ (resp. $\hat{\mathfrak{C}}$), we say that it is $(\mathfrak{N}, \phi)$*-admissible*. The following results throw some useful light on the actual meaning and role of the concept of $(\mathfrak{N}, \phi)$-admissibility; they are immediate consequences of Theorem II.1 and are thus given without proof.

Corollary II.1. *For any von Neumann subalgebra* $\mathfrak{C}$ *of* $\mathfrak{N}$ *the following conditions are equivalent:*

(i) $\mathfrak{C}$ *is* $(\mathfrak{N}, \phi)$*-admissible;*

(ii) $\mathfrak{C}$ *is contained in the fixed point algebra* $\mathfrak{N}_0$ *of* $\mathfrak{N}$ *under* $\alpha^\beta(\mathbb{R})$;

(iii) *every von Neumann subalgebra* $\mathfrak{D}$ *of* $\mathfrak{C}$ *is* $(\mathfrak{N}, \phi)$*-admissible;*

(iv) *the* $\mathfrak{D}$*-conditional expectation* $\mathscr{E}_\phi(\cdot|\mathfrak{D})$ *with respect to* ϕ *exists for every von Neumann subalgebra* $\mathfrak{D}$ *of* $\mathfrak{C}$.

Remark. $\mathfrak{C}$ is $(\mathfrak{N}, \phi)$-admissible implies thus the existence of the conditional expectation $\mathscr{E}_\phi(\cdot|\mathfrak{C})$; but the converse is in general not true, as we already noticed.

Corollary II.2. *For every finite partition* $\mathfrak{F}$ *of the identity in* $\mathfrak{N}$, *the following conditions are equivalent:*

(i) $\mathfrak{F}$ *is* $(\mathfrak{N}, \phi)$*-admissible;*

(ii) $\mathfrak{F}$ *belongs to* $\mathfrak{N}_0$;

(iii) $\phi = \Sigma_i \lambda_i \phi_i$, *where* λ_i *and* ϕ_i *are as in* Theorem II.1 (i);

(iv) *every partition* $\mathfrak{G}$ *of the identity, coarser than* $\mathfrak{F}$, *is* $(\mathfrak{N}, \phi)$*-admissible;*

(v) $\mathscr{E}_\phi(\cdot|\mathfrak{D})$ *exists for every von Neumann algebra* $\mathfrak{D}$ *in* $\mathfrak{F}''$;

(vi) $\mathfrak{F}''$ *is* $(\mathfrak{N}, \phi)$*-admissible.*

Remark. A von Neumann subalgebra $\mathfrak{C}$ of $\mathfrak{N}$ is thus $(\mathfrak{N}, \phi)$-admissible if and only if all finite partitions of the identity in $\mathfrak{C}$ are $(\mathfrak{N}, \phi)$-admissible.

Corollary II.3. *The following conditions are equivalent:*

(i) *every von Neumann subalgebra of* $\mathfrak{N}$ *is* $(\mathfrak{N}, \phi)$*-admissible;*

(ii) *every finite partition of the identity in* $\mathfrak{N}$ *is* $(\mathfrak{N}, \phi)$*-admissible;*

(iii) *the* $\mathfrak{C}$*-conditional expectation* $\mathscr{E}_\phi(\cdot|\mathfrak{C})$ *with respect to* ϕ *exists for every von Neumann subalgebra* $\mathfrak{C}$ *of* $\mathfrak{N}$;

(iv) $\mathfrak{N}_0 = \mathfrak{N}$;

(v) $\alpha^\beta(\mathbb{R}) = \{\mathrm{id}\}$;

(vi) ϕ *is a trace on* $\mathfrak{N}$.

Remarks. Since ϕ is assumed throughout to be a faithful normal state on $\mathfrak{N}$, the conditions of this corollary can be satisfied only when $\mathfrak{N}$ is a von Neumann algebra of finite type. These conditions are in particular satisfied in classical probability theory, in which case $\mathfrak{N}$ is abelian and the concept of $(\mathfrak{N}, \phi)$-admissibility becomes redundant. Although we do not need a further extension of the above framework for the purpose of the present paper, we might finally surmise that our assumption that ϕ be a state on $\mathfrak{N}$ could be relaxed to the weaker assumption that ϕ be a "weight" [2, 17] on $\mathfrak{N}$ (all other assumptions being kept the same).

The remainder of the present investigation will be conducted under the general assumptions that we are given an arbitrary von Neumann algebra $\mathfrak{N}$ acting on a separable Hilbert space $\mathfrak{H}$; and a vector Φ in $\mathfrak{H}$, normalized to 1, cyclic in $\mathfrak{H}$ with respect to $\mathfrak{N}$, and separating for $\mathfrak{N}$, so that $\langle \phi; N\rangle \equiv (N\Phi, \Phi)$ for all N in $\mathfrak{N}$ defines a faithful normal state ϕ on $\mathfrak{N}$, but not necessarily a trace. The concept of $(\mathfrak{N}, \phi)$-admissibility will thus be an essential tool, for our generalization to this framework, of the concept of conditional entropy.

III. Entropy of a Dynamical System

The concept of $(\mathfrak{N}, \phi)$-admissibility introduced in Section II will now be used to establish the existence of a generalized conditional entropy in the sense of Section I; specifically, we will prove the following result:

Theorem III.1. Let $\hat{\mathfrak{F}}$ and $\hat{\mathfrak{C}}$ be the $(\mathfrak{N}, \phi)$-admissible classes defined in Section II, and $\hat{H}_\phi$: $\hat{\mathfrak{F}} \times \hat{\mathfrak{C}} \to \mathbb{R}^+$ be defined by:

$$\hat{H}_\phi(\mathfrak{F}|\mathfrak{C}) \equiv \inf_{\substack{\mathfrak{B} \subseteq \mathfrak{C} \\ \mathfrak{B}\ \text{abelian}}} H_\phi(\mathfrak{F}|\mathfrak{B})$$

where for every $\mathfrak{F} = \{F_i\} \in \hat{\mathfrak{F}}$, and every abelian $\mathfrak{B}$ in $\hat{\mathfrak{C}}$:

$$H_\phi(\mathfrak{F}|\mathfrak{B}) \equiv \Sigma_i \langle \phi; h[\mathscr{E}_\phi(F_i|\mathfrak{B})]\rangle$$

with h: $x \in [0, 1] \mapsto -x \log x$. *Then,* $\hat{H}_\phi$ has the following properties:

(0) $\mathfrak{B}$ *abelian in* $\hat{\mathfrak{C}}$ *implies* $\hat{H}_\phi(\mathfrak{F}|\mathfrak{B}) = H_\phi(\mathfrak{F}|\mathfrak{B})$;

(i) $\hat{H}_\phi(\mathfrak{F}|\mathfrak{C})=0$ *if and only if* $\mathfrak{F}\subseteq\mathfrak{C}$;
(ii) $\mathfrak{F}_1\subseteq\mathfrak{F}_2$ *implies* $\hat{H}_\phi(\mathfrak{F}_1|\mathfrak{C})\leqq\hat{H}_\phi(\mathfrak{F}_2|\mathfrak{C})$;
(iii) $\mathfrak{C}_1\subseteq\mathfrak{C}_2$ *implies* $\hat{H}_\phi(\mathfrak{F}|\mathfrak{C}_1)\geqq\hat{H}_\phi(\mathfrak{F}|\mathfrak{C}_2)$.

Since the assumptions of our theory are such that it reduces to classical probability theory exactly when $\mathfrak{N}$ is abelian, we should recall (see Section II) that in this particular case every (abelian!) von Neumann subalgebra of $\mathfrak{N}$ is $(\mathfrak{N}, \phi)$-admissible; it is then easy to verify that our $\hat{H}_\phi$ reduces in this case to the classical conditional entropy (defined for instance in [1] and [12]). The aim of Theorem III.1 is thus to provide a generalization of the latter concept, which we will then use to define the entropy of a general dynamical system.

To prove the theorem, we need the following generalization of the classical Jensen's inequality [12]:

Lemma III.1. *Let A be a self-adjoint element in $\mathfrak{N}$, with $0\leqq A\leqq I$; $\mathfrak{B}$ be an abelian von Neumann subalgebra of $\mathfrak{N}$, stable with respect to the automorphism group $\alpha^\beta(\mathbb{R})$ canonically associated to ϕ; and h: $x\in[0,1]\mapsto -x\log x$. Then $\mathscr{E}_\phi(h[A]|\mathfrak{B})\leqq h[\mathscr{E}_\phi(A|\mathfrak{B})]$.*

Proof. To adapt the classical proof (see for instance [12]) to the present case, we first recall that $\mathfrak{B}$ abelian implies (see for instance [3]) that there exists: a locally compact space Z; a positive measure ν on Z, with support Z; and an isometric isomorphism π from the involutive normed algebra $\mathfrak{B}$ onto the involutive normed algebra $\mathscr{L}^\infty(Z,\nu)$. We first prove the lemma for the particular case where the spectrum of A is discrete and finite, i.e. where A admits the spectral resolution $A=\Sigma_{i=1}^n a_i E_i$. In this case we thus have to prove:

$$\Sigma_{i=1}^n h(a_i)\, B_i\leqq h[\Sigma_{i=1}^n a_i B_i] \tag{$*$}$$

where $\{a_i|i=1,2,\dots,n\}\subset[0,1]$ and $\{B_i\equiv\mathscr{E}_\phi(E_i|\mathfrak{B})\big|i=1,2,\dots,n\}$. From the defining properties of the conditional expectation $\mathscr{E}_\phi(\cdot|\mathfrak{B})$ we have: $B_i\in\mathfrak{B}$; $B_i\geqq 0$; and $\Sigma_{i=1}^n B_i=I$. Consequently the $\pi(B_i)\in\mathscr{L}^\infty(Z,\nu)$ satisfy (ν-almost everywhere): $\pi(B_i)(\zeta)\geqq 0$ and $\Sigma_{i=1}^n \pi(B_i)(\zeta)=1$. We can therefore use the concavity of h to conclude:

$$\begin{aligned}\pi\big(\Sigma_i h(a_i)\, B_i\big)(\zeta) &= \Sigma_i h(a_i)\,\pi(B_i)(\zeta)\\ &\leqq h[\Sigma_i a_i \pi(B_i)(\zeta)] = h[\pi(\Sigma_i a_i B_i)(\zeta)]\\ &= \pi(h[\Sigma_i a_i B_i])(\zeta),\end{aligned}$$

where the last equality follows from the continuity of h on $[0,1]$ and the fact that π is an isometric isomorphism. We have thus established ($*$), and thus the validity of the lemma in the particular case where A has a finite discrete spectrum. We next remark that an arbitrary A in $\mathfrak{N}$, with $0\leqq A\leqq I$, can be written as the limit, in the normtopology, of an increasing sequence $\{A_n|n\in\mathbb{Z}^+\}$ of mutually commuting A_n, each of which satisfies the assumptions of the particular case in which we just established the lemma. Since the mapping $\mathscr{E}_\phi(\cdot|\mathfrak{B})$: $\mathfrak{N}\to\mathfrak{B}$ is order preserving and of norm 1, $\{\mathscr{E}_\phi(A_n|\mathfrak{B})\big|n\in\mathbb{Z}^+\}$ is an increasing sequence of positive operators, uniformly bounded by I, and converging to $\mathscr{E}_\phi(A|\mathfrak{B})$ in the norm topology. The continuity of the function h: $[0,1]\to\mathbb{R}^+$ implies thus the convergence, in the norm topology, of $h[\mathscr{E}_\phi(A_n|\mathfrak{B})]$ to $h[\mathscr{E}_\phi(A|\mathfrak{B})]$; of $h[A_n]$ to $h[A]$, and thus of $\mathscr{E}_\phi(h[A_n]|\mathfrak{B})$ to $\mathscr{E}_\phi(h[A]|\mathfrak{B})$. Consequently, the validity of the lemma follows from its validity for the particular case considered in the first part of this proof, q.e.d.

Lemma III.2. *Let $\mathfrak{F}$ be a finite partition $\{F_i\}$ of the identity in $\mathfrak{N}$; $\mathfrak{C}$ be a $(\mathfrak{N}, \phi)$-admissible von Neumann subalgebra of $\mathfrak{N}$; and $\mathfrak{B}$ be an arbitrary abelian von Neumann subalgebra of $\mathfrak{C}$. Then: $H_\phi(\mathfrak{F}|\mathfrak{B}) \geqq \Sigma_i \langle \phi; h[\mathscr{E}_\phi(F_i|\mathfrak{C})]\rangle$.*

Proof. $0 \leqq F_i \leqq I$ and $\mathscr{E}_\phi(\cdot|\mathfrak{C})$ order preserving imply $0 \leqq \mathscr{E}_\phi(F_i|\mathfrak{C}) \leqq I$. We have thus from Lemma III.1:

$$\mathscr{E}_\phi(h[\mathscr{E}_\phi(F_i|\mathfrak{C})]\,|\,\mathfrak{B}) \leqq h[\mathscr{E}_\phi(\mathscr{E}_\phi(F_i|\mathfrak{C})|\mathfrak{B})].$$

The uniqueness of the conditional expectation implies that $\mathscr{E}_\phi(\mathscr{E}_\phi(N|\mathfrak{C})|\mathfrak{D}) = \mathscr{E}_\phi(N|\mathfrak{D})$ for all N in $\mathfrak{N}$ whenever $\mathfrak{D} \subseteq \mathfrak{C}$. We can therefore rewrite the above inequality as:

$$\mathscr{E}_\phi(h[\mathscr{E}_\phi(F_i|\mathfrak{C})]\,|\,\mathfrak{B}) \leqq h[\mathscr{E}_\phi(F_i|\mathfrak{B})].$$

Consequently:

$$\Sigma_i \langle \phi; \mathscr{E}_\phi(h[\mathscr{E}_\phi F_i|\mathfrak{C})]\,|\,\mathfrak{B})\rangle \leqq \Sigma_i \langle \phi; h[\mathscr{E}_\phi(F_i|\mathfrak{B})]\rangle.$$

The RHS of this inequality is precisely $H_\phi(\mathfrak{F}|\mathfrak{B})$, whereas its LHS is indeed $\Sigma_i \langle \phi; h[\mathscr{E}_\phi(F_i|\mathfrak{C})]\rangle$, q.e.d.

Lemma III.3. *Let F be an operator on $\mathfrak{H}$, with $0 \leqq F \leqq I$. Then $h[F]=0$ if and only if F is a projector.*

Proof. Let $\mathfrak{B}$ be the abelian von Neumann algebra generated by F. With the same construction as that used to prove Lemma 1, we have $0 \leqq \pi(F)(\zeta) \leqq 1$ ν-almost everywhere. Now, to say that $F = F^2$ is equivalent to saying that $\pi(F)(\zeta)^2 = \pi(F)(\zeta)$ ν-a.e., and thus $\pi(F)(\zeta) = 0$ or 1 ν-a.e. This in turn is equivalent to saying that $h[\pi(F)(\zeta)] = 0$ ν-a.e., i.e. $\pi(h[F])(\zeta) = 0$ ν-a.e., i.e. $h[F] = 0$, q.e.d.

Lemma III.4. *Let F be a projector in $\mathfrak{N}$, and $\mathfrak{C}$ be a von Neumann subalgebra of $\mathfrak{N}$, stable under $\alpha^\beta(\mathbb{R})$. Then $\mathscr{E}_\phi(F|\mathfrak{C})$ is a projector if and only if F belongs to $\mathfrak{C}$.*

Proof. Since $\mathscr{E}_\phi(\cdot|\mathfrak{C})$ is a projection from $\mathfrak{N}$ to $\mathfrak{C}$ we clearly have $\mathscr{E}_\phi(F|\mathfrak{C}) = F$ when F belongs to $\mathfrak{C}$, so that in this case $\mathscr{E}_\phi(F|\mathfrak{C})$ is evidently a projector. Suppose now that $\mathscr{E}_\phi(F|\mathfrak{C})$ is a projector, and form the positive operator

$$X = \mathscr{E}_\phi(F|\mathfrak{C})(I - F)\mathscr{E}_\phi(F|\mathfrak{C}).$$

We have then $\mathscr{E}_\phi(X|\mathfrak{C}) = \mathscr{E}_\phi(F|\mathfrak{C}) - \mathscr{E}_\phi(F|\mathfrak{C})^3$. Since $\mathscr{E}_\phi(\cdot|\mathfrak{C})$ is faithful, we have then $X = 0$ i.e. $\mathscr{E}_\phi(F|\mathfrak{C}) = \mathscr{E}_\phi(F|\mathfrak{C}) F \mathscr{E}_\phi(F|\mathfrak{C})$ which is to say that the operator $F - \mathscr{E}_\phi(F|\mathfrak{C})$ is positive. We can thus use the fact that ϕ is faithful to conclude, from $\langle \phi; F - \mathscr{E}_\phi(F|\mathfrak{C})\rangle = 0$, that $F = \mathscr{E}_\phi(F|\mathfrak{C})$, which is to say that F belongs to $\mathfrak{C}$, q.e.d.

Proof of Theorem III.1. Property (0): Let $\mathfrak{B}$ be a $(\mathfrak{N}, \phi)$-admissible abelian subalgebra of $\mathfrak{N}$, and $\mathfrak{B}_0$ be a von Neumann subalgebra of $\mathfrak{B}$. From Lemma III.2 we conclude that $H_\phi(\mathfrak{F}|\mathfrak{B}_0) \geqq H_\phi(\mathfrak{F}|\mathfrak{B})$ for all finite partitions of the identity in $\mathfrak{N}$. Hence, with $\hat{\mathfrak{B}}$ denoting the class of all abelian $(\mathfrak{N}, \phi)$-admissible von Neumann subalgebras of $\mathfrak{N}$, $\hat{H}_\phi$ is well defined on $\hat{\mathfrak{F}} \times \hat{\mathfrak{B}}$, and is equal to H_ϕ, thus proving property (0). We should further remark here that for every finite partition $\mathfrak{F}=\{F_i\}$ of the identity in $\mathfrak{N}$, $0\leqq F_i\leqq I$ for all $i=1,2,\ldots,n$ implies $0\leqq\mathscr{E}_\phi(F_i|\mathfrak{C})\leqq I$ for every von Neumann subalgebra $\mathfrak{C}$ in $\mathfrak{N}$, stable under $\alpha^\beta(\mathbb{R})$. Consequently $h[\mathscr{E}_\phi(F_i|\mathfrak{C})]$ are positive and thus $\Sigma_i\langle\phi;h[\mathscr{E}_\phi(F_i|\mathfrak{C})]\rangle$ is positive. We can therefore conclude from Lemma 2 that at fixed $\mathfrak{F}$ in $\hat{\mathfrak{F}}$ and $\mathfrak{C}$ in $\hat{\mathfrak{C}}$, the family $\{H_\phi(\mathfrak{F}|\mathfrak{B})\}$, obtained when $\mathfrak{B}$ runs over all abelian von Neumann subalgebras of $\mathfrak{C}$, is bounded below by a positive number, so that $\hat{H}_\phi(\mathfrak{F}|\mathfrak{C})$ is indeed well-defined for every $\mathfrak{F}$ in $\hat{\mathfrak{F}}$ and every $\mathfrak{C}$ in $\hat{\mathfrak{C}}$, and takes its values in $\mathbb{R}^+$. To prove property (i) we first notice that if $\mathfrak{F}\subseteq\mathfrak{C}$ there exists an abelian von Neumann algebra $\mathfrak{B}$ with $\mathfrak{F}\subseteq\mathfrak{B}\subseteq\mathfrak{C}$. We have therefore $\mathscr{E}_\phi(F_i|\mathfrak{B})=F_i$ for all $F_i\in\mathfrak{F}$; and thus by Lemma III.3, we have: $h[\mathscr{E}_\phi(F_i|\mathfrak{B})]=0$ i.e. $H_\phi(\mathfrak{F}|\mathfrak{B})=0$ and thus $\hat{H}_\phi(\mathfrak{F}|\mathfrak{C})=0$. Conversely, if $\hat{H}_\phi(\mathfrak{F}|\mathfrak{C})=0$ we have by Lemma III.2 $\Sigma_i\langle\phi;h[\mathscr{E}_\phi(F_i|\mathfrak{C})]\rangle=0$. Since all the terms in this sum are positive, and since ϕ is faithful, we conclude that $h[\mathscr{E}_\phi(F_i|\mathfrak{C})]=0$, and thus, by Lemma III.3, $\mathscr{E}_\phi(F_i|\mathfrak{C})$ are all projectors. This implies, by Lemma III.4, that all F_i's belong to $\mathfrak{C}$, i.e. $\mathfrak{F}\subseteq\mathfrak{C}$, thus concluding the proof of property (i). From the definition of $\hat{H}_\phi$ it is clear that the validity of property (ii) follows immediately from its validity on $\hat{\mathfrak{F}}\times\hat{\mathfrak{B}}$. Let thus $\mathfrak{B}$ be an abelian $(\mathfrak{N},\phi)$-admissible von Neumann subalgebra of $\mathfrak{N}$. By the technique used in Lemma 1, one checks easily that for every finite family $\{B_i\}$ of positive elements in $\mathfrak{B}$ with $\Sigma_i B_i\leqq I$, one has $h(\Sigma_i B_i)\leqq\Sigma_i h(B_i)$. Let now $\mathfrak{F}_1=\{F_i^{(1)}|i\in I\}$ and $\mathfrak{F}_2=\{F_j^{(2)}|j\in J\}$ be two finite partitions of the identity in $\mathfrak{N}$. $\mathfrak{F}_1\subseteq\mathfrak{F}_2$ amounts to saying that there exists a partition $\{J_i|i\in I\}$ of J such that $F_i^{(1)}=\Sigma_{j\in J_i}F_j^{(2)}$ for each i in I. We now have:

$$\begin{aligned}\hat{H}_\phi(\mathfrak{F}_1|\mathfrak{B})&=\Sigma_i\langle\phi;h[\mathscr{E}_\phi(\Sigma_{j\in J_i}F_j^{(2)}|\mathfrak{B})]\rangle\\&\leqq\Sigma_i\Sigma_{j\in J_i}\langle\phi;h[\mathscr{E}_\phi(F_j^{(2)}|\mathfrak{B})]\rangle=\hat{H}_\phi(\mathfrak{F}_2|\mathfrak{B})\end{aligned}$$

which proves property (ii) for all $\mathfrak{B}$ in $\hat{\mathfrak{B}}$, and thus for all $\mathfrak{C}$ in $\hat{\mathfrak{C}}$, as we already noticed. Property (iii) follows trivially from the very definition of $\hat{H}_\phi(\mathfrak{F}|\mathfrak{C})$ as the largest lower bound of the $H_\phi(\mathfrak{F}|\mathfrak{B})$'s when $\mathfrak{B}$ runs over all abelian von Neumann subalgebras $\mathfrak{B}$ of $\mathfrak{C}$, q.e.d.

Although we do not seem to need it for the present investigation, it would be interesting to prove some theorems of the martingale type in the extended framework presented up to this point. In particular, the presentation of the theory would probably benefit from a theorem to the effect that given $\mathfrak{F}$ and $\mathfrak{C}$ $(\mathfrak{N},\phi)$-admissible, there exists a maximal abelian von Neumann subalgebra $\mathfrak{B}$ of $\mathfrak{C}$ (depending on $\mathfrak{F}$) such that $\hat{H}_\phi(\mathfrak{F}|\mathfrak{B})=\hat{H}_\phi(\mathfrak{F}|\mathfrak{C})$. In the present exploratory investigation we however choose to concentrate instead here on the application of the theory to the generalization of the Kolmogorov-Sinai entropy for the class of dynamical systems described in Section I. To do so, we need the following result which we shall now prove.

Lemma III.5. *Let $\{\mathfrak{N},\Phi,\alpha\}$ be a dynamical system, $\mathscr{T}$ be a subset of $\mathbb{R}$; and $\mathfrak{C}$ be a $(\mathfrak{N},\phi)$-admissible von Neumann subalgebra of $\mathfrak{N}$. Then the von Neumann subalgebra $\mathfrak{C}(\mathscr{T})$ of $\mathfrak{N}$ generated by $\{\alpha(t)[C]|C\in\mathfrak{C},t\in\mathscr{T}\}$ is also $(\mathfrak{N},\phi)$-admissible.*

Proof. To prove that $\mathfrak{C}(\mathscr{T})$ is $(\mathfrak{N},\phi)$-admissible is equivalent (see Corollary II.1) to proving that $\mathfrak{C}(\mathscr{T})\subseteq\mathfrak{N}_0$. It is therefore sufficient to prove the lemma for $\mathfrak{C}(t)=\alpha(t)[\mathfrak{C}]$ and every t in $\mathbb{R}$. To prove that $\mathfrak{C}(t)$ is $(\mathfrak{N},\phi)$-admissible, it is sufficient (see Corollary II.1) to prove that $\alpha(t)[\mathfrak{N}_0]$ is $(\mathfrak{N}, \phi)$-admissible, which we achieve by showing (see Corollary II.1) that the $\mathfrak{D}$-conditional expectation $\mathscr{E}_\phi(\cdot|\mathfrak{D})$ with respect to ϕ exists for every von Neumann subalgebra $\mathfrak{D}$ in $\alpha(t)[\mathfrak{N}_0]$. For any such algebra $\alpha(-t)[\mathfrak{D}]$ is contained in $\mathfrak{N}_0$, and thus the $\alpha(-t)[\mathfrak{D}]$-conditional expectation $\mathscr{E}_\phi(\cdot|\alpha(-t)[\mathfrak{D}])$ with respect to ϕ exists. Straightforward computations show then that:

$$\mathscr{P} \equiv \alpha(t)\left[\mathscr{E}_\phi(\alpha(-t)[\cdot]\,|\,\alpha(-t)[\mathfrak{D}])\right]$$

satisfies: (i) $\mathscr{P}: \mathfrak{N} \to \mathfrak{D}$; (ii) $\mathscr{P}$ is σ-weakly continuous; (iii) $\mathscr{P}$ is faithful; (iv) $\mathscr{P}(D)=D$ for every D in $\mathfrak{D}$, and thus $\mathscr{P}^2=\mathscr{P}$; (v) $\|\mathscr{P}\|=1$; and (vi) $\langle\phi; N\rangle = \langle\phi; \mathscr{P}(N)\rangle$ $\forall N$ in $\mathfrak{N}$. Hence the $\mathfrak{D}$-conditional expectation $\mathscr{E}_\phi(\cdot|\mathfrak{D})=\mathscr{P}$ with respect to ϕ indeed exists, q.e.d.

Corollary. *$\mathfrak{N}_0$ is stable under $\alpha(\mathbb{R})$.*

Proof. $\mathfrak{N}_0$ is $(\mathfrak{N}, \phi)$-admissible; hence $\alpha(t)[\mathfrak{N}_0]$ is $(\mathfrak{N}, \phi)$-admissible, i.e. $\alpha(t)[\mathfrak{N}_0] \subseteq \mathfrak{N}_0$, q.e.d.

For any $(\mathfrak{N}, \phi)$-admissible partition $\mathfrak{F}$, let $\mathfrak{C}_\mathfrak{F}$ be the (abelian) von Neumann algebra $\mathfrak{F}''$. We denote by $\mathfrak{C}_\mathfrak{F}^{(n)}$ the (not necessarily abelian!) von Neumann algebra $\mathfrak{C}_\mathfrak{F}(\{0, 1, \ldots, n-1\})$ generated by $\{\alpha(m)[F_i]\,|\,F_i \in \mathfrak{F}, m=0, 1, \ldots, n-1\}$. From Lemma III.5 we know then that $\mathfrak{C}_\mathfrak{F}^{(n)}$ is $(\mathfrak{N}, \phi)$-admissible, so that we can now define

$$\hat{H}_\phi^{(n)}(\mathfrak{F}|\alpha) \equiv \hat{H}_\phi(\alpha(n)[\mathfrak{F}]\,|\,\mathfrak{C}_\mathfrak{F}^{(n)}).$$

Clearly $\hat{H}_\phi(\alpha(n)[\mathfrak{F}]\,|\,\mathfrak{C}_\mathfrak{F}^{(n)}) = \hat{H}_\phi(\mathfrak{F}\,|\,\mathfrak{C}_\mathfrak{F}(\{-1, -2, \ldots, -n\}))$, which by Theorem III.1 (iii) is a decreasing sequence in n, bounded below by zero. Hence

$$\hat{H}_\phi(\mathfrak{F}|\alpha) \equiv \lim_{n\to\infty} \hat{H}_\phi^{(n)}(\mathfrak{F}|\alpha)$$

exists; we call this quantity the *entropy of $\mathfrak{F}$ with respect to α*. Finally we define the *entropy of the dynamical system $\{\mathfrak{N}, \Phi, \alpha\}$* as:

$$\hat{H}_\phi(\alpha) = \sup_{\mathfrak{F}\in\mathfrak{F}} \hat{H}_\phi(\mathfrak{F}|\alpha).$$

We should notice here that in the particular case where $\mathfrak{N}$ is abelian, $\hat{H}_\phi(\alpha)$ reduces to the Kolmogorov-Sinai entropy [8, 16]. One also verifies easily that $\hat{H}_\phi(\alpha)$ vanishes identically if the "true" evolution $\alpha(\mathbb{R})$ coincides with the "free" evolution $\alpha^\beta(\mathbb{R})$, or if $\alpha^\beta(\mathbb{R})$ acts ergodically on $\mathfrak{N}$ (i.e. $\mathfrak{N}_0 = \mathbb{C}I$). We shall however see in the last section of this paper that there exist dynamical systems where the following conditions are simultaneously satisfied: (i) $\mathfrak{N}$ is not abelian; (ii) $\mathfrak{N}_0$ is not abelian; (iii) $\alpha(\mathbb{R})$ acts ergodically on $\mathfrak{N}$; and (iv) $\hat{H}_\phi(\alpha) > 0$. This result will be obtained by considering the non-abelian generalization of the concept of K-flow to be introduced in the next section. To prove it, we will need the following preliminary result:

Lemma III.6. *Let $\{\mathfrak{N}, \Phi, \alpha\}$ be a dynamical system with $\hat{H}_\phi(\alpha)=0$; and $\mathfrak{C}$ be a $(\mathfrak{N}, \phi)$-admissible von Neumann subalgebra of $\mathfrak{N}$. Then $\mathfrak{C}(\mathbb{Z}_n^-) = \mathfrak{C}(\mathbb{Z}_m^-)$ for all n, m in $\mathbb{Z}$, where $\mathfrak{C}(\mathbb{Z}_n^-)$ denotes the von Neumann algebra generated by*

$$\{\alpha(k)[\mathfrak{C}]\,|\,k\in\mathbb{Z}, k \leq n\}.$$

Proof. It is sufficient to prove the lemma for abelian $(\mathfrak{N}, \phi)$-admissible von Neumann subalgebras of $\mathfrak{N}$, since every von Neumann algebra is generated by its abelian von Neumann subalgebras, and $\mathfrak{B} \subseteq \mathfrak{C} \in \hat{\mathfrak{C}}$ implies $\mathfrak{B} \in \hat{\mathfrak{C}}$. Actually the same remark shows that it is sufficient to prove that $\mathfrak{C}_{\mathfrak{F}}(\mathbb{Z}_n^-) = \mathfrak{C}_{\mathfrak{F}}(\mathbb{Z}_m^-)$ for every $\mathfrak{F}$ in $\hat{\mathfrak{F}}$ and every (n, m) in $\mathbb{Z} \times \mathbb{Z}$. To prove this, it is clearly sufficient to prove that for every $\mathfrak{F}$ in $\hat{\mathfrak{F}}$ and every n in $\mathbb{Z}$ $\mathfrak{C}_{\mathfrak{F}}(\mathbb{Z}_n^-) = \mathfrak{C}_{\mathfrak{F}}(\mathbb{Z}_{n-1}^-)$, i.e. $\alpha(n)[\mathfrak{C}_{\mathfrak{F}}(\mathbb{Z}_0^-)] = \alpha(n)[\mathfrak{C}_{\mathfrak{F}}(\mathbb{Z}_{-1}^-)]$; it is thus sufficient to prove that $\mathfrak{C}_{\mathfrak{F}}(\mathbb{Z}_0^-) = \mathfrak{C}_{\mathfrak{F}}(\mathbb{Z}_{-1}^-)$. To prove this, we notice that $\hat{H}_\phi(\alpha) = 0$ implies that for every $\mathfrak{F}$ in $\hat{\mathfrak{F}}$,

$$\lim_{n \to \infty} \hat{H}_\phi(\mathfrak{F} \mid \mathfrak{C}_{\mathfrak{F}}(\{-1, -2, \ldots, -n\})) = 0.$$

Since on the other hand $\{\mathfrak{C}_{\mathfrak{F}}(\{-1, -2, \ldots, -n\})\}$ is an increasing sequence of von Neumann algebras bounded above by $\mathfrak{C}_{\mathfrak{F}}(\mathbb{Z}_{-1}^-)$, with all these algebras in $\hat{\mathfrak{C}}$, we conclude from Theorem III.1 (iii) that $\hat{H}_\phi(\mathfrak{F} \mid \mathfrak{C}_{\mathfrak{F}}(\mathbb{Z}_{-1}^-)) = 0$, and thus, from Theorem 1 (i) that $\mathfrak{F} \subseteq \mathfrak{C}_{\mathfrak{F}}(\mathbb{Z}_{-1}^-)$. Consequently $\mathfrak{C}_{\mathfrak{F}}(\mathbb{Z}_0^-) \subseteq \mathfrak{C}_{\mathfrak{F}}(\mathbb{Z}_{-1}^-) \subseteq \mathfrak{C}_{\mathfrak{F}}(\mathbb{Z}_0^-)$, q.e.d.

IV. Non-Abelian *K*-Flows

The following generalization of the classical Kolmogorov-Sinai concept of a *K*-flow has been proposed [5]. Let $\{\mathfrak{N}, \Phi, \alpha\}$ be a dynamical system in the sense of Section I. The classical *K*-flow condition that there exists a "refining partition" is extended to the present context by the condition that there exists a von Neumann subalgebra $\mathfrak{A}$ of $\mathfrak{N}$ such that: (i) $\alpha(-t)[\mathfrak{A}] \subseteq \mathfrak{A}$ for all $t \geqq 0$; (ii) $\mathfrak{N}$ is equal to the von Neumann algebra $\bigvee_{t \in \mathbb{R}} \alpha(t)[\mathfrak{A}]$ generated by $\{\alpha(t)[A] \mid A \in \mathfrak{A}, t \in \mathbb{R}\}$; (iii) $\bigcap_{t \in \mathbb{R}} [\alpha(t)[\mathfrak{A}]\, \Phi] = \mathbb{C}\, \Phi$, where for each $t \in \mathbb{R}$, $[\alpha(t)[\mathfrak{A}]\, \Phi]$ is the closed subspace of $\mathfrak{H}$ generated by $\{\alpha(t)[A]\, \Phi \mid A \in \mathfrak{A}\}$. (A classical *K*-flow is then obtained in the particular case where $\mathfrak{N}$, or $\mathfrak{A}$, is abelian.) The fact that $\mathfrak{N}$ is not necessarily abelian is compensated by imposing the condition that $\mathfrak{A}$ be stable under $\alpha^\beta(\mathbb{R})$ (a condition which is automatically satisfied in the classical theory, where $\alpha^\beta(\mathbb{R}) = \{\mathrm{id}\}$). Finally, a genuine extension outside of the realm of the classical theory is obtained by imposing the additional requirements that: (i) ϕ is not a trace on $\mathfrak{N}$; and (ii) for every Z in $\mathfrak{N} \cap \mathfrak{N}'$ and all t in $\mathbb{R}$: $\alpha(t)[Z] = Z$. Clearly, we are now in a situation where $\mathfrak{N}$ cannot be abelian anymore. A dynamical system satisfying all the above conditions is thus called a *non-abelian K-flow*. One can prove [5] that these systems have the following properties: (a) $\mathfrak{N}$ is a type III factor; (b) $\alpha(\mathbb{R})$ acts in an ergodic manner on $\mathfrak{N}$, i.e. $\alpha(t)[N] = N$ for all t in $\mathbb{R}$, and N in $\mathfrak{N}$, imply $N = cI$ with c in $\mathbb{C}$; (c) for any invariant mean η on $\mathbb{R}$, $\alpha(\mathbb{R})$ acts in a η-abelian manner on $\mathfrak{N}$; i.e. for every N_1, N_2, N_3, N_4 in $\mathfrak{N}$: $\eta\langle \phi; N_1(\alpha(t)[N_2]\, N_3 - N_3\, \alpha(t)[N_2])\, N_4 \rangle = 0$; (d) $\alpha(\mathbb{R})$ is strongly mixing on $\mathfrak{N}$, i.e. $\lim_{t \to \infty} \langle \phi; \alpha(t)[N_1]\, N_2 \rangle = \langle \phi; N_1 \rangle \langle \phi; N_2 \rangle$ for all N_1, N_2 in $\mathfrak{N}$; (e) there exists a strongly continuous group $U(\mathbb{R})$ of unitary operators acting on $\mathfrak{H}$ such that: (i) $U(t)\, \Phi = \Phi$ for all t in $\mathbb{R}$, (ii) $\alpha(t)[N] = U(t)\, N\, U(-t)$ for all t in $\mathbb{R}$ and all N in $\mathfrak{N}$, and (iii) $U(\mathbb{R})$ has Lebesgue spectrum, i.e. there exists a $U(\mathbb{R})$-invariant decomposition $\mathfrak{H} = \oplus_{n=0}^{\infty} \mathfrak{H}^{(n)}$ with $\mathfrak{H}^{(0)} = \mathbb{C}\, \Phi$, such that for every $n \geqq 1$ the restriction $U^{(n)}(\mathbb{R})$ of $U(\mathbb{R})$ to $\mathfrak{H}^{(n)}$ is unitarily equivalent to $V(\mathbb{R}) = \{V(t) \mid t \in \mathbb{R}\}$ acting on $\mathscr{L}^2(\mathbb{R}, dx)$ as $(V(t)\, \Psi)(x) = \Psi(x - t)$. One can furthermore show [5] that $\alpha^\beta(\mathbb{R})$

commutes with $\alpha(\mathbb{R})$, and is periodic. The ergodic action of $\alpha(\mathbb{R})$ on $\mathfrak{N}$, joined to the latter fact, implies that we can use Takesaki's theory [19] of homogeneous periodic states on a von Neumann algebra to conclude that: (i) $\mathfrak{N}_0$ is a von Neumann algebra of type II_1; and (ii) every maximal abelian von Neumann subalgebra of $\mathfrak{N}_0$ is also maximal abelian as a von Neumann subalgebra of $\mathfrak{N}$. We have thus here a class of dynamical systems for which the maximal $(\mathfrak{N}, \phi)$-admissible von Neumann subalgebra $\mathfrak{N}_0$ of $\mathfrak{N}$ is quite a large (non-abelian!) subalgebra of $\mathfrak{N}$.

We might mention in closing this section that the relevance of the above structure in the physical context of quantum transport theory has been pointed out in [6] where an example of a non-abelian K-flow has been explicitly constructed.

V. Entropy of Non-Abelian K-Flows

The aim of this section is to prove the following extension of the classical Kolmogorov-Sinai theorem [8, 16].

Theorem V.1. *The entropy $\hat{H}_\phi(\alpha)$ of a non-abelian K-flow is strictly positive.*

Proof. Since $\mathfrak{N}_0$ is $(\mathfrak{N}, \phi)$-admissible, we know in particular that $\mathscr{E}_\phi(\cdot|\mathfrak{N}_0)$ exists. Since moreover $\mathfrak{N}_0$ is the fixed point von Neumann algebra of $\mathfrak{N}$ under $\alpha^\beta(\mathbb{R})$, and since $\alpha^\beta(\mathbb{R})$ is periodic (say of period T) we have (see also [19]) that for every N in $\mathfrak{N}$:

$$\mathscr{E}_\phi(N|\mathfrak{N}_0)=\frac{1}{T}\int_0^T dt\, \alpha^\beta(t)[N]$$

(the convergence of the integral being understood in the weak-operator topology). Consequently, for every von Neumann subalgebra $\mathfrak{C}$ of $\mathfrak{N}$, stable under $\alpha^\beta(\mathbb{R})$, the image of $\mathfrak{C}$ through $\mathscr{E}_\phi(\cdot|\mathfrak{N}_0)$ is contained in $\mathfrak{C}$, i.e. $\mathscr{E}_\phi(\mathfrak{C}|\mathfrak{N}_0)\subseteq\mathfrak{C}$. The relations: $\mathscr{E}_\phi(\mathfrak{C}|\mathfrak{N}_0)\subseteq\mathfrak{N}_0$ and $\mathfrak{C}\cap\mathfrak{N}_0\subseteq\mathscr{E}_\phi(\mathfrak{C}|\mathfrak{N}_0)$ are trivially verified, and thus $\mathscr{E}_\phi(\mathfrak{C}|\mathfrak{N}_0)$ is the von Neumann algebra $\mathfrak{C}\cap\mathfrak{N}_0$. In particular if $\{\mathfrak{N}, \Phi, \alpha\}$ is a non-abelian K-flow with self-refining von Neumann subalgebra $\mathfrak{A}$ (see Section IV), we denote by $\mathfrak{A}_0$ the $(\mathfrak{N}, \phi)$-admissible von Neumann algebra $\mathfrak{A}\cap\mathfrak{N}_0=\mathscr{E}_\phi(\mathfrak{A}|\mathfrak{N}_0)$. Upon using the fact that the conditional expectation $\mathscr{E}_\phi(\cdot|\mathfrak{N}_0)$ is σ-weakly continuous, we conclude that $\mathfrak{A}_0$ inherits from $\mathfrak{A}$ the following properties: (i) $\alpha(-t)[\mathfrak{A}_0]\subseteq\mathfrak{A}_0$ for all $t\geqq 0$; (ii) $\bigvee_{n\in\mathbb{Z}}\alpha(n)[\mathfrak{A}_0]=\mathfrak{N}_0$; and (iii) $\bigcap_{n\in\mathbb{Z}}\alpha(n)[\mathfrak{A}_0]=\mathbb{C}I$. Suppose now that $\hat{H}_\phi(\alpha)$ were to vanish. By Lemma III.6, this would imply that $\alpha(n)[\mathfrak{A}_0]=\mathfrak{A}_0(\mathbb{Z}_n^-)=\mathfrak{A}_0(\mathbb{Z}_0^-)=\mathfrak{A}_0$ for all n in $\mathbb{Z}$ which contradicts properties (ii) and (iii) above since we know that $\mathfrak{N}_0\neq\mathbb{C}I$ (actually $\mathfrak{N}_0$ is of type II_1!). Hence $\hat{H}_\phi(\alpha)$ cannot vanish, q.e.d.

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(Received March, 5, 1974)