Z. Wahrscheinlichkeitstheorie verw. Geb. 10, 236–251 (1968)

Compactification of the Discrete State Space of a Markov Process

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Received March 15, 1968

Summary. The countable state space of a Markov chain whose stationary transition probabilities satisfy the continuity condition (1.5) is compactified to get a state space on which the corresponding processes can be made right continuous with left limits, and strongly Markovian. There is a form of quasi left continuity, modified by the possible presence of branch points. Excessive functions are investigated.

1. Introduction

Let $(p_{ij}(.))$ be a stationary Markov chain transition function and let $p(.,.)$ be a corresponding absolute probability function, both on the parameter interval $(0, \infty)$. That is

$$p_{ij}(t) \geqq 0\,, \quad \sum_j p_{ij}(t) = 1\,, \quad p_{ij}(s+t) = \sum_k p_{ik}(s)\, p_{kj}(t) \tag{1.1}$$

and

$$p(t,i) \geqq 0\,, \quad \sum_i p(t,i) = 1\,, \quad p(s+t), j) = \sum_i p(s,i)\, p_{ij}(t)\,. \tag{1.2}$$

Here the indices i, j range through the set of integers and s, t are strictly positive real numbers. Then there is a Markov process $\{x(t),\ t > 0\}$ with state space the set of integers, for which

(1.3) $$P\{x(t) = i\} = p(t,i)$$

and, if $p(s,i) > 0$,

(1.4) $$P\{x(s+t) = j \mid x(s) = i\} = p_{ij}(t)\,.$$

In the following we shall always accept the standard continuity condition

(1.5) $$\lim_{t \to 0} p_{ii}(t) = 1$$

for all i, which implies that the functions $p_{ij}(.)$ and $p(., i)$ are continuous for all i, j and that

(1.6) $$p \lim_{t \to s} x(t) = x(s)\,, \quad s > 0\,.$$

In order to discuss sample function continuity and related properties of the process the standard procedure is to apply the notion of separability (relative to the closed sets). In the present context this means the following. Let K be a compact Hausdorff space with a countable topological basis and suppose that some countable dense subset is identified with the set of integers. That is, the new state space K is a compactification of the set of integers. Then the $x(t)$ process described above is a process with state space K and there is accordingly a separable standard modification of the process. The new process is a Markov process with the same absolute probability and transition functions as the old process, as far as integral states are concerned. The most commonly used compactification is the Alexandrov one (see [1] for example) in which the set of integers is given the discrete topology and K is constructed by adjoining a single point ∞ with the usual conventions.

To get the chains under discussion into the context of modern Markov process theory K should be chosen so that almost all sample functions of the separable process can be made right continuous with left limits. In addition the process should have the strong Markov property. To achieve the latter one must have an extended version of the given transition matrix function, adapted to K. That is one must have a transition probability $p(t, \xi, d\eta)$ for $t > 0$ (probability of a transition from ξ into $d\eta$ in time t) on K or at least on some subset K_0 so large that it supports the sample functions and their left limits, satisfying the Chapman Kolmogorov equations, with $p(t, i, j) = p_{ij}(t)$ for i, j integers. It will be shown how to define a state space K with the stated properties, and $p(.,.,j)$ (j an integer) will be continuous on $(0, \infty) \times K_0$. The process need not be a 'Hunt process', one satisfying 'hypothesis A', because of the possible existence of branch points. Note also that one cannot expect to have $p(.,.,j)$ defined and continuous on $(0, \infty) \times K$. In fact if the transition functions were that smooth the series $\sum_j p(.,.,j)$ would converge uniformly to 1 on $[1, 2] \times K$, by Dini's theorem, whereas there is not uniform convergence when the matrix $(p_{ij}(t))$ is the identity matrix for all t.

In discussing any theorem involving process separability the compactification must be specified and for a given process quite different compactifications may be appropriate for different purposes. RAY [6] treated a certain compactification. His treatment was corrected and developed further by KUNITA and WATANABE [2] and generalized by MEYER [4]. These authors studied more general state spaces than countable ones and used semigroup methods. Their campactifications, like that of NEVEU [5], who had a more specialized purpose, are essentially the same and reduce to very nearly that of the present paper when the state space is countable. WILLIAMS [7] uses what is apparently an essentially different compactification in which the sample functions do not have left limits. CHUNG [1] has the most detailed treatment for countable state spaces using the Alexandrov one point compactification of the state space (considered discrete). In the present paper probabilistic rather than semigroup methods are used. The final process on K is not obtained by first finding a transition function on K and using it to define a process but by defining the process on K as a separable standard modification of the given process whose state space has been identified with a dense subset of K. Only countable state spaces are considered here but the method is applicable more generally.

In the following sections if μ_n and μ are measures of Borel subsets of a metric space, $\lim_{n\to\infty} \mu_n = \mu$ will always mean that $\lim_{n\to\infty} \int f d\mu_n = \int f d\mu$ for every bounded continuous function on the space. We shall use the fact that if K_0 is a Borel subset of a metric space K and if μ_n and μ are supported by K_0 then $\lim_{n\to\infty} \mu_n = \mu$ in the sense of convergence of measures on K_0 if and only if the limit relation is true in the sense of convergence of measures on K. The 'if' follows from the fact that if f is a bounded continuous function on K_0, f has both an upper semicontinuous and a lower semicontinuous bounded extension to K; the 'only if' is trivial.

Let $\sum_j a_{ij}$ be a convergent series of positive terms and suppose that $\lim_{i\to\infty} a_{ij} = a_j$ and $\lim_{i\to\infty} \sum_j a_{ij} = \sum_j a_j$. Then we shall frequently use the fact (with a justification containing the words 'uniform summability') that $\sum_j a_{ij}$ converges uniformly as i varies and therefore that if $\sup |b_j| < \infty$ then $\lim_{i\to\infty} \sum_j a_{ij} b_j = \sum_j a_j b_j$.

For any function $q(.)$ from the integers to the positive reals we define $q(A) = \sum_{j\in A} q(j)$ for any set A of integers (or, if the set of integers is immersed in a metric space, for any Borel set A). If f is a function on the integers (or on the larger metric space) we define

$$q(f) = \int f(\eta)\, q(d\eta) = \sum_j q(j) f(j).$$

2. Compactification of the Set of Integers

Define $r_{ij}(\alpha)$ for $\alpha > 0$ by

$$r_{ij}(\alpha) = \alpha \int_0^\infty e^{-\alpha t} p_{ij}(t)\, dt = \alpha \int_0^\infty e^{-\alpha t}\, d\pi(t, i, j) \tag{2.1}$$

where

$$\pi(t, i, j) = \int_0^t p_{ij}(s)\, ds. \tag{2.2}$$

Let U be the class of functions $i \rightsquigarrow r_{ij}(\alpha)$ for fixed j, α with α rational and strictly positive. Then $i \rightsquigarrow \{u(i), u \in U\}$ maps the set of integers into the product space $\hat{K}$: $[0, 1] \times [0, 1] \times \cdots$. This product space in one of the usual metrics is compact. Let K be the closure in $\hat{K}$ of the image of the set of integers. If i, k go into the same point of $\hat{K}$,

$$\int_0^\infty e^{-\alpha t} p_{ij}(t)\, dt = \int_0^\infty e^{-\alpha t} p_{kj}(t)\, dt$$

for all j and for every strictly positive rational α. There is therefore equality for strictly positive real α and $p_{ij}(t) = p_{kj}(t)$ for all $t > 0$. Hence $i = k$ because of (1.5). That is we have mapped the set of integers univalently onto a dense subset of the compact metric space K. The space K is the desired compactification and the integers will be identified with their images in K. Since the coordinate functions on $\hat{K}$ separate this space their restrictions to K separate K. Each such restriction $\xi \rightsquigarrow r_{\xi j}(\alpha)$ (j, α fixed, α rational) is a continuous extension of the corresponding function $i \rightsquigarrow r_{ij}(\alpha)$. If $\xi \in K$

$$\left| \frac{r_{\xi j}(\alpha)}{\alpha} - \frac{r_{\xi j}(\beta)}{\beta} \right| = \lim_{i\to\xi} \int_0^\infty (e^{-\alpha t} - e^{-\beta t})\, p_{ij}(t)\, dt \leqq \frac{1}{\alpha} - \frac{1}{\beta} \quad (0 < \alpha < \beta) \tag{2.3}$$

for rational α and β. Hence $i \rightsquigarrow r_{ij}(\alpha)$ has a continuous extension to K for all real strictly positive α.

CHUNG [1] introduced a 'fine' topology on the integers. MEYER [4] introduced a compactification of the integers (closely related to the one used in this paper) involving a topology whose restriction to the integers he showed to be CHUNG's fine topology. We shall show that the restriction to the integers of of our K topology is also CHUNG's fine topology. MEYER showed that for fixed j and strictly positive t the function $i \rightsquigarrow p_{ij}(t)$ is continuous in CHUNG's fine topology. Our corresponding result (see Theorem 5.3) is that if $p(t, ., j)$ is the extension to K_0 of $p_{.j}(t)$, the function $(t, \xi) \rightsquigarrow p(t, \xi, j)$ (fixed j) is continuous at each point (t, ξ) with $t > 0$ and ξ in K_0.

3. Sample Function Properties

Suppose that $\{x(t), t > 0\}$ is an integer valued Markov process determined by the given transition matrix function and some absolute probability function. Then a trivial calculation derives the well known fact that for any integer j and strictly positive α the process

$$\{e^{-\alpha t} r_{x(t)j}(\alpha),\ t > 0\} \tag{3.1}$$

is a positive not necessarily separable supermartingale. We shall suppose from now on that the $x(t)$ process, considered as a process with the state space K, has been made separable by a standard modification. Since $f(.) = r_{.j}(\alpha)$ is continuous on K, the supermartingale (3.1) is now separable. Therefore almost no sample function of this supermartingale has an oscillatory discontinuity and almost every sample function has a limit at 0. Since the countable class of functions f with α rational separates K, almost every sample function of the $x(t)$ process has right limits on $[0, \infty)$ and left limits on $(0, \infty)$. In view of (1.6)

$$P\{x(t+) = x(t) = x(t-)\} = 1 \tag{3.2}$$

for all $t > 0$. Defining $x(0)$ as $x(0+)$ and replacing $x(t)$ by $x(t+)$ we have now proved the following theorem, in which we use the standard convention that a stochastic process is said to be right continuous [have left limits] if almost every sample function has this property at all points.

Theorem 3.1. *There is a Markov process $\{x(t), t \geqq 0\}$ with state space K, right continuous with left limits, having the given transition probability function and a specified absolute probability function.*

Note that $x(0)$, unlike the other random variables of the process, is not necessarily almost surely integer valued.

Define $p(t, .)$ for $t \geqq 0$ as the distribution on K of $x(t)$, so that $p(t, .)$ is supported by the integers if $t > 0$. If f is an arbitrary bounded function on K the function $p(., f)$:

$$t \rightsquigarrow \sum_j p(t, j) f(j) = \int f(\eta)\, p(t, d\eta) \qquad (t > 0)$$

is continuous (by a uniform summability argument). In particular this means that the function $t \rightsquigarrow p(t, .)$ from $(0, \infty)$ into Borel measures on K is continuous. Actually in view of (3.2) and the definition of $x(0)$ as $x(0+)$, $t \rightsquigarrow p(t, .)$ is even continuous on $[0, \infty)$.

Throughout this paper we shall consider processes as given by Theorem 3.1. For any such process, defined on a complete measure space by hypothesis, $\mathscr{F}(t)$ will denote the smallest σ-algebra of sets of the measure space containing the null sets with respect to which $x(s)$ is measurable for $s \leqq t$. Going to a standard modification of the process does not change $\mathscr{F}(t)$. Hence Chung's theorem [1, p. 166] that $\mathscr{F}(t) = \mathscr{F}(t+)$ is applicable. A positive extended real valued random variable τ is called a stopping time if $\{\tau \leqq t\}$ is in $\mathscr{F}(t)$ for every t and $\mathscr{F}(\tau)$ is the σ-algebra of sets whose intersection with $\{\tau \leqq t\}$ is in $\mathscr{F}(t)$ for all t.

4. The Support of Paths

If $r(.)$, $\pi(.)$ are functions on $(0, \infty)$, $[0, \infty)$, with $\pi(0) = 0$ and $0 \leqq \pi(t) - \pi(s) \leqq t - s$ for $s < t$, the transformation $\pi \rightsquigarrow r$ defined by

$$r(\alpha) = \alpha \int_0^\infty e^{-\alpha t}\, d\pi(t)\,, \qquad \alpha > 0\,, \tag{4.1}$$

is a transformation with a single valued inverse, and in fact the inverse is continuous in the sense that if $\pi_n \rightsquigarrow r_n$ and if $r_n \to r$ pointwise then some π exists such that $\pi \rightsquigarrow r$ and $\pi_n \to \pi$ pointwise. We conclude that $\pi(t, ., j)$ defined by (2.2) has a continuous extension (also denoted by $\pi(t, ., j)$) to K for every t, j because $r_{.j}(\alpha)$ has such an extension for every j, α. Moreover

$$r_{\xi j}(\alpha) = \alpha \int_0^\infty e^{-\alpha t}\, d\pi(t, \xi, j)\,, \qquad \xi \in K\,. \tag{4.2}$$

Define the lower semicontinuous function $r.(\alpha)$ on K by

$$r_\xi(\alpha) = \sum_j r_{\xi j}(\alpha) = \alpha \int_0^\infty e^{-\alpha t}\, d \sum_j \pi(t, \xi, j)\,. \tag{4.3}$$

Then $0 \leqq r_\xi(\alpha) \leqq 1$ and $r_\xi(\alpha) = 1$ if ξ is an integer.

Theorem 4.1. *For each ξ the function $r_\xi(.)$ is monotone decreasing and is identically 1 if it is ever 1. In the latter case, and only then, $\sum_j \pi(t, \xi, j) = t$ for all $t \geqq 0$.*

The resolvent equation

$$(\beta - \alpha) \sum_j r_{ij}(\beta)\, r_{jk}(\alpha) = \beta\, r_{ik}(\alpha) - \alpha\, r_{ik}(\beta) \tag{4.4}$$

implies that

$$(\beta - \alpha \sum_j r_{\xi j}(\beta)\, r_{jk}(\alpha) \leqq \beta\, r_{\xi k}(\alpha) - \alpha\, r_{\xi k}(\beta) \quad \text{if} \quad \alpha < \beta \tag{4.5}$$

and therefore that

$$(\beta - \alpha)\, r_\xi(\beta) \leqq \beta\, r_\xi(\alpha) - \alpha\, r_\xi(\beta) \quad \text{if} \quad \alpha < \beta\,. \tag{4.6}$$

Hence $r_\xi(.)$ is monotone decreasing. If this function is ever 1 it is clear from (4.3) that $\sum_j \pi(t, \xi, j) = t$ for $t \geqq 0$ which in turn implies that $r_\xi(\alpha) = 1$ for $\alpha > 0$, as was to be proved.

Define K_0 as the set of those ξ with $r_\xi(.) = 1$. Then if ξ is in K_0, $\sum_j \pi(t, \xi, j) = t$ for all t. The set K_0 includes the integers; it is a G_δ subset of K because $r.(\alpha)$ is lower semicontinuous. If $\pi(t, \xi, .)$ is the Borel measure defined by $j \rightsquigarrow \pi(t, \xi, j)$, (see Section 1) we shall use the fact, which follows from a uniform summability argument, that if f is a bounded function on the integers the restriction to K_0 of the function $\xi \rightsquigarrow \sum_j \pi(t, \xi, j)\, f(j)$ (t fixed) is continuous. This fact implies that the map $\xi \rightsquigarrow \pi(t, \xi, .)$ is continuous from K_0 to the set of Borel measures on K_0.

Lemma 4.2. *Let $\{y_j(t), 0 \leqq t < \infty\}$ be a positive right continuous supermartingale for $j \geqq 1$ and let $y(t) = \sum_j y_j(t)$. If $\sum_j E\{y_j(0)\} < \infty$, the process $\{y(t), t \geqq 0\}$ is a right continuous supermartingale and the series of sample functions $\sum_j y_j(.)$ almost surely converges uniformly on $[0, \infty)$.*

If $m \leqq n$ the process $\left\{\sum_m^n y_j(t),\ t \geqq 0\right\}$ is a positive right continuous supermartingale so by a standard inequality

$$P\left\{\sup_t \sum_m^n y_j(t) \geqq 1/k\right\} \leqq k E\left\{\sum_m^n y_j(0)\right\} \leqq k \sum_m^\infty E\{y_j(0)\}. \tag{4.7}$$

Then if $m = m_k$ is chosen so large that the last term in this inequality is $\leqq k^{-2}$

$$P\left\{\sup_t \sum_{m_k}^\infty y_j(t) \geqq 1/k\right\} \leqq k^{-2} \tag{4.8}$$

and, applying the Borel Cantelli lemma the asserted uniform convergence is verified. The rest of the lemma follows trivially.

Theorem 4.3. *Almost every path, together with its left limits, of a process as described in Theorem 3.1 lies in K_0.*

The process

$$\{e^{-\alpha t} r_{x(t)}(\alpha),\ t \geqq 0\} = \left\{\sum_j e^{-\alpha t} r_{x(t)j}(\alpha),\ t \geqq 0\right\} \tag{4.9}$$

is the sum of positive right continuous supermartingales. According to Lemma 4.2 the sum process is then itself right continuous and for almost every sample function, in view of the uniform convergence, any left limit at a point is the sum of the corresponding left limits. Now the sum is $e^{-\alpha t}$ if and only if $x(t)$ has its value in K_0. Since for fixed $t > 0$ $x(t)$ is almost surely an integer, the sum supermartingale is almost surely identically $e^{-\alpha t}$ for strictly positive rational parameter values and therefore almost surely identically $e^{-\alpha t}$ for all $t \geqq 0$. The left limits of the sum, equal to the sums of the left limits for almost all sample functions, are almost surely $e^{-\alpha t}$ simultaneously for all t. That is, $x(t)$ and $x(t-)$ are almost surely in K_0 simultaneously for all t, as was to be proved.

5. Extension of the Transition Functions

If A is a Borel subset of K, the functions

$$\pi(t, ., A), \quad \pi(t, ., K - A) = t - \pi(t, ., A)$$

are both lower semicontinuous on K_0. Hence $\pi(t, ., A)$ is continuous on K_0. Since $\{\pi(., \xi, A),\ \xi \in K\}$ is equicontinuous, the function $\pi(., ., A)$ is continuous on $[0, \infty) \times K_0$. In particular the function $(t, \xi) \rightsquigarrow \pi(t, \xi, .)$ from $[0, \infty) \times K_0$ into Borel measures on K (or K_0) is continuous.

Theorem 5.1. *For each ξ in K_0 and integer j,* $\pi(., \xi, j)$ has a continuous derivative *$p(., \xi, j)$ on $(0, \infty)$. If ξ is an integer, $p(., \xi, j) = p_{\xi j}(t)$. Moreover*

$$p(t, \xi, j) \geqq 0, \quad \sum_j p(t, \xi, j) = 1, \tag{5.1}$$

$$p(s + t, \xi, j) = \sum_i p(s, \xi, i)\, p(t, i, j). \tag{5.2}$$

If ξ_1, ξ_2 are points of K_0 and if $p(.,\xi_1,.) = p(.,\xi_2,.)$ then $\xi_1 = \xi_2$.

It will be shown that $\pi(.,\xi,j)$ even has a continuous derivative on $[0,\infty)$ but we shall define $p(0,\xi,j)$ not as $\pi'(0,\xi,j)$ but as the value of a certain measure $p(0,\xi,.)$ for the singleton $\{j\}$. These two values will be shown to be equal (Theorem 6.2).

We shall need an integrated version of the Chapman-Kolmogorov equations:

$$\pi(s+t,\xi,j) - \pi(t,\xi,j) = \sum_i \pi(s,\xi,i)\, p_{ij}(t)\,, \quad \xi \in K_0\,. \tag{5.3}$$

This equation is obviously true if ξ is an integer, and is therefore true for any ξ in K_0 by a continuity argument using uniform summability. Since (5.3) is satisfied, it is known ([1], p. 204) from a general theorem on functional equations of this form that $\pi(.,\xi,j)$ has a continuous derivative on $[0,\infty)$ and we denote this derivative on $(0,\infty)$ by $p(.,\xi,j)$. According to the same reference (5.2) is true. Since $\sum_j \pi(t,\xi,j) = t$ for ξ in K_0, $\sum_j p(t,\xi,j) \leqq 1$ on $(0,\infty)$, with equality (fixed ξ) for almost every t (Lebesgue measure). There is equality everywhere because by (5.2) the sum in question defines a monotone function. The last assertion of the theorem is a consequence of the fact that if ξ_1 and ξ_2 are points of K and if $r_{\xi_1 \cdot}(.) = r_{\xi_2 \cdot}(.)$ then $\xi_1 = \xi_2$.

Theorem 5.2. *The sequence $\{\xi_n, n \geqq 1\}$ in K_0 converges to the integer i if and only if either of the following equivalent conditions is satisfied.*

(a) *There is a sequence $\{\delta_n, n \geqq 1\}$ of strictly positive numbers such that*

$$\lim_{n\to\infty} \delta_n = 0\,, \quad \lim_{n\to\infty} p(\delta_n,\xi_n,i) = 1\,. \tag{5.4}$$

(b) *For all $t > 0$ (equivalently for arbitrarily small strictly positive values of t) and every integer j,*

$$\lim_{n\to\infty} p(t,\xi_n,j) = p(t,i,j)\,. \tag{5.5}$$

If (a) is satisfied,

$$\lim_{n\to\infty} p(t,\xi_n,j) = \lim_{n\to\infty} \sum_k p(\delta_n,\xi_n,k)\, p(t-\delta_n,k,j) = p(t,i,j) \tag{5.6}$$

for $t > 0$ and every integer j. Hence $\lim_{n\to\infty} r_{\xi_n j}(\alpha) = r_{ij}(\alpha)$ for all $\alpha > 0$, which implies that $\lim_{n\to\infty} \xi_n = i$. Thus condition (a) implies the stronger of conditions (b) which in turn implies that $\lim_{n\to\infty} \xi_n = i$. Conversely $\lim \xi_n = i$ implies that

$$\lim_{n\to\infty} \pi(t,\xi_n,i) = \pi(t,i,i)$$

for $t \geqq 0$. Choose integers $a_1 < a_2 < \cdots$ in such a way that

$$m\,\pi(1/m,\xi_n,i) = m\int_0^{1/m} p(s,\xi_n,i)\,ds \geqq m\,\pi(1/m,i,i) - 1/m = c_m \tag{5.7}$$

if $n \geqq a_m$. Then $\lim_{n\to\infty} c_m = 1$. For n satisfying $a_m \leqq n < a_{m+1}$ choose δ_n satisfying $0 < \delta_n \leqq 1/m$, $p(\delta_n,\xi_n,i) \geqq c_m$ to get a sequence satisfying (5.4). Finally (5.5) for $t = t_0$ and all j implies (5.5) for $t > t_0$ and all j. Hence the weaker form of (b) implies the stronger.

Theorem 5.2 implies that the restriction of the K topology to the integers is CHUNG's fine topology referred to in Section 1. (See also MEYER [4].)

Theorem 5.3. *The function $p(.,\ .,\ j)$ is continuous on $(0, \infty) \times K_0$.*

Integrating in (5.2) after interchanging s and t we obtain

$$\pi(s + t', \xi, j) - \pi(t', \xi, j) \geq p(t', \xi, j)\,\pi(s, j, j). \tag{5.8}$$

If $(t', \xi) \to (t, \eta)$, (5.8) becomes

$$\pi(s + t, \eta, j) - \pi(t, \eta, j) \geq \limsup\, p(t', \xi, j)\,\pi(s, j, j) \tag{5.9}$$

and therefore

$$p(t, \eta, j) \geq \limsup\, p(t', \xi, j). \tag{5.10}$$

Applying (5.2), if $s < t'$, $p(s, \xi, j) \leq p(t', \xi, j)\,/\,p(t' - s, j, j)$, so that if $\delta < t'$

$$[\pi(t', \xi, j) - \pi(t' - \delta, \xi, j)]\,/\,\delta \leq p(t', \xi, j)\ \sup_{s<\delta}[p(s, j, j)]^{-1}. \tag{5.11}$$

When $(t', \xi) \to (t, \eta)$ and then $\delta \to 0$ this inequality yields

$$p(t, \eta, j) \leq \liminf\, p(t, \xi, j), \tag{5.12}$$

and this inequality combined with (5.10) yields the theorem.

Note added in page proof. The proofs in the following sections do not use this theorem, whose conclusion was strengthened from upper semicontinuity to continuity (following [5]) in page proof. Some proofs, for example that of Theorem 7.1, can now be simplified, using p instead of π.

6. Transition and Absolute Probability Functions Near t = 0

An absolute probability function on K_0 is defined as a function $t \rightsquigarrow p(t, .)$ from $(0, \infty)$ into the space of probability measures on K_0 such that

$$p(s + t, A) = \int p(t, \xi, A)\, p(s, d\xi)\,, \quad 0 < s, t\,. \tag{6.1}$$

But according to (6.1) each of these measures is supported by the set of integers and we have therefore not effectively enlarged the class of absolute probability functions over that considered in Section 1. As explained in Sections 3 and 4 the measure $p(0, .)$ is defined by $\lim_{n \to 0} p(t, .)$ (convergence of measures on K) and is supported by K_0. If ξ is in K_0, $p(., \xi, .)$ is an absolute probability function and it follows that $\lim_{t \to 0} p(t, \xi, .)$ exists in the sense of convergence of measures on K and is supported by K_0. We define $p(0, \xi, .)$ as the limit measure. Then applying the remarks in Section 3 on absolute probability functions, if f is a bounded [bounded continuous] function on K_0 the function $t \rightsquigarrow \int f(\eta)\, p(t, \xi, d\eta) = p(t, \xi, f)$ is continuous on $(0, \infty)$ [$[0, \infty)$]. In particular for each ξ in K_0 the map $t \rightsquigarrow p(t, \xi, .)$ from $[0, \infty)$ into Borel measures on K_0 is continuous.

Theorem 6.1. *If ξ is in K_0, if $p(.,\ .)$ is an absolute probability function, and if $s \geqq 0$, $t \geqq 0$ then*

$$p(s + t, \xi, .) = \int p(t, \eta, .)\, p(s, \xi, d\eta) \tag{6.2}$$

and

$$p(s+t,\,.)=\int p(t,\eta,\,.)\,p(s,d\eta)\,. \tag{6.3}$$

If ξ_1, ξ_2 are points of K_0 and if $p(0,\xi_1,\,.)=p(0,\xi_2,\,.)$ then $\xi_1=\xi_2$.

Since (6.2) is a special case of (6.3) we only consider the latter, which is already known to be true when s and t are strictly positive. If f is continuous on K, $p(.,f)$ is continuous on $[0,\infty)$. Eq. (6.3) can be written in the equivalent form

$$p(s+t,f)=\int p(t,\eta,f)\,p(s,d\eta) \tag{6.4}$$

with f continuous on K. When $t\to 0$ in (6.4) with $s>0$ the resulting equation is equivalent to (6.3) with $s>0$, $t=0$. We now use an integrated version of (6.4): defining $\pi(t,j)=\int_0^t p(s,j)\,ds$,

$$\pi(s+t,f)-\pi(s,f)=\int\pi(t,\eta,f)\,p(s,d\eta)\,. \tag{6.5}$$

When $t>0$ and $s\to 0$ (6.5) yields

$$\pi(t,f)=\int\pi(t,\eta,f)\,p(0,d\eta)\,, \tag{6.6}$$

which implies that

$$p(t,f)=\int p(t,\eta,f)\,p(0,d\eta)\,, \tag{6.7}$$

and this equation is equivalent to (6.3) with $s=0$, $t>0$. When $t\to 0$ in (6.7) the equation becomes equivalent to (6.3) with $s=t=0$. Finally if $p(t,\xi_1,\,.)=p(t,\xi_2,.)$ when $t=0$ this equation will be true for all t by (6.2); hence $\xi_1=\xi_2$ by Theorem 5.1.

For $s=0$, (6.3) states that every absolute probability function is an integral average of those determined by the transition function.

If $\xi\in K_0$ and if $p(0,\xi,\xi)<1$, ξ will be called a branch point. The set of branch points will be denoted by K_b.

Theorem 6.2. *Suppose that $(t,\zeta')\in[0,\infty)\times K_0$ and that $(t,\zeta')\to(0,\zeta)$.*

(a) *If $\zeta\in K_0-K_b$ then* $\lim p(t,\zeta',\,.)=p(0,\zeta,\,.)$ *in the sense of convergence of measures on K.*

(b) *If $\zeta\in K_0$ then*

$$\lim_{t\to 0}p(t,\zeta,j)=p(0,\zeta,j)=\limsup p(t,\zeta',j)\,. \tag{6.8}$$

Suppose that $(t_n,\zeta_n)\to(0,\zeta)$ with $t_n\geqq 0$ and ζ_n, ζ in K_0 and suppose also that $\lim_{n\to\infty}p(t_n,\zeta_n,\,.)=\mu$ exists in the sense of convergence of measures on K. Integrating in (5.2) after interchanging s and t we obtain

$$\pi(s+t,\xi,j)-\pi(t,\xi,j)=\int\pi(s,\eta,j)\,p(\;,\xi\;d\eta)\,. \tag{6.9}$$

Replacing (t,ξ) by (t_n,ζ_n) we obtain, when $n\to\infty$,

$$\pi(s,\zeta,j)=\int\pi(s,\eta,j)\,\mu(d\eta) \tag{6.10}$$

Summing over j we find that $\sum\pi(s,\eta,j)=s$ for μ almost every η and therefore that the sum is identically s for μ almost every η. That is, μ is supported by K_0. Then (6.10) implies

$$p(s,\zeta,j)=\int p(s,\eta,j)\,\mu(d\eta)\,,\quad s>0\,, \tag{6.11}$$

and therefore, since $\lim_{s\to 0} p(s, \eta, .) = p(0, \eta, .)$ in the sense of convergence of measures on K whenever η is in K_0,

$$p(0, \zeta, A) = \int p(0, \eta, A)\,\mu(d\eta) \tag{6.12}$$

for every Borel set A. To prove (a) suppose that ζ is in $K_0 - K_b$ and choose $A = \{\zeta\}$. The left side of (6.12) becomes 1 and we conclude that $p(o, \eta, \zeta) = 1$, μ almost everywhere. This is impossible unless $\mu(\zeta) = 1$ according to the last assertion of Theorem 6.1. Thus every limit measure μ is supported by $\{\zeta\}$ and therefore (a) is true. To prove (b) choose (t_n, ζ_n) with the additional condition that $\lim_{n\to\infty} p(t_n, \zeta_n, j)$ exists and is the maximum of $p(0, \zeta, j)$ and the superior limit in (6.8). (This may require that $(t_n, \zeta_n) = (0, \zeta)$ for large n.) The fact that a singleton is a closed set implies that

$$\mu(j) \geqq \lim_{n\to\infty} p(t_n, \zeta_n, j)\,, \tag{6.13}$$

(6.12) implies that

$$p(0, \zeta, j) \geqq \mu(j) \tag{6.14}$$

and (6.11) implies that

$$\liminf_{t\to 0} p(t, \zeta, j) \geqq \mu(j)\,. \tag{6.15}$$

According to these three relations

$$p(0, \zeta, j) \geqq \mu(j) \geqq \limsup p(t, \zeta', j) \vee p(0, \zeta, j) \geqq \limsup_{t\to 0} p(t, \zeta, j) \geqq \liminf_{t\to 0} p(t, \zeta, j) \geqq \mu(j) \tag{6.16}$$

from which (6.8) follows at once.

7. The Strong Markov Property and Discontinuities at Stopping Times

Let $\{x(t), \mathscr{F}(t), t \geqq 0\}$ be defined as in Section 3. If $\mathscr{F}_n$ is a σ-algebra, $\bigvee_n \mathscr{F}_n$ denotes the smallest σ-algebra containing every $\mathscr{F}_n$.

Theorem 7.1. *The process $\{x(t), \mathscr{F}(t), t \geqq 0\}$ has the strong Markov property in the sense that if τ is a stopping time, if A is a Borel subset of K_0, and if $t \geqq 0$,*

$$P\{\tau < \infty, x(\tau + t) \in A \,|\, \mathscr{F}(\tau)\} = p(t, x(\tau), A) \tag{7.1}$$

almost everywhere where $\tau < \infty$. Moreover if $\{\tau_n, n \geqq 1\}$ is an increasing sequence of stopping times with limit τ and if A and t are as above,

$$P\{\tau < \infty, x(\tau + t) \in A \,|\, \bigvee_n \mathscr{F}(\tau_n)\} = p(t, x(\tau -), A) \tag{7.2}$$

almost everywhere on the set where $\tau_n < \tau < \infty$ for all n.

Since the proof of the strong Markov property is standard except that $p(., ., .)$ is replaced by $\pi(., ., .)$ in a few places, the proof is only sketched. We can assume that τ is bounded. Let $\{\tau_n, n \geqq 1\}$ be a decreasing sequence of discrete bounded stopping times with limit τ and let f be continuous on K. The usual argument shows that if $s > 0$

$$E\{f[x(\tau_n' + s)] \,|\, \mathscr{F}(\tau_n')\} = p(s, x(\tau_n'), f) \tag{7.3}$$

with probability 1 and it follows that

$$E\{f[x(\tau_n' + s)] \mid \mathscr{F}(\tau)\} = E\{p(s, x(\tau_n'), f) \mid \mathscr{F}(\tau)\} \tag{7.4}$$

with probability 1. Then

$$E\left\{\int_0^t f[x(\tau_n' + s)]\, ds \mid \mathscr{F}(\tau)\right\} = E\{\pi(t, x(\tau_n'), f) \mid \mathscr{F}(\tau)\} \tag{7.5}$$

with probability 1. When $n \to \infty$ this equation becomes

$$E\left\{\int_0^t f[x(\tau + s)]\, ds \mid \mathscr{F}(\tau)\right\} = E\{\pi(t, x(\tau), f) \mid \mathscr{F}(\tau)\} = \pi(t, x(\tau), f) \tag{7.6}$$

with probability 1, and on taking the derivative with respect to t we obtain an equation equivalent to (7.1), valid for $t \geqq 0$. To prove (7.2) we remark first that (3.1) with $t \geqq 0$ is a right continuous supermartingale relative to the family of σ-algebras $\{\mathscr{F}(t),\ t \geqq 0\}$. By a standard supermartingale stopping time theorem

$$E\{e^{-\alpha\tau} r_{x(\tau)j}(\alpha) \mid \mathscr{F}(\tau_n)\} \leqq e^{-\alpha\tau_n} r_{x(\tau_n)j}(\alpha) \tag{7.7}$$

almost everywhere where τ_n is finite. Here the obvious interpretation is to be made when an exponent is $-\infty$. When $n \to \infty$ we deduce

$$E\{e^{-\alpha\tau} r_{x(\tau)j}(\alpha) \mid \vee_n \mathscr{F}(\tau_n)\} \leqq e^{-\alpha\tau} r_{x(\tau-)j}(\alpha) \tag{7.8}$$

almost everywhere on the set Λ where $\tau_n < \tau < \infty$ for all n, that is

$$E\{1_{\tau<\infty} r_{x(\tau)j}(\alpha) \mid \vee_n \mathscr{F}(\tau_n)\} \leqq r_{x(\tau-)j}(\alpha) \tag{7.9}$$

almost everywhere on Λ. Summing over j yields 1 on both sides because both $x(\tau)$ and $x(\tau-)$ have their values in K_0. There is therefore equality almost everywhere on Λ in (7.9). The exceptional null set can be chosen to be independent of α and j. Hence if f is bounded and continuous on K_0

$$E\{1_{\tau<\infty}\, \alpha \int_0^\infty e^{-\alpha s} p(s, x(\tau), f)\, ds \mid \vee_n \mathscr{F}(\tau_n)\} = \alpha \int_0^\infty e^{-\alpha s} p(s, x(\tau-), f)\, ds \tag{7.10}$$

almost everywhere on Λ. When $\alpha \to \infty$ in (7.10) we obtain

$$E\{1_{\tau<\infty} p(0, x(\tau), f) \mid \vee_n \mathscr{F}(\tau_n)\} = p(0, x(\tau-), f) \tag{7.11}$$

almost everywhere on Λ. Since this equation is true for f continuous and bounded on K_0 it is true for f Borel measurable and bounded on K_0. If $t \geqq 0$ is fixed and f is replaced by $p(t, ., f)$ (7.11) becomes the same equation with 0 replaced by t. Finally if in this modification of (7.11) f is replaced by the indicator function of a Borel set A and if (7.1) is applied we obtain the desired (7.2).

Eq. (7.1) implies that $x(\tau + t)$ is almost surely integral valued when $t > 0$, and under this restriction on t the set A in (7.1) and (7.2) can be taken as an integer singleton without loss of force.

According to the next theorem we can write $x(\tau)$ instead of $x(\tau-)$ in (7.2) when the value of $x(\tau-)$ is not a branch point.

Theorem 7.2. *(Quasi left continuity). If $\{\tau_n, n \geqq 1\}$ is an increasing sequence of stopping times with limit τ, $x(\tau) = x(\tau-)$ almost everywhere where $\tau_n < \tau < \infty$ for all n and $x(\tau-)$ is not a branch point.*

We can assume that τ is finite valued (or replace τ by $\tau \wedge c$ and let $c \to \infty$ later). Let Λ_0 be the set of points where $\tau_n < \tau$ for all n and where $x(\tau-)$ is not a branch point. According to (7.2) if f is continuous on K

$$E\{f[x(\tau)] \mid \vee_n \mathscr{F}(\tau_n)\} = f[x(\tau-)] \tag{7.12}$$

almost everywhere on Λ_0. Then $[f[x(\tau)] - f[x(\tau-)]]^2$ has integral 0 over Λ_0 for every f, and this fact implies that $x(\tau) = x(\tau-)$ almost everywhere on Λ_0, as was to be proved.

8. Branch Points

Lemma 8.1. *The set K_b of branch points is a Borel set and*

$$p(s, \xi, K_b) = 0 \tag{8.1}$$

for $s \geqq 0$, ξ in K_0.

The set $K_0 - K_b$ is the set of those points ξ in K_0 for which $f(\xi) = \lim_{t\to 0} p(t, \xi, f)$ for every function f continuous on K or equivalently for every f in a countable dense subset of $C(K)$ (supremum norm). Hence K_b is a Borel set. If f is continuous on K and if $\{x(t), t \geqq 0\}$ is a process as in Theorem 3.1,

$$f[x(0)] = \lim_{t\to 0} E\{f[x(t)] \mid x(0)\} = \lim_{t\to 0} p(t, x(0), f) \tag{8.2}$$

with probability 1. Thus if the absolute probability function of the process is $p(., \xi, .)$ for some ξ in K_0, $f(\eta) = p(0, \eta, f)$ $p(0, \xi, d\eta)$ almost everywhere. If this result is applied to a sequence of functions f dense in $C(K)$ we find that $p(0, \eta, \eta) = 1$ for $p(0, \xi, d\eta)$ almost every η, that is, (8.1) is true for $s = 0$. Eq. (6.2) with $t = 0$ yields (8.1) for $s > 0$.

Lemma 8.2. *If $\{x(t), \mathscr{F}(t), t \geqq 0\}$ is a process as described in Section 3 and if τ is a stopping time, $x(\tau)$ (on the set where $\tau < \infty$) is almost never in K_b.*

This fact follows at once from (7.1) with $t = 0$ and $A = K_b$, since the left side has the value 1 almost everywhere where $\tau < \infty$ and $x(\tau)$ is in K_b, whereas the right side is 0 when $\tau < \infty$ according to the preceding lemma.

Theorem 8.3. *If $\{x(t), \mathscr{F}(t), t \geqq 0\}$ is a process as described in Section 3, almost no path meets K_b.*

Let δ be the probability that a path meets K_b. In the terminology of Meyer [3, p. 157] the process is well measurable (since it is right continuous with left limits). Hence according to a theorem of Meyer [3, p. 162] there is a stopping time τ such that $x(\tau)$ lies in K_b when $\tau < \infty$, and $P\{\tau < \infty\} \geqq \delta/2$. But then according to Lemma 8.2, $\tau = \infty$ almost everywhere and it follows that $\delta = 0$, as was to be proved.

9. Excessive Functions

If f is a positive extended real valued function on the integers and if $\alpha \geqq 0$ the function u on K_0 defined by

$$u(\xi) = \int_0^\infty e^{-\alpha t} p(t, \xi, f)\, dt = \sum_j r_{\xi j}(\alpha) f(j)/\alpha \tag{9.1}$$

will be called the α-potential of f. The function u is lower semicontinuous. (The second expression for u is of course valid only if $\alpha > 0$.) Let u be a positive extended real valued function defined either on the set of integers or on K_0, satisfying

$$u(\xi) \geqq u_t(\xi) = e^{-\alpha t} p(t, \xi, u) \tag{9.2}$$

on its domain for some $\alpha \geqq 0$ and all $t > 0$. Such a function will be called α-supermedian. If u is α-supermedian u_t is also α-supermedian, that is for fixed ξ, $u_.(\xi)$ is monotone decreasing. Let $u_0 = \lim_{t\to 0} u_t$. Then $u_0 \leqq u$, and if the two functions are identical u will be called α-excessive. If u is supermedian on either domain and $t \geqq 0$, u_t is α-excessive. If u is α-supermedian and if the domain is the set of integers, u is necessarily α-excessive; if the domain is K_0, $u_0 = u$ on the integers. In fact in either case if i is an integer

$$u_0(i) \geqq \lim_{t\to 0} u(i)\, p(t, i, i) = u(i)\,. \tag{9.3}$$

If u is α-supermedian on K_0 the restriction of u to the set of integers is α-excessive. If two α-excessive functions on K_0 are equal on the integers the functions are identical. The limit of an increasing sequence of α-supermedian [α-excessive] functions is α-supermedian [α-excessive].

If u is α-supermedian and $t > 0$, define $\hat{u}_t$ on the domain of u by

$$\hat{u}_t(\xi) = \int_0^t e^{-\alpha s} p(s, \xi, u)\, ds/t = \sum_j r_{\xi j}(\alpha)\,[u(j) - e^{-\alpha t} p(t, j, u)]/\alpha t\,, \tag{9.4}$$

where the last expression is valid if $\int_0^\infty e^{-\alpha s} p(s, \xi, u)\, ds < \infty$, and where $r_{\xi j}(\alpha)/\alpha$ is to be replaced by its defining integral if $\alpha = 0$. Then $\hat{u}_t$ is α-excessive and increases to u_0 when t decreases to 0. Moreover $\pi(t, ., j)$ is continuous on K and it follows from integration by parts that $\int_0^t e^{-\alpha s} p(s, ., j)\, ds$ is continuous on K_0. It is now clear that $\hat{u}_t$ is lower semicontinuous for fixed t and that therefore u_0, and accordingly every α-excessive function, is lower semicontinuous.

Theorem 9.1. *If u is an α-supermedian function on K_0 and is lower semicontinuous then $u = u_0$ on $K_0 - K_b$.*

Since u is lower semicontinuous

$$u_0(\xi) = \lim_{t\to 0} p(t, \xi, u) \geqq p(0, \xi, u)\,. \tag{9.5}$$

If ξ is not a branch point this inequality reduces to $u_0(\xi) \geqq u(\xi)$ and since the reverse inequality is true the theorem is proved.

Theorem 9.2. *If u and v are α-excessive functions on the integers, $u \wedge v$ is also. If u and v are α-excessive functions on K_0, $u \wedge v$ is α-supermedian and coincides on $K_0 - K_b$ with the α-excessive function $(u \wedge v)_0$.*

If u and v are α-supermedian, $u \wedge v$ is also. Thus the first assertion of the theorem is true. If the domain is K_0, $u \wedge v$ is α-supermedian and lower semicontinuous. Hence $u \wedge v = (u \wedge v)_0$ on $K_0 - K_b$ by the preceding theorem.

According to their definition α-excessive functions on K_0 are determined by their values on the integers. The following theorem exhibits this fact explicitly and sharpens the lower semicontinuity property of these functions.

Theorem 9.3. *Let u be an α-excessive function on K_0.*

(a) *For every ξ in K_0, $u(\xi) = p(0, \xi, u)$.*

(b) *The restriction of u to $K_0 - K_b$ is continuous at each integer.*

(c) *If $\xi \in K_0 - K_b$, $u(\xi) = \liminf_{i \to \xi} u(i)$ (where i is restricted to the integers).*

Note that (a) is trivial unless ξ is a branch point and that (c) is weaker than (b) if ξ is an integer.

The truth of (a) follows from

$$p(0, \xi, u) = \lim_{t \to 0} \int e^{-\alpha t} p(t, \eta, u)\, p(0, \xi, d\eta) = \lim_{t \to 0} p(t, \xi, u) = u(\xi). \tag{9.6}$$

To prove (b) suppose first that u is bounded. Let i be an integer and choose ξ_n in K_0 in such a way that $\lim_{n \to \infty} \xi_n = i$, $\lim_{n \to \infty} u(\xi_n) = \limsup_{\xi \to i} u(\xi)$. There is then a sequence $\{\delta_n, n \geqq 1\}$ satisfying (5.4) Moreover, as a glance at the proof of Theorem 5.2 shows, δ_n can be chosen to make the sequence converge to 0 arbitrarily fast. We can therefore suppose that δ_n is so small that

$$u(\xi_n) \leqq e^{-\alpha \delta_n} p(\delta_n, \xi_n, u) + 1/n. \tag{9.7}$$

We conclude, using the boundedness of u, that

$$\limsup_{\xi \to i} u(\xi) = \lim_{n \to \infty} u(\xi_n) \leqq u(i), \tag{9.8}$$

and therefore that u is upper semicontinuous at i. Since u is lower semicontinuous, u must be continuous at i. If u is not bounded and if m is a positive integer the restrictions to $K_0 - K_b$ of $u \wedge m$ and of the bounded α-excessive function $(u \wedge m)_0$ are identical, according to Theorem 9.2. Hence the restriction to $K_0 - K_b$ of $u \wedge m$ and therefore also that of u are continuous at each integer. In proving (c) we can suppose that ξ is not an integer, because otherwise (c) is weaker than (b), and let m_A be the infimum of u on the set of integers in an open set A containing ξ. Then

$$u(\xi) = \lim_{t \to 0} u_t(\xi) \geqq \lim_{t \to 0} m_A\, p(t, \xi, A) = m_A \tag{9.9}$$

and this inequality combined with the lower semicontinuity of u implies the truth of (c).

Theorem 9.4. *An α-excessive function on the integers is continuous.*

The proof of Theorem 9.3 (b) is applicable, with a simplification in the unbounded case made possible by the fact that if u is α-excessive on the integers $u \wedge m$ is also, for m a positive constant.

If u is an α-excessive function on the integers, and if u' is defined as u on the integers, as ∞ elsewhere on K_0, u' is α-supermedian and u_0' is therefore a function which is an α-excessive extension of u to K_0. Such an extension is trivially unique.

If u is an α-excessive function on the integers, $\{u \wedge m, m \geqq 1\}$ is an increasing sequence of bounded α-excessive functions with limit u. If u is an α-excessive function on K_0, $\{(u \wedge m)_0,\ m \geqq 1\}$ is an increasing sequence of α-excessive functions, with limit u because the limit is equal to u on the integers. When $\alpha > 0$ the next theorem strengthens this remark because the α-potential of a positive function f is α-excessive, and is bounded if f is bounded and if $\alpha > 0$.

Theorem 9.5. *If $\alpha > 0$ an α-excessive function is the limit of an increasing sequence of α-potentials of positive bounded functions.*

We treat only the case when the domain of the given function is K_0. The truth of the theorem when the domain is the set of integers then follows trivially. If u is a bounded and α-supermedian function on K_0, $\hat{u}_t$ is the α-potential of the bounded function $[u - e^{-\alpha t} p(t, ., u)]/t$, and if u is α-excessive $\hat{u}_t$ increases to u when $t \to 0$. If u is not bounded let m be a positive integer and define $u^{(m)}$ as the α-supermedian function $u \wedge m$. Then $\hat{u}^{(m)}_{1/m}$ is an α-excessive function,

$$\hat{u}^{(m)}_{1/m}(\xi) = m \int_0^{1/m} e^{-\alpha s} p(s, \xi, u \wedge m)\, ds\,, \tag{9.10}$$

and the sequence $\{\hat{u}^{(m)}_{1/m},\ m \geqq 1\}$ is a monotone increasing sequence of potentials of bounded functions. The limit of the sequence is at most u and at least

$$\lim_{m\to\infty} n \int_0^{1/n} e^{-\alpha s} p(s, \xi, u \wedge m)\, ds = \hat{u}_{1/n}(\xi) \tag{9.11}$$

for every positive integer n. The limit is therefore u and the proof of the theorem is complete.

Theorem 9.6. *If u is an α-excessive function on K_0, the $u[x(t)]$ process is right continuous with left limits for any $x(t)$ process as described in Theorem* 3.1.

Suppose first that u is bounded. The sum $\sum_j \int_0^t e^{-\alpha s} p(s, ., j)\, ds$ is a sum of continuous functions on K_0 (for fixed t) with a continuous limit. Hence (uniform summability) $\hat{u}_s$ is continuous as well as α-excessive on K_0. For such a function u, $\{e^{-\alpha t}\hat{u}_s[x(t)],\ t \geqq 0\}$ is therefore a right continuous supermartingale which increases when $s \to 0$ to $\{e^{-\alpha t} u[x(t)],\ t \geqq 0\}$ and, by a theorem of Meyer [3] the latter process must therefore also be right continuous. This process is therefore a separable supermartingale and as such has left limits. If u is not bounded, $(u \wedge m)_0$ is bounded, α-excessive and equal to $u \wedge m$ on $K_0 - K_b$, for m a positive integer. Since the $(u \wedge m)_0 [x(t)]$ process is right continuous with left limits and since almost no $x(t)$ process sample path meets a branch point, the $(u \wedge m)[x(t)]$ process must also be right continuous with left limits, for every m, and therefore the $u[x(t)]$ process is also right continuous with left limits, as was to be proved.

This section, which does not pretend to completeness, shows that the complications in the study of α-excessive functions created by the possible presence of branch points are not serious.

10. Example

Under the hypothesis (1.5), $p'_{ij}(0) = q_{ij} < \infty$ (if $i \neq j$) and $p'_{ii}(0) = -q_i \geqq -\infty$ exist. Define $q_{ii} = 0$. Suppose that, for all i, $q_i = \sum_j q_{ij} < \infty$, and to avoid trivialities suppose that, for all i, q_i is strictly positive. Then (q_{ij}/q_i) is a stochastic matrix. If some initial distribution on the integers is specified and a corresponding stochastic process is defined, using the given transition function, separable for the Alexandrov compactification, the sample functions almost all have successive jumps, the transitions at the jumps being governed by the above stochastic matrix, at times $\tau_1, \tau_2, \ldots$. Let $\tau = \lim_{n\to\infty} \tau_n$ and suppose that τ is almost surely finite, to make the sample function analysis more interesting. Then $x(\tau-)$ need not exist. If however the compactification studied in this paper is used, $x(\tau-)$ exists almost surely for a process as described in Theorem 3.1, and the distribution of $x(\tau)$ relative to $\bigvee_n \mathscr{F}(\tau_n)$ is $p(0, x(\tau-), .)$. If the value of $x(\tau-)$ is not a branch point, $x(\tau)$ and $x(\tau-)$ are almost surely equal. Suppose that there is a distribution μ on the integers such that, regardless of the past before time τ, $x(\tau)$ has the distribution μ. Then almost surely $p(0, x(\tau-), .) = \mu(.)$. According to the last assertion of Theorem 6.1 two different values of ξ in K_0 give different measures $p(0, \xi, .)$. Hence the distribution of $x(\tau-)$ must be supported by a singleton $\{\xi\}$. In other words the limit $x(\tau-)$ is the same point ξ for almost all paths, no matter what the jump matrix (q_{ij}/q_i) is. If ξ is not a branch point, $x(\tau)$ and $x(\tau-)$ are almost surely equal, from which we conclude that ξ is an integer and $\mu(\xi) = 1$. Thus unless μ is supported by a singleton the unique limit ξ must be a branch point. In the Martin compactification of the set of integers, as determined by the jump matrix, it is still true that the limit $x = \lim_{n\to\infty} x(\tau_n)$ exists almost surely, as a point of the Martin boundary in this context, but this compactification and therefore the properties of x depend only on the jump matrix and not on the character of the process after time τ. If in the preceding discussion μ is allowed to depend on the value of x on the Martin boundary and is never supported by a singleton, a process is obtained for which, roughly, to each possible value of x corresponds a branch point of K_0 and different points of the Martin boundary correspond to different branch points if and only if the corresponding choices of μ are different.

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