Z. Wahrscheinlichkeitstheorie verw. Gebiete
58, 125 – 138 (1981)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

On the Speed of Convergence in the Random Central Limit Theorem for φ-Mixing Processes

E. Schneider*

Mathematisches Institut der Universität Köln, Weyertal 86-90, D-5000 Köln 41,
Federal Republic of Germany

1. Introduction

Let X_n, $n \in \mathbb{N}$, be a sequence of random variables on a probability space $(\Omega, \mathscr{B}, P)$. The σ-algebra generated by X_n with $a \leqq n \leqq b$ is denoted by $\mathscr{M}_a^b$.

Suppose there exists a sequence $\varphi(k)$, $k \in \mathbb{N}$, of real numbers with $1 \geqq \varphi(k) \downarrow 0$ for $k \to \infty$ such that

$$|P(E_1 \cap E_2) - P(E_1)\, P(E_2)| \leqq \varphi(k)\, P(E_1)$$

for all $l, k \in \mathbb{N}$, $E_1 \in \mathscr{M}_1^l$, $E_2 \in \mathscr{M}_{l+k}^\infty$, then the sequence X_n, $n \in \mathbb{N}$, is called φ-mixing.

In this paper uniform and non-uniform bounds in the random central limit theorem for φ-mixing processes are derived. The bounds are very near to those given by Landers and Rogge [9, 10] in the independent case. Of course the results are based on corresponding theorems for non-random summation. These theorems are obtained by modifying and developing methods of Tihomirov [16], Erickson [4], Babu, Ghosh and Singh [1].

The following notations are used:

If X is a random variable the measure induced by X is denoted by $P * X$.

Put $\sigma_n^2 := \operatorname{Var} \sum_{i=1}^{n} X_i$, and for $x \in \mathbb{R}$

$$[x] := \max(1, \max\{n \in \mathbb{N} : n \leqq x\}),$$

$$\psi(x) := (2\pi)^{-1/2} \exp(-x^2/2), \qquad \Phi(x) := \int_{-\infty}^{x} \psi(t)\, dt.$$

Throughout the paper D denotes a generic constant.

* Some results presented in this paper are a part of the author's doctoral dissertation, written at the University of Cologne

0044-3719/81/0058/0125/$02.80

2. Non-Random Summation

For stationary processes the following theorem has been proved by A.N. Tihomirov ([16], Theorem 3). Therefore, we have only to show how in Tihomirov's paper stationarity can be substituted by our weaker conditions.

Theorem 1. *Let X_n, $n \in \mathbb{N}$, be a φ-mixing sequence with*

$$EX_n = 0, \qquad n \in \mathbb{N} \tag{1}$$

$$\sup_{n \in \mathbb{N}} E|X_n|^3 < \infty \tag{2}$$

$$\liminf_{n \in \mathbb{N}} \sigma_n^2 / n > 0 \tag{3}$$

$$\varphi(n) \leqq D \exp(-\lambda n), \qquad n \in \mathbb{N}, \qquad \textit{for some} \quad \lambda > 0. \tag{4}$$

Then for all $n > 1$

$$\sup_{t \in \mathbb{R}} \left| P\left\{ \sum_{i=1}^{n} X_i < \sigma_n t \right\} - \Phi(t) \right| \leqq D n^{-1/2} \log n. \tag{5}$$

Proof. Using our assumptions (2) and (3) and Lemma 2 we can renounce stationarity in Tihomirov's proof. Difficulties arise only in the proof of Tihomirov's Lemma 3.2. (Like in [16] we first assume m-dependence.)

We show now how we can avoid this Lemma. We adopt the notation of Tihomirov. Lemma 3.2 and 3.3 are used only to give a bound for

$$L := \left| \sum_{j=1}^{n} EX_j \prod_{l=1}^{r-1} \xi_j^{(l)} e^{itS_j^{(r)}} - \sum_{j=1}^{n} EX_j \prod_{l=1}^{r-1} \xi_j^{(l)} f_n(t) \right|.$$

But this can be achieved directly without using stationarity. If we set $a_j = EX_j \prod_{l=1}^{r-1} \xi_j^{(l)}$ we have by [17] Lemma 3.1

$$|a_j| \leqq D \left(\frac{|t| D \sqrt{m}}{\sigma_n} \right)^{r-1}$$

and therefore

$$\begin{aligned} L &= \left| E \exp(itS_n) \sum_{j=1}^{n} a_j \eta_j^{(r)} \right| \\ &\leqq \left| E \exp(itS_n) \sum_{j=1}^{n} a_j E\eta_j^{(r)} \right| + \left| E \exp(itS_n) \sum_{j=1}^{n} a_j (\eta_j^{(r)} - E\eta_j^{(r)}) \right| \\ &\leqq D |f_n(t)| \cdot n \cdot \left(\frac{|t| D \sqrt{m}}{\sigma_n} \right)^{r-1} \cdot \left| \frac{t}{\sigma_n} \right| \sqrt{rm} + E \left| \sum_{j=1}^{n} a_j (\eta_j^{(r)} - E\eta_j^{(r)}) \right|. \end{aligned}$$

Here the second summand is bounded by (cf. [16], p. 806, 807)

$$\left(\sum_{j=1}^{n} \sum_{|p-j| \leqq 3rm} \operatorname{cov}(a_j \eta_j^{(r)}, a_p \eta_p^{(r)}) \right)^{1/2} \leqq D \left(\frac{|t| D \sqrt{m}}{\sigma_n} \right)^{r-1} \sqrt{n} \cdot \left| \frac{t}{\sigma_n} \right| rm.$$

Like in [16] it is easy to deduce the corresponding bound for φ-mixing variables.

The next theorem is a specification of a result of Erickson [4].

Theorem 2. *Let* X_n, $n\in\mathbb{N}$, *be a* φ*-mixing sequence fulfilling* (1) *and* (4). *Further let*

$$\sup_{n\in\mathbb{N}} E|X_n|^s<\infty \qquad \text{for some real } s>2, \tag{6}$$

$$\inf_{n\in\mathbb{N}} \sigma_n^2/n>0. \tag{7}$$

Set $X_i^{(n)}=X_i I\{|X_i|\leqq\sqrt{n}\}$.

Then for all $n>1$, $t\in\mathbb{R}$

$$\left|P\left\{\sum_{i=1}^{n} X_i<\sigma_n t\right\}-P\left\{\sum_{i=1}^{n} X_i^{(n)}<\sigma_n t\right\}\right|\leqq D n^{1-s/2}(1+|t|)^{-s}(\log n)^s. \tag{8}$$

Erickson only gives a proof for the d-dependent case. It is not very difficult to extend his proof, if the following hints are observed. (cf. [4], Sect. 6).

Instead of [4] (2.2) set

$$K(x):=1-P(U>x, \cap B_k^c).$$

The terms T_2, T_4, T_6 in the bound given in [4] Proposition 3.1 disappear, if the last equation in the proof of the proposition is replaced by

$$|s_k|^m I(d_k)-|s_{k-1}|^m I(d_{k-1})=|s_k|^m I(d_{k-1}^c) I(b_k)+(|s_k|^m-|s_{k-1}|^m) I(d_{k-1}).$$

This shortens the proof a great deal.

Use our Lemma 2(i) instead of Erickson's Proposition 5.2 and apply Lemma 1 where Erickson uses the d-dependence. A detailed proof of Theorem 2 can be found in [14].

With the aid of Theorem 2 we obtain a nonuniform bound in the central limit theorem by altering a proof of Babu, Ghosh, Singh [1].

Theorem 3. *Let* X_n, $n\in\mathbb{N}$, *be a* φ*-mixing sequence satisfying* (1), (4), (6) *and* (7).

Then there exists a constant $d>0$ *so that for all* $n>1$, $t\in\mathbb{R}$, *with* $t^2\geqq d\log n$

$$\left|P\left\{\sum_{i=1}^{n} X_i<\sigma_n t\right\}-\Phi(t)\right|\leqq D n^{1-s/2}|t|^{-s}(\log n)^s. \tag{9}$$

Proof. Let $c=s-2$, $c'=\min(1,c)$ and define $X_i^{(n)}$ like in Theorem 2. Assume w.l.o.g. that $t>0$.

We have by [5], p. 175, Lemma 2

$$\Phi(-t)\leqq D n^{-c/2} t^{-2-c} \qquad \text{if } t^2>(c+1)\log n \tag{10}$$

and by (1) and (7)

$$\left|\sum_{i=1}^{n} EX_i^{(n)}/\sigma_n\right|\leqq D n^{-c/2}. \tag{11}$$

Using these inequalities, Theorem 2 and (7) we see that it suffices to show

$$P\left\{\sum_{i=1}^{n}(X_i^{(n)}-EX_i^{(n)})\geqq 3tn^{1/2}\right\}\leqq Dn^{-c/2}t^{-2-c} \tag{12}$$

for all $n>1$, $t^2\geqq d\log n$ where $d>1$ can be chosen later. (The factor $(\log n)^s$ in (9) is only caused by Theorem 2.)

To prove (12) we use Lemma 2(i). We proceed in the same way as Babu, Ghosh and Singh in the proof of Lemma 3 [1]. So we adopt their notation and only indicate the changes to be made.

We set

$$X_i'=X_i^{(n)}-EX_i^{(n)},$$
$$y=12(c+1)c'^{-1}t^{-1}n^{-1/2},$$
$$\xi_j^*=\xi_j I\{\xi_j<1/y\}.$$

Since $\sum_{i=1}^{n} X_i'\leqq n^{3/2}$ we can assume that $t\leqq n$ and so we can replace [1] (3.5) by

$$P\{U_n^*>tn^{1/2}\}\leqq \exp(-(2c+2)\log n)E\exp(zU_n^*)$$
$$\leqq n^{-c/2}t^{-c-2}n^{-c/2}\prod_{j=1}^{k}s_j \qquad \text{(use [1], Lemma 2)}$$

where

$$z:=z(n,t)=(2c+2)t^{-1}n^{-1/2}\log n$$

and

$$s_j:=2\exp((c'/6)\log n)\varphi(p)+E\exp(z\xi_j^*).$$

Obviously the first summand of s_j is smaller than Dk^{-1}, and the second is according to Lemma 4 bounded by

$$P\{|\xi_j|\geqq 1/y\}+E(I\{|\xi_j|<1/y\}\exp(z\xi_j))$$
$$\leqq y^{2c+2}E|\xi_j|^{2c+2}+1+z^2E\xi_j^2/2+E|\xi_j|^{2+c'}y^{2+c'}\exp(2z/y).$$

Here the first and the last summand are bounded by Dk^{-1}.

Since $t^2\geqq d\log n$ we have

$$z^2\leqq (D/d)n^{-1}\log n$$

yielding

$$z^2E\xi_j^2\leqq (D/d)k^{-1}\log k.$$

Therefore (use $x^n\leqq\exp(n(x-1))$)

$$\prod_{j=1}^{k}s_j\leqq(1+Dk^{-1}+(D/d)k^{-1}\log k)^k\leqq k^{D/d}\leqq Dn^{c/2}$$

if d is chosen large enough.

Now we have

$$P\{U_n^*>tn^{1/2}\}\leqq Dn^{-c/2}t^{-c-2}$$

and this yields the assertion like in [1].

Combining Theorem 1 and Theorem 3 we get the following result (cf. [6], (2.4), (2.5) and [16], Theorem 4)

Theorem 4. *Let the assumptions of Theorem 3 be fulfilled for some* $s \geqq 3$.
Then for all $t \in \mathbb{R}$, $n > 1$

$$\left| P\left\{ \sum_{i=1}^{n} X_i < \sigma_n t \right\} - \Phi(t) \right| \leqq D n^{-1/2} (1+|t|)^{-s} (\log n)^{1+s'/2}$$

where $s' = s$ *if* $s > 3$ *and* $s' = s+1$ *if* $s = 3$.

3. Random Summation

We are now ready to prove two theorems about the speed of convergence in the random central limit theorem for φ-mixing processes. The only result in this direction is due to B.L.S. Prakasa Rao [12]. His technique is different from ours and the order of convergence he reaches is far from the order in the independent case.

First we give a φ-mixing version of a theorem of Landers and Rogge [9]. Their result gives the exact rate of convergence under independence (see [8]). Examining their proof, we see that one of the main tools, namely Lemma 7 of [8], cannot be transferred to φ-mixing processes because the proof heavily uses independence and stationarity. (The last fact has not been noticed by Rychlik [13] and so his proof is not correct in this place, see [13], p. 233.)

Lemma 7 of [8] allows to replace $\max\limits_{p<n\leqq q} \sum\limits_{i=p+1}^{n} X_i$ by $\sum\limits_{i=p+1}^{q} X_i$ in a certain situation.

This lemma can be avoided by retaining the maximum and using Serfling's [15] inequality for maxima of sums in the proof of Landers' and Rogge's Lemma 8 (see [8], p. 282, (*)).

The order of approximation we obtain differs from the order in the independent case only by a logarithmic factor. If the assumptions are strengthened a little bit, the factor disappears.

Theorem 5. *Let the assumptions of Theorem 1 be fulfilled. Let* ε_n, $n \in \mathbb{N}$, *be a sequence with* $n^{-1} \leqq \varepsilon_n < 1$ *and* $\varepsilon_n \xrightarrow[n\to\infty]{} 0$.

Let τ_n, $n \in \mathbb{N}$, *be positive integer valued random variables. Assume that* τ *is a positive random variable independent of* $(X_n)_{n\in\mathbb{N}}$ *so that one of the two following conditions is fulfilled.*

(i)
$$P\{\tau < c_0/(n\varepsilon_n)\} \leqq D\delta_n, \quad n \in \mathbb{N}, \tag{13}$$

for some constant $c_0 > 0$ *where* $\delta_n := \sqrt{\varepsilon_n}(\log \varepsilon_n)^2$.
(ii) *There exists* $\varepsilon > 0$ *so that* $\varepsilon_n \geqq n^{-1+\varepsilon}$, $n \in \mathbb{N}$, *and*

$$P\{\tau < c_0/(n\varepsilon_n^{1+\varepsilon})\} \leqq D\delta_n, \quad n \in \mathbb{N}, \tag{14}$$

for some constant $c_0 > 0$ *where* $\delta_n := \sqrt{\varepsilon_n}$.

Suppose further that for some $c_1>0$

$$P\{|\tau_n/(n\tau)-1|>c_1\,\varepsilon_n\}\leqq D\,\delta_n,\qquad n\in\mathbb{N}.$$

Then

$$\sup_{t\in\mathbb{R}}\left|P\left\{\sum_{i=1}^{\tau_n} X_i<\sigma_{[n\tau]}\,t\right\}-\Phi(t)\right|\leqq D\,\delta_n,\qquad n\in\mathbb{N}.$$

Proof. Let $n\in\mathbb{N}$ be sufficiently large and $t\in\mathbb{R}$. In case (i) we set $\gamma_n:=[c_0/\varepsilon_n]$ and in case (ii) $\gamma_n:=[c_0/\varepsilon_n^{1+\varepsilon}]$.

Therefore in either case

$$P\{n\tau<\gamma_n\}\leqq D\,\delta_n.$$

In case (i) we have

$$\varepsilon_n^{1/2}\log(D/\varepsilon_n)\leqq D\,\varepsilon_n^{1/2}\,|\log\varepsilon_n| \tag{15}$$

and in case (ii)

$$\varepsilon_n^{(1+\varepsilon)/2}\log(D/\varepsilon_n^{1+\varepsilon})\leqq D\,\varepsilon_n^{1/2}. \tag{16}$$

Thus

$$\gamma_n^{-1/2}\log\gamma_n\leqq D\,\delta_n.$$

Using Theorem 1 one gets, like in [9], p. 1021,

$$\begin{aligned}&\left|P\left\{\sum_{i=1}^{[n\tau]} X_i<\sigma_{[n\tau]}\,t\right\}-\Phi(t)\right|\\ &\quad\leqq P\{n\tau<\gamma_n\}+\sum_{l=\gamma_n}^{\infty} D P\{[n\tau]=l\}\,l^{-1/2}\log l\leqq D\,\delta_n.\end{aligned} \tag{17}$$

Set $S_0=0$, $S_j=\sum_{i=1}^{j} X_i$, $j\in\mathbb{N}$,

$$p_x=[x(1-c_1\,\varepsilon_n)],\qquad q_x=[x(1+c_1\,\varepsilon_n)],\qquad x>0.$$

(17) yields like in [9], the assertion, if it is shown that

$$P\{\min_{p_{n\tau}\leqq j\leqq q_{n\tau}} S_j<\sigma_{[n\tau]}\,t\}-P\{\max_{p_{n\tau}\leqq j\leqq q_{n\tau}} S_j<\sigma_{[n\tau]}\,t\}\leqq D\,\delta_n.$$

This difference is bounded from above by

$$P\{n\tau<\gamma_n\}+\int_{\gamma_n}^{\infty} P\{\min_{p_x\leqq j\leqq q_x} S_j<\sigma_{[x]}\,t\}-P\{\max_{p_x\leqq j\leqq q_x} S_j<\sigma_{[x]}\,t\}\,P*(n\tau)\,dx.$$

For $p<q$, $r\in\mathbb{R}$ we have

$$P\{\min_{p\leqq j\leqq q} S_j<r\}-P\{\max_{p\leqq j\leqq q} S_j<r\}=P\{S_p<r\leqq\max_{p\leqq j\leqq q} S_j\}+P\{\min_{p\leqq j\leqq q} S_j<r\leqq S_p\}.$$

Since we can replace X_i by $-X_i$ it apparently suffices to show for $r\in\mathbb{R}$, $x\geqq\gamma_n$, $p:=p_x$, $q:=q_x$ that

$$P\{S_p \leqq r \leqq \max_{p \leqq j \leqq q} S_j\} \leqq D\delta_n,$$

where D does not depend on n, r, x.

Set

$$m := [-(2\lambda)^{-1} \log \varepsilon_n], \quad \text{then} \quad \varphi(m) \leqq D\sqrt{\varepsilon_n}.$$

Let $k \in \mathbb{N}$, $\eta > 0$ to be specified later, fulfilling

$$(2k+1)m \leqq p/2. \tag{18}$$

We now show

$$P\{S_p \leqq r \leqq \max_{p \leqq j \leqq q} S_j\} \leqq Dk(\sqrt{\varepsilon_n} + p^{-1/2}(\eta + \log p)) + (D(-\log \varepsilon_n)^{3/2}/\eta^3)^{k+1} \tag{19}$$

where D is independent of η and k, too.

Setting

$$L_j := S_{p-(j-1)m} - S_{p-jm}, \qquad H := \max_{p \leqq j \leqq q} (S_j - S_p),$$

one obtains

$$\begin{aligned}
P\{S_p \leqq r \leqq \max_{p \leqq j \leqq q} S_j\} &\leqq P\{r - H \leqq S_p \leqq r\} \\
&\leqq P\{|L_j| > \eta \text{ for all } j = 1, \ldots, 2k+1\} + \sum_{j=1}^{2k+1} P\{r - H \leqq S_p \leqq r, |L_j| \leqq \eta\} \\
&\leqq P\left(\bigcap_{i=0}^{k} \{|L_{2i+1}| > \eta\}\right) + \sum_{j=1}^{2k+1} P\{r - \eta - H \leqq S_{p-jm} + S_p - S_{p-(j-1)m} \leqq r + \eta\}.
\end{aligned}$$

Using the φ-mixing property and Lemma 2(ii), the first probability can be bounded by

$$k\varphi(m) + \prod_{i=0}^{k} P\{|L_{2i+1}| > \eta\} \leqq Dk\sqrt{\varepsilon_n} + (D(-\log \varepsilon_n)^{3/2}/\eta^3)^{k+1}.$$

Let $j \in \{1, \ldots, 2k+1\}$ and set $Z := S_p - S_{p-(j-1)m}$.

According to (3) and (18)

$$\sigma_{p-jm}^2 \geqq D(p - jm) \geqq Dp.$$

Since S_{p-jm} is $\mathcal{M}_1^{p-jm}$-measurable and (Z, H) is $\mathcal{M}_{p-(j-1)m+1}^q$-measurable we obtain, using Lemma 3 and Theorem 1,

$$\begin{aligned}
&P\{r - \eta - H \leqq S_{p-jm} + Z \leqq r + \eta\} - \varphi(m) \\
&\quad \leqq \int P\{r - \eta - h - z \leqq S_{p-jm} \leqq r + \eta - z\}\, P*(Z, H)\, dz\, dh \\
&\quad \leqq D(p - jm)^{-1/2} \log(p - jm) \\
&\qquad + \int |\Phi((r + \eta - z)/\sigma_{p-jm}) - \Phi((r - \eta - h - z)/\sigma_{p-jm})|\, P*(Z, H)\, dz\, dh \\
&\quad \leqq D(p/2)^{-1/2} \log p + (2\eta + EH)/\sigma_{p-jm} \\
&\quad \leqq Dp^{-1/2}(\eta + \log p + EH).
\end{aligned}$$

As $EH \leqq D(q-p)^{1/2}$ according to Lemma 2(iii) and $(q-p)/p \leqq D\varepsilon_n$ we get (19).

Now we choose k and η.

In case (i) we set $k = [-\log \varepsilon_n]$. Since

$$(2k+1)m \leqq D(\log \varepsilon_n)^2 \quad \text{and} \quad p \geqq D\gamma_n \geqq D/\varepsilon_n,$$

(18) is valid.

Let $\eta := \alpha^{1/3}(-\log \varepsilon_n)^{1/2}$ where α is chosen such that

$$(D(-\log \varepsilon_n)^{3/2}/\eta^3)^{k+1} \leqq (D/\alpha)^{-\log \varepsilon_n} = \varepsilon_n^{-\log(D/\alpha)} = \varepsilon_n^{1/2}.$$

Using (15) it is now easy to see that the bound in (19) has the order $\sqrt{\varepsilon_n}(\log \varepsilon_n)^2$.

In case (ii) we choose $k \in \mathbb{N}$ so large that $3(k+1)\varepsilon > 1$ and set $\eta := \varepsilon_n^{-\varepsilon/2}$. Then (18) is fulfilled and

$$(D(-\log \varepsilon_n)^{3/2}/\eta^3)^{k+1} \leqq D(-\log \varepsilon_n)^{3(k+1)/2}\, \varepsilon_n^{3(k+1)\varepsilon/2} \leqq D\varepsilon_n^{1/2}.$$

Using this and (16), we see that (19) yields the desired order of convergence.

Remark. a) If τ is constant, the condition (13) respectively (14) is fulfilled, and in case (i) we obtain the order

$$\delta_n = |\log \varepsilon_n| (\sqrt{\varepsilon_n} + n^{-1/2} \log n), \qquad \text{if we set } \gamma_n = [n\tau/2].$$

b) With the additional assumption $\sup_{n \in \mathbb{N}} \|X_n\|_\infty =: M < \infty$ we obtain in case (i) the order $\sqrt{\varepsilon_n}|\log \varepsilon_n|$ if we set $k=0$ and $\eta = c|\log \varepsilon_n|$ where c is a constant so that $mM \leqq \eta$. If τ is furthermore constant, we obtain the order $\sqrt{\varepsilon_n} + n^{-1/2} \log n$.

In the following theorem a non uniform bound is derived, corresponding to another result of Landers and Rogge [10]. For the sake of brevity it is assumed that τ is constant. It is not difficult to weaken this assumption like in the preceding theorem (cf. [7]). The moment condition required by Landers and Rogge is somewhat surprising. But it was shown by A. Klein [7] that the theorem becomes wrong, if only the existence of lower moments is assumed.

Theorem 6. *Let X_n, $n \in \mathbb{N}$, be a φ-mixing sequence fulfilling* (1), (4), (7) *and*

$$\sup_{n \in \mathbb{N}} E|X_n|^{s+1} < \infty \qquad \textit{for some} \quad s \geqq 2.$$

Let ε_n, $n \in \mathbb{N}$, be a sequence with $n^{-1} \leqq \varepsilon_n < 1$ and $\varepsilon_n \xrightarrow[n \to \infty]{} 0$.

Set $\delta_n := (\sqrt{\varepsilon_n} + n^{-1/2}(\log n)^{1+(s'+1)/2})|\log \varepsilon_n|$ where $s' = s$ if $s > 2$ and $s' = s+1$ if $s = 2$.

If $\varepsilon_n \geqq n^{-1+\varepsilon}$, $n \in \mathbb{N}$, for some $\varepsilon > 0$ also $\delta_n := \sqrt{\varepsilon_n}$ is allowed.

Let τ_n, $n \in \mathbb{N}$, be positive integer valued random variables with

$$P\{|\tau_n/(n\tau) - 1| > t\varepsilon_n\} \leqq D\delta_n t^{-s}, \qquad n \in \mathbb{N},\ t \geqq t_0,$$

for some constants $\tau > 0$, $t_0 > 0$.

Then for all $n>1$, $t\in\mathbb{R}$

$$\left|P\left\{\sum_{i=1}^{\tau_n} X_i < t\sigma_{[n\tau]}\right\} - \Phi(t)\right| \leqq D\delta_n(1+|t|)^{-s}(\log(2+|t|))^{s+1}. \tag{20}$$

Proof. Define S_n like before. Let

$$\begin{aligned} p &= p(n,t) = [n\tau(1-\varepsilon_n|t|)] \\ q &= q(n,t) = [n\tau(1+\varepsilon_n|t|)] \\ I_n(t) &= \{k\in\mathbb{N}: p\leqq k\leqq q\}. \end{aligned}$$

With a view to the remark after Theorem 5 we can assume that $|t|\geqq t_1$ for some constant $t_1\geqq t_0$. Then

$$Dn\varepsilon_n|t| \leqq q-p \leqq Dn\varepsilon_n|t|. \tag{21}$$

Set $m:=[\lambda^{-1}\log(n^{1/2}|t|^s)]$, then $\varphi(m)\leqq Dn^{-1/2}|t|^{-s}$.

According to (7) and Lemma 2 we have $Dn\leqq\sigma_n^2\leqq Dn$.

(i) Let $n>1$, $t\in\mathbb{R}$ and $t_1\leqq|t|\leqq n^{1/(2s)}$.

First we proceed like in the proof of Theorem 1 in [10] (2)–(6). Instead of Petrov's theorem we use Theorem 4. Like in the proof of Theorem 5 we see that it suffices to estimate

$$R:=P\{S_p\leqq t\sigma_{[n\tau]}\leqq \max_{p\leqq j\leqq q} S_j\}.$$

Since $|t|\leqq n^{1/(2s)}$ we have

$$m\leqq D\log n. \tag{22}$$

We now consider two cases.

Case 1. $|t|\leqq\varepsilon_n^{-(s-1)/s}$.

Let $0<\eta\leqq|t|\,\sigma_{[n\tau]}/6$, $k\in\mathbb{N}$ to be specified later.

We show that there exists a constant D independent of n, t, η, k with

$$\begin{aligned} R\leqq Dk(n^{-1/2}(\log n)^{1+(s'+1)/2}+\sqrt{\varepsilon_n}+k^{s/2}/n^{1/2} \\ +\eta/(|t|\sqrt{n}))/|t|^s+(D((\log n)^{1/2}/\eta)^s)^{k+1}. \end{aligned} \tag{23}$$

For $H:=\max_{p\leqq j\leqq q}(S_j-S_p)$ we have according to Lemma 2(iii) and (21)

$$P\{H\geqq|t|\,\sigma_{[n\tau]}/6\}\leqq D(q-p)^{s/2}/(|t|\,\sigma_{[n\tau]})^s\leqq D\sqrt{\varepsilon_n}/|t|^s \tag{24}$$

since $|t|\leqq\varepsilon_n^{-(s-1)/s}$.

If $p\leqq(2k+1)m$ we have by Lemma 2(ii) and (22)

$$P\{|S_p|\geqq|t|\,\sigma_{[n\tau]}/2\}\leqq Dp^{s/2}/(|t|\,\sigma_{[n\tau]})^s\leqq Dk^{s/2}/(|t|^s n^{1/2}). \tag{25}$$

For $t<0$ this yields (23).

For $t>0$

$$R\leqq P\{|S_p|\geqq t\sigma_{[n\tau]}/2\}+P\{H\geqq t\sigma_{[n\tau]}/2\}.$$

Then (23) follows from (24) and (25).

Let $p>(2k+1)m$.

Setting $L_j:=S_{p-(j-1)m}-S_{p-jm}$ one obtains

$$R \leqq P\left(\bigcap_{i=0}^{k}\{|L_{2i+1}|>\eta\}\right) + \sum_{j=1}^{2k+1} P\{t\sigma_{[n\tau]}-H-\eta \leqq S_{p-jm}+S_p-S_{p-(j-1)m} \leqq t\sigma_{[n\tau]}+\eta\}. \tag{26}$$

Like in the foregoing proof we can bound the first probability by $Dkn^{-1/2}|t|^{-s}+(D((\log n)^{1/2}/\eta)^s)^{k+1}$.

Let $j\in\{1,\ldots,2k+1\}$, $Z:=S_p-S_{p-(j-1)m}$

$$F_n(t):=\{\omega: H(\omega)\geqq|t|\,\sigma_{[n\tau]}/6 \text{ or } |Z(\omega)|\geqq|t|\,\sigma_{[n\tau]}/6\}$$

$$h_1(\omega):=t\sigma_{[n\tau]}-H(\omega)-\eta-Z(\omega), \qquad h_2(\omega):=t\sigma_{[n\tau]}+\eta-Z(\omega).$$

For $\omega\in F_n(t)^c$ (the complement of $F_n(t)$)

$$|h_1(\omega)|\geqq|t|\,\sigma_{[n\tau]}/2, \qquad |h_2(\omega)|\geqq|t|\,\sigma_{[n\tau]}/2. \tag{27}$$

Thus in view of Lemma 3 and Theorem 4 the j-th summand in (26) is bounded by

$$\begin{aligned}
&\varphi(m)+P(F_n(t))+\int_{F_n(t)^c} P\{h_1(\omega)\leqq S_{p-jm}\leqq h_2(\omega)\}\,Pd\omega\\
&\quad\leqq Dn^{-1/2}|t|^{-s}+P(F_n(t))\\
&\qquad+D(p-jm)^{-1/2}(|t|\,\sigma_{[n\tau]}/\sigma_{p-jm})^{-s-1}(\log(p-jm))^{1+(s'+1)/2}\\
&\qquad+\int_{F_n(t)^c}|\Phi(h_2(\omega)/\sigma_{p-jm})-\Phi(h_1(\omega)/\sigma_{p-jm})\,Pd\omega.
\end{aligned}$$

It is easy to bound the first three summands here (use (24)). Now we estimate the integral. According to (27) the integrand is bounded by

$$\begin{aligned}
&D((h_2(\omega)-h_1(\omega))/\sigma_{p-jm})\\
&\qquad\cdot\max\{\exp(-h_1(\omega)^2/(2\sigma_{p-jm}^2)), \exp(-h_2(\omega)^2/(2\sigma_{p-jm}^2))\}\\
&\quad\leqq D((h_2(\omega)-h_1(\omega))/\sigma_{p-jm})(\sigma_{p-jm}/(|t|\,\sigma_{[n\tau]}))^{s+1}\\
&\quad\leqq D(\eta+H(\omega))/(|t|^{s+1}n^{1/2}).
\end{aligned}$$

By using Lemma 2(iii) and (21) the proof of (23) is accomplished.

(23) yields the desired bound if we set $k:=[-\log\varepsilon_n]$, $\eta:=\gamma|t|\log n$ where γ is chosen so that $0<\gamma\leqq\sigma_{[n\tau]}/(6\log n)$ for all $n>1$. If $\varepsilon_n\geqq Dn^{-1+\varepsilon}$, $n\in\mathbb{N}$, we set $\eta:=\gamma\sigma_{[n\tau]}^{\varepsilon}|t|$ with $0<\gamma\leqq\sigma_{[n\tau]}^{1-\varepsilon}/6$, $n\in\mathbb{N}$, and choose $k\in\mathbb{N}$ so large that $s(k+1)\varepsilon>1$.

Case 2. $|t|\geqq\varepsilon_n^{-(s-1)/s}$.

Using this inequality, Theorem 4, Lemma 2 of Feller [5], p. 175 and the fact that $q\leqq Dn|t|\leqq Dn^{1+1/(2s)}$ we obtain for all $j\leqq q$

$$
\begin{aligned}
P\{|S_j|\geqq|t|\,\sigma_{[n\tau]}/8\} &\leqq 2(1-\Phi(|t|\,\sigma_{[n\tau]}/(8\sigma_j)))\\
&\quad + Dj^{-1/2}(\log j)^{1+(s'+1)/2}(|t|\,\sigma_{[n\tau]}/8\sigma_j))^{-s-1}\\
&\leqq D\left(\frac{\sigma_j}{|t|\,\sigma_{[n\tau]}}\right)^{2s} + D(\log n)^{1+(s'+1)/2}\,\frac{\sqrt{n}^{s(1+1/(2s))}}{(|t|\sqrt{n})^{s+1}}\\
&\leqq D\left(\frac{\sqrt{q}}{|t|\sqrt{n}}\right)^{2s} + D/|t|^{s+1}\\
&\leqq D\sqrt{\varepsilon_n}/|t|^s.
\end{aligned}
\tag{28}
$$

For $t<0$ (28) implies directly

$$R\leqq D\sqrt{\varepsilon_n}/|t|^s.$$

If $t>0$ we set

$$A_k:=\{\omega: S_j(\omega)<t\sigma_{[n\tau]},\ p\leqq j<k,\ S_k\geqq t\sigma_{[n\tau]}\}.$$

Then

$$R\leqq P\{t\sigma_{[n\tau]}/2\leqq S_q\}+\sum_{k=p}^{q-1}P(A_k\cap\{S_q<t\sigma_{[n\tau]}/2\}).$$

(28) yields the estimation for the first term.

Set $d=d(k)=\min(m, q-k)$. Then the second is bounded by

$$
\begin{aligned}
&\sum_{k=p}^{q-1}P(A_k\cap\{S_k-S_q>t\sigma_{[n\tau]}/2\})\\
&\quad\leqq P\left(\bigcup_{k=p}^{q-1}\{S_{k+d}-S_k<-t\sigma_{[n\tau]}/4\}\right)+\sum_{k=p}^{q-1}P(A_k\cap\{S_{k+d}-S_q>t\sigma_{[n\tau]}/4\}).
\end{aligned}
$$

The first term can be estimated by using Lemma 2(ii), (21) and (22). The second is smaller than

$$\sum_{k=p}^{q-1}\varphi(m)\,P(A_k)+P(A_k)\,P\{S_q-S_{k+d}<-t\sigma_{[n\tau]}/4\}\leqq D\sqrt{\varepsilon_n}/|t|^s \quad \text{by (28).}$$

(ii) Let $n>1$, $t\in\mathbb{R}$ with $|t|\geqq t_1\geqq 2$, $|t|\geqq n^{1/2s}$.

Then

$$q(n,t)\leqq n\tau+n\tau\varepsilon_n|t|+1\leqq Dn|t|\leqq D|t|^{1+2s} \tag{29}$$

$$m\leqq D\log|t|. \tag{30}$$

W.l.o.g. we assume $t>0$ and then proceed like Landers and Rogge [10], p. 102, (21)–(23). Apparently it suffices to show

$$P\{\max_{p\leqq k\leqq q}S_k\geqq t\sigma_{[n\tau]}\}\leqq Dn^{-1/2}t^{-s}(\log t)^{s+1}.$$

For $k\in\mathbb{N}$, $p\leqq k\leqq q$ define $d=d(k)$ and A_k like in (i).

Since in view of Lemma 2 and (29)

$$P\{|S_q-S_{k+d}|\geqq t\sigma_{[n\tau]}/4\}\leqq D/t^{s/2}$$

we obtain (for t_1 sufficiently large)

$$P\{S_{k+d}-S_q \leqq t\sigma_{[n\tau]}/4\} \geqq 1/2.$$

Therefore by (29), (30)

$$\begin{aligned} P\{\max_{p\leqq k\leqq q} S_k \geqq t\sigma_{[n\tau]}\}/2 &\leqq \sum_{k=p}^{q} P(A_k)\, P\{S_{k+d}-S_q \leqq t\sigma_{[n\tau]}/4\} \\ &\leqq \sum_{k=p}^{q} \varphi(m)\, P(A_k) + P(A_k \cap \{S_{k+d}-S_q \leqq t\sigma_{[n\tau]}/4\}) \\ &\leqq \varphi(m) + \sum_{k=p}^{q} P\{S_k - S_{k+d} \geqq t\sigma_{[n\tau]}/4\} + P(A_k \cap \{S_k - S_q \leqq t\sigma_{[n\tau]}/2\}) \\ &\leqq D n^{-1/2} t^{-s} (\log t)^{(s+1)/2} + P\{S_q \geqq t\sigma_{[n\tau]}/2\}. \end{aligned}$$

(29) implies

$$t\sigma_{[n\tau]}/(2\sigma_q) \geqq D t \sqrt{n}/\sqrt{q} \geqq D\sqrt{t} \geqq D\sqrt{q}^{\,1/(1+2s)}.$$

This shows that Theorem 3 is applicable. It yields with (29) and Feller [5], p. 175

$$\begin{aligned} P\{S_q \geqq t\sigma_{[n\tau]}/2\} &\leqq D q^{1-(s+1)/2} (t\sigma_{[n\tau]}/(2\sigma_q))^{-s-1} (\log q)^{s+1} \\ &\quad + D(\sigma_q/(t\sigma_{[n\tau]}))^{4s} \leqq D n^{-1/2} t^{-s} (\log t)^{s+1}. \end{aligned}$$

Remark. If in Theorem 6 $s>3$ we can apply in its proof Theorem 4 to s instead of $s+1$. Then we can replace $(\log n)^{1+(s+1)/2}$ by $(\log n)^{1+s/2}$ in the definition of δ_n.

4. Lemmas

Lemma 1. *Let $\mathscr{B}_1$, $\mathscr{B}_2$ be sub-σ-algebras of $\mathscr{B}$ and $c>0$.*

If $|P(B_1 \cap B_2) - P(B_1)\, P(B_2)| \leqq c P(B_1)$ for all $B_i \in \mathscr{B}_i$, $i=1,2$ then for all $r_1>1$, $r_2>1$ with $r_1^{-1}+r_2^{-1}=1$, $f_i \in \mathscr{L}_{r_i}(\Omega, \mathscr{B}_i, P)$, $i=1,2$,

$$|Ef_1 f_2 - Ef_1\, Ef_2| \leqq 2c^{1/r_1} \|f_1\|_{r_1} \|f_2\|_{r_2}.$$

Proof. See [2], p. 170, Lemma 1.

Lemma 2. *Let X_n, $n\in\mathbb{N}$, be a φ-mixing sequence. Assume that*

$$EX_n = 0, \qquad n\in\mathbb{N}$$

$$\sum_{n=1}^{\infty} \varphi(n)^{1/2} < \infty$$

$$\sup_{n\in\mathbb{N}} E|X_n|^s \leqq N \quad \text{for some } s>2 \quad \text{and} \quad N>1.$$

For $d>1$ set $Y_i := Y_{d,i} := X_i 1_{\{|X_i|\leqq d\}}$.

(i) *Then for any real number* $v \geqq 2$ *there exists a constant* $C(v)>0$ *depending only on* φ, v, s *and* N *such that for all positive integers* $n \leqq d^2$

$$E\left|\sum_{i=1}^{n} Y_i\right|^v \leqq C(v)(n^{v/2}+nd^{v-s})$$

(ii) *For all* $v \in [2, s]$, $n \in \mathbb{N}$

$$E\left|\sum_{i=1}^{n} X_i\right|^v \leqq 2C(v)\, n^{v/2}.$$

(iii) *For any* $v \in [2, s]$ *there exists* $D(v)>0$ *such that for all* $n \in \mathbb{N}$, $j \in \mathbb{N} \cup \{0\}$

$$E \max_{1 \leqq l \leqq n}\left|\sum_{i=j+1}^{j+l} X_i\right|^v \leqq D(v)\, n^{v/2}.$$

Notice that the constant $C(v)$ *does not alter, if we turn to a subsequence of* X_n, $n \in \mathbb{N}$.

Proof. (i) see [1], Lemma 1.

(ii) follows from (i).

(iii) follows from (ii) on account of [15] Corollary B1.

Lemma 3. *Let* X_i *be a random variable with values in a measurable space* $(\Omega_i, \mathscr{B}_i)$, $i=1, 2$, *and* $0 \leqq c \leqq 1$.

If for all $B_1 \in \mathscr{B}_1$, $B_2 \in \mathscr{B}_2$

$$|P*(X_1, X_2)(B_1 \times B_2) - (P*X_1) \times (P*X_2)(B_1 \times B_2)| \leqq cP*X_1(B_1)$$

then

$$|P*(X_1, X_2)(D) - (P*X_1) \times (P*X_2)(D)| \leqq c$$

for every $D \in \mathscr{B}_1 \times \mathscr{B}_2$.

Proof. See [3] (3.5).

The last lemma is obtained by evaluation of the constant b in Michel's Lemma 3 [11].

Lemma 4. *Let* X *be a random variable with* $EX=0$ *and* $E|X|^{2+c} < \infty$ *for some* $c \in (0, 1]$. *Then for all* $z>0$, $h \geqq \|X\|_2$

$$E(I\{|X| \leqq h\} \exp(zX)) \leqq 1 + z^2 EX^2/2 + E|X|^{2+c} h^{-2-c} \exp(2hz).$$

References

1. Babu, G.J., Ghosh, M., Singh, K.: On rates of convergence to normality for Φ-mixing processes. Sankyā Ser. A. **40**, 278–293 (1978)
2. Billingsley, P.: Convergence of probability measures. New York: Wiley 1968
3. Eberlein, E.: An invariance principle for lattices of dependent random variables. Z. Wahrscheinlichkeitstheorie verw. Gebiete **50**, 119–133 (1979)
4. Erickson, R.V.: Truncation of dependent random variables. Theor. Probability Appl. **20**, 873–880 (1975)

5. Feller, W.: An introduction to probability theory and its applications. Vol. 1. 3. ed., New York: Wiley 1968
6. Hipp, C.: Convergence rates in the central limit theorem for stationary mixing sequences of random vectors. J. Multivariate Anal. **9**, 560–578 (1979)
7. Klein, A.: Nicht gleichmäßige Approximation beim zufälligen zentralen Grenzwertsatz mit nicht konstanter Grenzfunktion. Diplomarbeit, Köln 1978
8. Landers, D., Rogge, L.: The exact approximation order in the central-limit-theorem for random summation. Z. Wahrscheinlichkeitstheorie verw. Gebiete **36**, 269–283 (1976)
9. Landers, D., Rogge, L.: A counterexample in the approximation theory of random summation. Ann. Probability **5**, 1018–1023 (1977)
10. Landers, D., Rogge, L.: On nonuniform Gaussian approximation for random summation. Metrika **25**, 95–114 (1978)
11. Michel, R.: On the accuracy of nonuniform Gaussian approximation to the distribution functions of sums of independent and identically distributed random variables. Z. Wahrscheinlichkeitstheorie verw. Gebiete **35**, 337–347 (1976)
12. Rao, B.L.S.P.: Remark on the rate of convergence in the random central limit theorem for mixing sequences. Z. Wahrscheinlichkeitstheorie verw. Gebiete **31**, 157–160 (1975)
13. Rychlik, Z.: The order of approximation in the random central limit theorem. In: Probability theory on vector spaces (Lect. Notes in Math. **656**), 225–236. Berlin-Heidelberg-New York: Springer 1978
14. Schneider, E.: Genauigkeit der Normalapproximation bei φ-mischenden Prozessen. Dissertation, Köln 1979
15. Serfling, R.J.: Moment inequalities for the maximum cumulative sum. Ann. Math. Statist. **41**, 1227–1234 (1970)
16. Tihomirov, A.N.: On the rate of convergence in the central limit theorem for weakly dependent random variables. Teoriya veroyatn. i ee primen **25**, 800–818 (1980)

Received October 9, 1980, revised August 10, 1981