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Optional Supermartingales and the Andersen-Jessen Theorem*

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§1. Introduction and Results

Let $(\Omega, \mathscr{F}^0, P)$ be a probability space and Q a finite positive measure on $\mathscr{F}^0$. If $\mathscr{G}$ is any sub σ-field of $\mathscr{F}^0$, we write $Q_{\mathscr{G}}$ for the restriction of Q to $\mathscr{G}$. A random variable ξ will be called a *density of Q over $\mathscr{G}$*, understood relative to P, if

(i) ξ is $\mathscr{G}$-measurable and has its values in $[0, \infty]$ (written $\xi \in (\mathscr{G})_+$),
(ii) The Lebesgue decomposition of $Q_{\mathscr{G}}$ relative to $P_{\mathscr{G}}$ is

$$Q_{\mathscr{G}}(A) = E(\xi; A) + Q(A, \xi = \infty).$$

The "classical" theorem of Andersen and Jessen [2] deals with the following situation: $\{\mathscr{F}_n^0\}$ is an increasing sequence of σ-fields whose union generates $\mathscr{F}^0$, Q is a finite measure on $\mathscr{F}^0$, and X_n is a density of Q over $\mathscr{F}_n^0$ for each n (which easily implies that (X_n) is a supermartingale). The statement is

(1) **Theorem.** [2] (a) $X_\infty \equiv \lim_n X_n$ *exists* P-a.s. *and* Q-a.s.

(b) X_∞ *is a density of Q over* $\mathscr{F}^0$.

In particular, X_∞ is finite P-a.s. and will be finite Q-a.s. iff Q is P-absolutely continuous. An account of this theorem similar to that in [2] will be found in [8, pp. 369–374]; one finds there also a reversed-time version of the theorem for a decreasing sequence of σ-fields. We remark that, if N is a stopping time, then X_N, defined in the obvious way, will be a density of Q over $\mathscr{F}_N^0$.

Let $\{\mathscr{F}_t^0\}$, $t \geqq 0$, be an increasing family of sub σ-fields (briefly: a *filtration*) whose union generates $\mathscr{F}^0$; *no assumptions of right continuity or completeness are made*. The family of stopping times (finite or not) of $\{\mathscr{F}_t^0\}$ (respectively $\{\mathscr{F}_{t+}^0\}$) is denoted $\mathscr{S}^0$ (resp. $\mathscr{S}_+^0$). If Q is a finite measure on $\mathscr{F}^0$ we may again consider the family of densities X_t of Q over $\mathscr{F}_t^0$, but in order to have nice properties, it is necessary to choose the process more carefully. The main result of this paper is the following continuous-time analogue of the Andersen-Jessen theorem:

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(2) **Theorem.** *There exists a process $X=(X_t)$, unique up to $P+Q$-evanescence, such that*

(a) *X is optional (= well-measurable) and takes its values in $[0, \infty]$,*

(b) *X_T is a density of Q over $\mathscr{F}_T^0$ for each $T\in\mathscr{S}^0$,*

(c) *X has a left (resp. right) limit at every $t\in]0, \infty]$ (resp. $t\in[0, \infty[$) $P+Q$-a.s.,*

(d) *X remains at 0 (resp. at ∞) forever after touching 0 (resp. ∞) $P+Q$-a.s.,*

(e) *X_{T+} (resp. X_{T-}) is a density of Q over $\mathscr{F}_{T+}^0$ (resp. $\mathscr{F}_{T-}^0$) for every $T\in\mathscr{S}_+^0$ (resp. predictable $T\in\mathscr{S}_+^0$).*

We will say that Q *dominates* X in this case.

Partial results along the lines of Theorem (2) appear in [9] and [13]; these were found independently by the author. The theorem is new in full generality, with an important observation having been supplied by P.A. Meyer. Similarly some of the results used in the proof of (2), e.g. the result of (4) below, have been part of the (unpublished) folklore of martingale theory, and were also rediscovered in connection with the present work. Thus the main contributions of this paper are the synthesis of these components to obtain Theorem (2) and the formal exposition of the results on strong supermartingales.

As we shall see, the limits in (c) are finite P-a.s. and positive Q-a.s. X_T is to be interpreted as $X_\infty = \lim_{t\to\infty} X_t$ whenever $T=\infty$, and the meaning of "touching 0" is this: X touches 0 at t if at least one of X_{t-}, X_t, X_{t+} equals 0; similarly for "touching ∞".

Conditions (a) and (b) of Theorem (2) imply that X is a strong supermartingale in the sense of the following definition:

Definition. An optional process $Y=(Y_t)$ with values in $[0, \infty]$ is a *strong supermartingale*, understood relative to $\{\mathscr{F}_t^0\}$, P, unless otherwise stated, if Y has the *optional sampling property:* if $S, T\in\mathscr{S}^0$ are bounded, $S\leqq T$, then Y_S, Y_T are integrable and

(3) $Y_S\geqq E(Y_T\mid\mathscr{F}_S^0)$ a.s.

Similarly, Y is a *strong martingale* if equality holds in (3).

The same phrase will be used, with an obvious meaning, when P is a bounded, positive measure with mass different from 1. A simple argument shows that (3) holds without the boundedness or even finiteness of the stopping times, if we count $Y_T=0$ when $T=\infty$. A right continuous (nonnegative) supermartingale relative to a right continuous filtration is strong, and the following recent result of [4] shows that there are plenty of others: if $U=(U_t)$ is an arbitrary nonnegative supermartingale, then there is a *strong* supermartingale U^* such that $U_t^*=U_t$ a.s. for each t.

Properties (c) and (d) are consequences of the fact that X is a strong supermartingale, and $1/X$ is a strong Q-supermartingale. It is worth isolating the relevant results on strong supermartingales; these are of independent interest and are analogues of well-known properties in the "classical" case of right continuous supermartingales under the "conditions habituelles" on the filtration.

The first result strengthens a theorem of Mertens [10] in dropping the hypothesis of a right continuous filtration; also, the new version [5] of the optional section theorem allows a more transparent proof. P.A. Meyer has informed me that the general result (at least for left limits) has been given independently, with a different proof, by Th. Eisele in Heidelberg.

(4) **Theorem.** *Let Y be a strong supermartingale; then, a.s., Y has finite limits on the left at every* $t \in]0, \infty]$ *and on the right at every* $t \in [0, \infty[$.

Both the nonnegativity and the supermartingale property may be weakened: all that is needed is that Y_{T_n} converge a.s. for every increasing sequence of stopping times T_n.

The next result extends the "minimum principle" of Meyer [11, VI.15] to strong supermartingales.

(5) **Theorem.** *Let Y be a strong supermartingale and*

$$\tau(\omega) = \inf\{t: \liminf_{s \to t} Y_s(\omega) = 0\} \qquad (\inf \emptyset = \infty);$$

then $A_\alpha =]\!]\tau, \infty[\![\cap \{Y > \alpha\}$ *is an evanescent set for all* $\alpha \geqq 0$.

The "lim inf" is two-sided and includes the value of Y *at* t; in view of (4), τ coincides a.s. with the "first" time that at least one of Y_{t+}, Y_t, or Y_{t-} equals 0. Thus we may paraphrase (5) as follows: *a strong supermartingale "dies" after touching zero.*

As a trivial corollary of (5), we obtain a lemma of Doob [7, pp. 220–221]:

(6) **Corollary.** *Let* $(Y_n(t))$ *be a sequence of right continuous, nonnegative supermartingales, and put* $Y(t) = \inf_n Y_n(t)$; *then, letting*

$$\sigma = \inf\{t: Y(t) = 0\}, \qquad P\{\sup_{t>0} Y(\sigma + t) = 0\} = 1.$$

Indeed, it suffices that each Y_n be a strong supermartingale.

The proofs of (3)–(5) are in § 2; then, in § 3, we discuss briefly the question of the existence of a dominating measure when X is given. This turns out to be almost the same as the existence of Föllmer measures.

Terminology from the general theory of processes is in [5, 6]; in particular, "optional" refers to $\{\mathscr{F}_t^0\}$ unless otherwise indicated. As a technical device we will need the usual filtration $\{\mathscr{F}_t\}$ obtained by adjoining all P-null sets to $\mathscr{F}_{t+}^0$; the corresponding family of stopping times is denoted $\mathscr{S}$.

The Andersen-Jessen theorem completes certain so-called "probabilistic Fatou theorems" [1, 3], which assert P-a.s. convergence, with the assertion of Q-a.s. convergence. A similar situation obtains with respect to two classical theorems in analysis: (a) (Fatou's theorem) a positive harmonic function $u(z)$ in the unit disc, with boundary measure μ, converges for (Lebesgue) a.e. $\theta \in [0, 2\pi[$ as $z \to e^{i\theta}$ along a nontangential path; (b) (Fejér's theorem) the Cesàro averages of the Fourier-Stieltjes series of a measure μ on $[0, 2\pi[$ converge a.e. In both cases we have in addition that *the convergence takes place* μ*-a.e. and the limit function is a density (in the above sense) of* μ *relative to Lebesgue measure.*

I would like to thank P.A. Meyer for pointing out [3] and [13] and for many helpful suggestions, and M. Yor for calling my attention to [9].

§ 2. Proofs

We begin with the results on strong supermartingales.

Proof of (4). We treat the case of left limits first. Let

$$\bar{Y}_t = \limsup_{s \uparrow\uparrow t} Y_s, \qquad \underline{Y}_t = \liminf_{s \uparrow\uparrow t} Y_s;$$

$$D_{ab} = \{(t, \omega): \underline{Y}_t(\omega) < a < b < \bar{Y}_t(\omega)\}.$$

According to [5, IV.90], D_{ab} is a predictable set relative to $\{\mathscr{F}_t\}$. If the theorem were false, D_{ab} would fail to be evanescent for some $a<b$, hence, given $0<\varepsilon<P(\pi(D_{ab}))$, there would be (section theorem) a predictable $T\in\mathscr{S}$ such that

$$[\![T]\!] \subset D_{ab}, \qquad P(\pi(D_{ab})) \leqq P(T<\infty)+\varepsilon.$$

Thus $\alpha = P\{T<\infty\}>0$. Note that $\{T<\infty\}=\pi[\![T]\!]\subset\{\omega: (T(\omega), \omega)\in D_{ab}\}$. By [5, IV.78] there is a $T'\in\mathscr{S}^0$, predictable relative to $\{\mathscr{F}_t^0\}$, and such that $T'=T$ a.s.; thus we may and do replace T by T' in this discussion, henceforth dropping the prime.

The set $B_0=\{Y>b\}\cap[\![0, T[\![$ is optional and

$$\pi(B_0)=\{T<\infty\}+\pi(B_0)\cap\{T=\infty\},$$

the + indicating disjoint union. Hence $P(\pi(B_0))=\alpha+\beta_0$, say. Let $0<a<1$, the exact value to be specified later. By the optional section theorem there is a $T_1\in\mathscr{S}^0$ such that

$$[\![T_1]\!]\subset B_0, \qquad \alpha+\beta_0\leqq P(T_1<\infty)+a\alpha,$$

from which follows $\alpha_1\equiv P\{T_1<\infty, T<\infty\}\geqq\alpha(1-a)$. Next we consider the optional set $B_1=\{Y<a\}\cap]\!]T_1\wedge T, T[\![$:

$$P(\pi(B_1))=P\{T<\infty, T_1<\infty\}+P(\pi(B_1), T=\infty \text{ or } T_1=\infty)$$
$$\equiv\alpha_1+\beta_1.$$

Thus we have $T_2\in\mathscr{S}^0$ such that

$$[\![T_2]\!]\subset B_1, \qquad \alpha_1+\beta_1\leqq P(T_2<\infty)+a^2\alpha,$$

hence $\alpha_2=P(T_2<\infty, T_1<\infty, T<\infty)\geqq\alpha(1-a-a^2)$. Continuing in the same manner, we obtain a sequence $T_n\in\mathscr{S}^0$ such that (i) $T_n<\infty$ implies $T_{n-1}<T_n<T$, (ii) $P(T_n<\infty, \ldots, T_1<\infty, T<\infty)\geqq\alpha(1-a-a^2-\cdots-a^n)$, (iii) $Y_{T_{2n}}<a$ if $T_{2n}<\infty$, $Y_{T_{2n+1}}>b$ if $T_{2n+1}<\infty$. The set $C=\{T<\infty, T_n<\infty$ for every $n\}$ clearly has $P(C)\geqq\dfrac{1-2a}{1-a}\alpha>0$ if $a<1/2$. Moreover the sequence T_n is increasing. The sequence Y_{T_n}, being a positive supermartingale, should converge a.s., but this is impossible on C.

The proof concerning right limits is similar, so we will only point out the one slightly different aspect. Define D'_{ab} as D_{ab} above, but with $\downarrow\downarrow$ instead of $\uparrow\uparrow$. Then D'_{ab}

is progressive [7, IV.90] and so T, the debut of D'_{ab}, is in $\mathscr{S}$. As earlier, we may replace T by an $\{\mathscr{F}^0_{t+}\}$-stopping time equal to it a.s., which we again denote by T. We then observe that $[\![T]\!] \subset D'_{ab}$ up to an evanescent set, and finally, using [5, IV.62] and the section theorem, construct a *decreasing* sequence $T_n \in \mathscr{S}^0$ such that $\lim_n T_n = T$ a.s., and, on a set of positive probability, $T_n < \infty$ for every n, $Y_{T_{2n}} < a$, $Y_{T_{2n+1}} > b$, which contradicts the a.s. convergence of the reversed supermartingale Y_{T_n}.

Proof of (5). Since Y is optional, it is $\{\mathscr{F}^0_t\}$-progressive, hence also $\{\mathscr{F}_t\}$-progressive, and one derives easily from [5, IV.33] that $\liminf_{s\to t} Y_s$ is $\{\mathscr{F}_t\}$-progressive and so $\tau \in \mathscr{S}$ [5, IV.50]. Thus [5, IV.59] there exists $\tau' \in \mathscr{S}^0_+$ such that $\tau = \tau'$ a.s. Let $\pi: \mathbb{R}_+ \times \Omega \to \Omega$ be the usual projection. Then $\pi(A_\alpha) \in \mathscr{F}$, and defining A'_α with τ' in place of τ, $\pi(A'_\alpha) \in \mathscr{F}$, and $\pi(A_\alpha) = \pi(A'_\alpha)$ up to a P-null set. So we need only prove that $P(\pi(A'_\alpha)) = 0$ for $0 < \alpha < 1$.

First we observe that A'_α is an optional set (see [5, IV.62]). Were the conclusion false, the optional section theorem [5] would give a $\sigma \in \mathscr{S}^0$ such that

$$[\![\sigma]\!] \subset A'_\alpha, \qquad P(\pi(A'_\alpha)) \leqq P(\sigma < \infty) + \varepsilon$$

where $0 < \varepsilon < P(\pi(A'_\alpha))$, so that $P(\sigma < \infty) > 0$.

Let $0 < \beta < \alpha P(\sigma < \infty)/3$, and put

$$B_\beta =]\!]0, \sigma[\![\cap \{Y < \beta\}.$$

Then B_β is optional and one checks easily that $\{\sigma < \infty\} \subset \pi(B_\beta)$ except possibly for a P-null set, and so $P(\pi(B_\beta)) > 0$. Let $0 < \gamma < \alpha P(\sigma < \infty)/3$ and choose $0 < \delta < \alpha P(\sigma < \infty)/3$ such that

$$P(M) \leqq \delta \quad \text{implies} \quad E(Y_\sigma; M) < \gamma,$$

which is possible because $Y_\sigma \in L^1$. The optional section theorem now gives us $\rho \in \mathscr{S}^0$ such that

$$[\![\rho]\!] \subset B_\beta, \qquad P(\pi(B_\beta)) \leqq P(\rho < \infty) + \delta,$$

and this yields in turn $P(\sigma < \infty, \rho = \infty) \leqq \delta$. The optional sampling property gives

$$E(Y_{\rho \wedge \sigma}) \geqq E(Y_{\rho \vee \sigma})$$

which translates to $E(Y_\rho; \rho < \infty) + E(Y_\sigma; \sigma < \infty, \rho = \infty) \geqq E(Y_\sigma; \sigma < \infty, \rho < \infty)$.

The left member is majorized by $\beta P(\rho < \infty) + \gamma < \frac{2}{3}\alpha P(\sigma < \infty)$, while the right member dominates $\alpha P(\sigma < \infty) - \alpha P(\sigma < \infty, \rho = \infty)$, hence

$$\tfrac{2}{3}\alpha P(\sigma < \infty) + \delta > \tfrac{2}{3}\alpha P(\sigma < \infty) + \alpha P(\sigma < \infty, \rho = \infty) \geqq \alpha P(\sigma < \infty),$$

which is impossible with our earlier choice of δ.

We turn now to the proof of Theorem (2), part of which is modeled on [9]. Here is a preliminary result.

Let z be a positive, bounded random variable. The optional projection theorem [6] yields an optional process $Z = (Z_t)$ such that

(7) $E(z I_{\{T < \infty\}} \mid \mathscr{F}^0_T) = Z_T I_{\{T < \infty\}}, \qquad T \in \mathscr{S}^0;$

i.e. Z is the optional projection of the constant-in-time process z. Defining $Z_T = z$ when $T = \infty$, Z becomes a uniformly integrable strong martingale (see (3)). By (4) we have a.s. the existence of $Z_{T\pm}$ whenever $T < \infty$, and an easy argument establishes

(i) $E(zI_{\{T<\infty\}} \mid \mathscr{F}^0_{T+}) = Z_{T+}\, I_{\{T<\infty\}}$, $T \in \mathscr{S}^0_+$,
(ii) $E(zI_{\{T<\infty\}} \mid \mathscr{F}^0_{T-}) = Z_{T-}\, I_{\{T<\infty\}}$, T predictable.

Similar results hold when the underlying finite measure is not a probability measure. Note that the notion of predictable stopping time is the same whether we start with $\mathscr{S}^0$ or $\mathscr{S}^0_+$ [5].

Suppose now that $\mathscr{G}$ is a sub σ-field of $\mathscr{F}^0$ and put $M = P + Q$. Obviously $P_{\mathscr{G}} \ll M_{\mathscr{G}}$, so we may write $\eta_{\mathscr{G}} = dP_{\mathscr{G}}/dM_{\mathscr{G}}$. One checks easily that $1/\eta_{\mathscr{G}} - 1$ is a density of Q over $\mathscr{G}$, unique up to M-equivalence.

We apply the above observation as follows: let $z = dP/dM$, and let Z be the optional projection as in (7), but relative to M. It is immediate from (7), (i), (ii) above that $Z_T = dP_T/dM_T$, and $Z_{T\pm} = dP_{T\pm}/dM_{T\pm}$ under the appropriate restrictions on T, where, e.g. M_{T+} means $M_{\mathscr{F}^0_{T+}}$. Parts (a), (b), (c), (e) are now obtained upon taking $X_t = 1/Z_t - 1$.

Suppose $S, T \in \mathscr{S}^0$, $S \leqq T$; then

$$Q_S(A) = Q_T(A) \geqq E(X_T; A), \qquad A \in \mathscr{F}_S^0,$$

hence $E(X_T \mid \mathscr{F}_S^0) \leqq X_S$, since X_S is the "largest" $\mathscr{F}_S^0$-measurable function whose indefinite integral is majorized by Q_S. Thus:

(8) **Lemma.** *X is a strong supermartingale.*

We observe that $1/X_T$ is a density of P over $\mathscr{F}^0_T$ with respect to Q, so the same proof shows that *$1/X$ is a strong Q-supermartingale*. Together with (4) this proves (c) anew, and Theorem (5), applied to X and $1/X$ respectively, yields (d).

Remarks. (a) It is easy to see that the predictability in (e) cannot be dropped: under the "conditions habituelles" let T be a totally inaccessible stopping time and X a uniformly integrable, right continuous martingale having an upward jump of size 1 at T; see [11, VII.46]. (b) The argument for (e) can be adapted to give a simple proof of Theorem (1).

As a corollary, we obtain a slight generalization (with a much simpler proof) of a result in [3]. Recall Q_t means $Q_{\mathscr{F}^0_t}$.

(9) **Corollary.** *There is an optional, evanescent set N such that*

(10) $$Q_T(A) = E(X_T; A) + Q(AN_T), \qquad T \in \mathscr{S}^0 \textit{ finite},$$

where N_t is the section of N at t.

The set $N = \{X = \infty\}$ is evanescent since $E(X_T; T < \infty) < \infty$ for $T \in \mathscr{S}^0$, and satisfies (9). This argument applies to $1/X$, Q and shows

(11) **Corollary.** *The limits in* (2c) *are finite P-a.s. and positive Q-a.s.*

Let τ_0 be the debut of N; in the right continuous case (10) becomes

(12) $$Q_T(A) = Q(A, \tau_0 > T) + Q(A, \tau_0 \leqq T), \qquad T \textit{ finite}.$$

This form of "progressive Lebesgue decomposition" of Q is due to Kunita [9] (cf. [13]) for $T=$const.

We conclude by sketching how the pieces of a Riesz-type decomposition of X match those of a corresponding decomposition of the dominating measure Q. Expressions such as $Q(U\mid\mathscr{F}_t^0)$ are understood in the sense of (7). An optional process Y with values in $[0,\infty]$ is *strongly dominated* by a finite measure R if $R_T(A)=E(Y_T;A)$ for all bounded $T\in\mathscr{S}^0$. This implies that Y is a strong martingale. It is easy to see that, if a strong martingale Y is dominated by R, then it is *strongly* dominated by $R^*(A)=R(A, Y_t<\infty$ for all $t)$.

Theorem. *There are decompositions* $X_t=u_t+v_t+w_t$, $Q=Q^u+Q^v+Q^w$ *such that*

(i) $u=(u_t)$ *is a strong potential (not necessarily class (D)) dominated by* Q^u;

(ii) v *is a strong martingale,* $v_\infty=\lim_{t\to\infty} v_t=0$ *a.s., strongly dominated by* Q^v;

(iii) w *is a uniformly integrable strong martingale, strongly dominated by* Q^w.

Let τ_0 be the debut of $N=\{X=\infty\}$,

$$U=\{\tau_0<\infty\},\qquad V=\{X_\infty=\infty\},\qquad W=\Omega\setminus(U\cup V),$$

and put $Q^u(A)=Q(AU)$, etc. From (2) we find that Q^w is the absolutely continuous component of Q and $w_t=E(X_\infty\mid\mathscr{F}_t^0)$ (optional projection), and it is easy to prove that

$$u_t=X_t\,Q(U\mid\mathscr{F}_t^0),\qquad v_t=X_t\,Q(V\mid\mathscr{F}_t^0)$$

are the required processes. Intuitively we may describe Q^u and Q^v as, respectively, the "locally singular" and "asymptotically singular" parts of Q.

One may show that the potential u has the following property (which does *not* imply class (D)): if $T_n\in\mathscr{S}^0$, and $T_n\uparrow\infty$ Q-a.s., then $\lim_n E(u_{T_n})=0$. Finally we note that u splits further into a local martingale and a class (D) potential, which effects a corresponding splitting of Q^u: let $R_n=\inf\{t\colon X_t>n\}$ and put

$$U'=\{X_{R_n}<\infty \text{ for every } n\},\qquad U''=U\setminus U''.$$

Defining $Q^{u'}$, $Q^{u''}$ in the obvious way, one checks that $u_t'=X_t\,Q(U'\mid\mathscr{F}_t^0)$ is a local martingale and $u_t''=X_t\,Q(U''\mid\mathscr{F}_t^0)$ is a class (D) potential.

§3. Existence of a Dominating Measure

The main result here is that, roughly speaking, a dominating measure exists iff a Föllmer measure in the sense of [1] exists. We work with the filtration $\{\mathscr{F}_{t+}^0\}$ and assume X *is a right continuous, nonnegative* $\{\mathscr{F}_{t+}^0\}$*-supermartingale;* according to [6] there is no loss of generality in assuming that X is optional, which we do.

Recall that *the Föllmer measure of* X, if it exists, is the unique measure Φ on the predictable σ-field $\mathscr{P}$ in $]0,\infty]\times\Omega$ satisfying

$$\Phi(]t,\infty]\times A)=E(X_t;A),\qquad A\in\mathscr{F}_{t+}^0. \tag{13}$$

Suppose a dominating measure Q exists. By (10) and (12) we have

(14) $Q_T(A)=E(X_T; A)+Q(A, \tau_0 \leqq T)$, $\quad$ *T finite.*

Define, for $U \in (\mathscr{P})_+$, i.e. $U=(U_t)$ a nonnegative, predictable process,

(15) $\int U\,d\Phi = \int\limits_{\tau_0>0} U_{\tau_0}\,dQ.$

The measure Φ so defined is immediately verified to be the Föllmer measure of X, using (14). Thus, following [13], we have

(16) **Theorem.** *If a dominating measure exists, so does a Föllmer measure.*

Going in the opposite direction we consider separately the cases in which X is a class (D) potential and a local martingale, respectively. The general case is then obtained by addition. Further, by subtracting a uniformly integrable martingale (for which domination and Föllmer measures are both trivial) we may assume the latter is also a potential but, of course, not of class (D). Notice that, under the present conditions, it suffices to check that X_T is a density of Q over $\mathscr{F}^0_{T+}$ for each $T \in \mathscr{S}^0_+$ in order to have *all* the conclusions of (2).

1° *Class (D) potentials.* There always exists a Föllmer measure Φ in the sense of (13) in this case, namely

$$\int U\,d\Phi = E\int_0^\infty U_s\,dA_s, \qquad U \in (\mathscr{P})_+,$$

where $A=(A_t)$ is the predictable, integrable, increasing process which generates X.

Let Ω be a space on which are defined killing operators k_t and a lifetime ζ subject to some "natural" axioms set down in [3] or [5, Ch. IV].

We define (as in [3]) a measure Q on Ω by

(17) $\int\limits_{\Omega} \xi\,dQ = E\int\limits_0^\infty \xi \circ k_s\,dA_s.$

If $\xi \in (\mathscr{F}^0_{t+})$, then $\xi \circ k_s = \xi$ whenever $s>t$, whence

$$\int \xi\,dQ = E(\xi X_t) + E\int_0^t \xi \circ k_s\,dA_s.$$

The second term on the right defines a measure which is carried by $\{\zeta \leqq t\} \in \mathscr{F}^0_{t+}$.

Clearly $P\{\zeta<\infty\}=0$ implies that Q is a dominating measure.

Suppose Ω is a space of trajectories, with or without lifetime. Adding a new deathpoint * not in the state space and defining killing operators

$$k^*_t\,\omega(s) = \begin{cases} \omega(s) & \text{if } s<t \\ * & \text{if } s \geqq t, \end{cases}$$

we may consider Ω as a set of probability 1 in a space Ω^* of trajectories in which, with the obvious notation, $P^*(\zeta^* < \infty) = 0$. The construction (20) then gives a measure Q^* which dominates X on the larger space.[1]

2° *Local martingale-potentials.* Keeping the assumption of right continuity, we now also require that Ω *be a Souslin space* [5].

(18) **Theorem.** *Suppose* Φ *exists as in* (13); *then a dominating measure* Q *exists.*

Let $T_n \in \mathscr{S}_+^0$ $(T_0 = 0)$ be an increasing sequence of reducing times for X, $T_\infty = \lim_n T_n$, so $P\{T_\infty < \infty\} = 0$. By [1],

(19) $\Phi(]\!]T, \infty]\!]) = E(X_T; T < \infty), \qquad T \in \mathscr{S}_+^0.$

Using $\Phi(]\!]T_n, T_{n+1}]\!]) = 0$, we find easily that Φ *lives on* $[\![T_\infty]\!]$.

A slight extension of [5, IV.45] shows that there is a measure Q on $\mathscr{F}^0$ such that Φ is the image measure of Q under the mapping $\omega \to (T_\infty(\omega), \omega)$; then (19) gives

(20) $Q(T < T_\infty) = E(X_T; T < \infty), \qquad T \in \mathscr{S}_+^0,$

which yields (14) with T_∞ replacing τ_0.

We conclude with a remark on the notion of Föllmer measure given in [3]. There one finds, leaving aside all the technical details, a measure μ such that

(21) $\mu(A, T < \zeta) = E(X_T; A, T < \zeta), \qquad A \in \mathscr{F}_{T+}^0, \; T \in \mathscr{S}_+^0,$

corresponding to an a.s. right continuous, nonnegative supermartingale X such that $X_t = 0$ for $t \geqq \zeta$. In the class (D) potential case, μ is the same as Q in (17).

Suppose X is a *local martingale* and let T_n, T_∞ be as in the proof of (18). Then (21) gives $\mu(T_n < \zeta) = E(X_0) = \mu(\Omega)$, whence $T_n < \zeta$ for all n and $T_\infty \leqq \zeta$ μ-a.s., and $\mu(T_\infty < \zeta) = 0$. Thus

$$\mu(A, T < T_\infty) = E(X_T; A, T < \infty), \qquad A \in \mathscr{F}_{T+}^0, \qquad T \in \mathscr{S}_+^0,$$

and so μ *dominates* X.

Finally, let $Z_t = X_t^{-1} I_{\{t < \zeta\}}$. A monotone class argument at (21) shows that Z *is a (strong)* μ*-supermartingale.* It follows that *there is a* $P + \mu$*-modification of* X *which is right continuous and has left limits* $P + \mu$-a.s.

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[1] Here and in (18) we need not have *a priori* that $\{X_T = \infty\}$ carries the singular part of Q_T; however, this can be accomplished by modifying X on an evanescent set.

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