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Non-Equilibrium Behaviour of a Many Particle Process: Density Profile and Local Equilibria

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Summary. One considers a simple exclusion particle jump process on $\mathbb{Z}$, where the underlying one particle motion is a degenerate random walk that moves only to the right. One starts with the configuration in which the left halfline is completely occupied and the right one free. It is shown that the number of particles at time t between site $[ut]$ and $[vt]$, divided by t, converges a.s. to $\int_u^v f(w)\,dw$, where f might be called the density profile. It is explicitely determined and shown to be an affine function. Secondly we prove that the distribution of the process looked at by an observer travelling at constant speed u, converges weakly to the Bernoulli measure with density $f(u)$, as the time tends to infinity.

1. Introduction

Consider on the space $E=\{0,1\}^Z$ with elements $x=(x_i)$, $i\in Z$, the Markovian process of an asymmetric simple random walk with exclusion. For simplicity we confine ourselves to the extreme case of a walk which only moves to the right; i.e. the generator of the process is given by

$$A\,g(x)=\sum_{i\in Z}(g(\tau_{i,i+1}x)-g(x))\,1_{\{x_i=1,\,x_{i+1}=0\}} \tag{1}$$

for g depending on finitely many coordinates. ($\tau_{ij}x$ is obtained from x by permuting the coordinates at i and j and keeping the rest fixed.) Denote the semigroup which A generates by (T_t); the process variable is denoted by $X(t)$, its coordinates by $X(k,t)$, $k\in Z$.

We let the process start at the fixed initial configuration $X(0)=\bar{x}$, where

$$\bar{x}_k=1_{\{k\leqq 0\}},\qquad k\in Z.$$

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In the sequel, all probabilities and expectations are to be understood with respect to a process with this initial value. In [3] it has been shown that the distribution of $X(t)$, as t goes to infinity, converges weakly to the Bernoulli measure with density 1/2; i.e. at any finite set $\{i_1, \ldots, i_n\}$ of sites the random variables $X(i_j, t)$, $j=1, \ldots, n$, become asymptotically independent with expectation 1/2.

Here we are interested in *global* properties of the system, not only its behaviour at a *fixed* set of sites; there are two aspects of this question which are intimately connected:

(i) First we look at the one particle correlations $\mathscr{E}X(k,t)$, $k\in Z$, over the whole space. We say that the system expands at a linear speed and admits $f(u)$, $u\in R$, as its *density profile*, if for all $u\in R$

$$\lim_{t\to\infty} \mathscr{E}X(k,t)=f(u), \tag{2}$$

whenever $\lim_{t\to\infty} k/t=u$.

(For this particular model, a more general definition allowing also other speeds of expansion is not needed.)

We say that a (strong) *law of large numbers* holds, if

$$\lim \frac{1}{t} \sum_{ut<k<vt} X(k,t)=\int_u^v f(w)\,dw \quad \text{a.s.} \tag{3}$$

for any $u<v$.

A complete answer to all questions in this context in given in Theorem 1 (Sect. 3). It states the existence of a density profile which satisfies the law of large numbers; even an explicite calculation of f is possible; one has a linear decay (in space) of density:

$$\begin{aligned} f(u)&=\tfrac{1}{2}(1-u) && \text{for } |u|\leqq 1, \text{ and}\\ f(u)&=1\,(0) && \text{for } u<-1 \ (u>1). \end{aligned} \tag{4}$$

(ii) The second aspect concerns the limiting behaviour of the particle process seen by a travelling observer, i.e. the distributions of $X([ut]+k,t)$, $k\in Z$ as t tends to infinity, for every fixed $u\in R$, not only $u=0$ as in [3]. ($[a]$: greatest integer less or equal to a.) We say that *propagation of chaos* holds, if all these measures tend weakly to a Bernoulli measure, with a density depending on u (it is $f(u)$, of course). In the physical context, propagation of chaos means asymptotic independence of any finite number of components in a system getting larger and larger. That is exactly what is proved here (Th. 2, Sect. 4); at least locally, in a fixed distance from ut, asymptotic independence of the $X(k,t)$ holds. (The only difference to the situation familiar in many physical contexts is that usually a scale parameter is introduced; for each value of this parameter one has a different dynamics (and tries to find a possible limit law), whereas here in this model we have to deal only with one dynamics. See also Remark 2 below.) Since all Bernoulli measures are invariant under the semigroup (T_t), we see that the system consists of many subsystems in *different* local equilibria, what is not unexpected if one thinks of transport phenomena like heat conduction through a widely extended system. (Compare [1]; in particular the second model there is quite similar to the process considered here.)

The organization of the paper is as follows: In Sect. 2 we state some facts about stochastic order relations between the measures $\mu(k,t)$, $k \in Z$, $t \geqq 0$. ($\mu(k,t)$ denotes the distribution of $X(k+l,t)$, $l \in Z$.) From there we deduce the existence of a density profile and the law of large numbers. Further it is shown that any limit of $\mu([ut],t)$, $t \to \infty$, is an exchangeable measure (i.e. mixture of Bernoulli measures).

Section 3 gives identical lower and upper estimates on the function $h(u)$: $= \int_u^1 f(v)\,dv$ and determines that function.

In Sect. 4 we show that the only limits of $\mu([ut],t)$ are Bernoulli measures. One surprising circumstance, perhaps, lies in the fact that one first has to identify f in order to show the propagation of chaos property. From a "physical" point of view the opposite way would have been more natural: *if* one has almost independence of the variables $X(k,t)$, $X(k+1,t)$ the differential equation for the evolution of the one point correlations becomes (almost) autonomous (as it is the case in the Boltzmann equation)

$$\frac{d}{dt}\rho(k) \approx [\rho(k-1)(1-\rho(k)) - \rho(k)(1-\rho(k+1))], \tag{5}$$

where $\rho(k)$ stands for $\mathscr{E}X(k,t)$; so, at least asymptotically, for large t, one would be able to compute $\mathscr{E}X(k,t) \approx f(k/t)$ and to identify f. We did not try to argue along these lines since we could not show propagation of chaos *a priori*.

Remark 1. Theorem 1 has a geometric interpretation; it determines the asymptotic shape of a subset of R_+^2 (in the sense of Richardson [5]) which grows according to the following rule:

Divide R_+^2 into squares of unit side length. Each square is initially white. A white square becomes black in time $(t, t+dt)$ with probability dt if two of the adjacent squares are black; a black square stays black for ever. The "boundary" $R^2 \setminus R_+^2$ is kept black so that the process can start and does not terminate at finite time. Denote by $c(k,l)$ the square with (k,l) as its right upper corner; then all possible black coloured sets in R_+^2 are of the form

$$\bigcup_{l \geqq 1,\, k(l) \geqq k \geqq 1} c(k,l)$$

where $k(\cdot)$ is decreasing and zero for large l. If one interpretes $k(l)$ as displacement of the k-th particle from its initial position, one gets a one to one correspondence between particle and growth process.

If we denote by B_t the black coloured set in R_+^2 at time t, Theorem 1 then says that the rescaled set $\frac{1}{t} \cdot B_t$ converges to a compact set B a.s. (in the Hausdorff-metric). The set B has as its boundary the two line segments from the origin to $(0,1)$ and to $(1,0)$ and the arc of the parabola $\sqrt{s_1} + \sqrt{s_2} = 1$ between these two points (s_1, s_2: Cartesian coordinates in R^2).

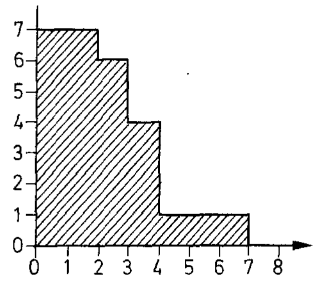

Fig. 1. A typical configuration of the growth process

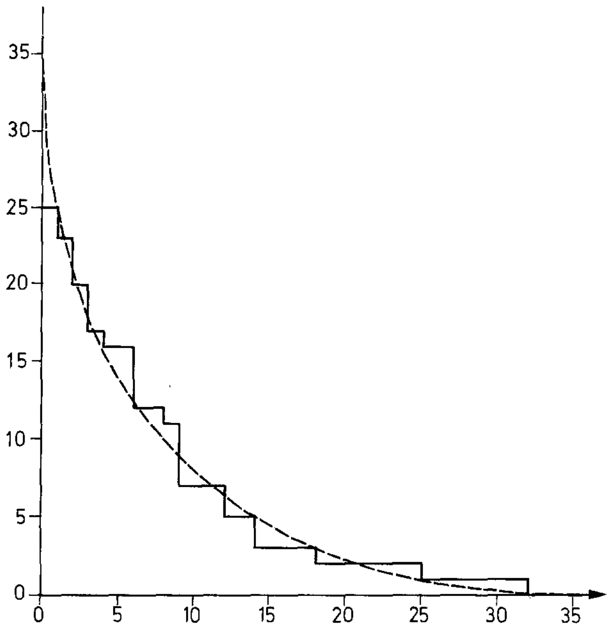

Fig. 2. A simulation of the growth process stopped at an area of 216 unit squares. Dashed line: the parabola $\sqrt{s_1}+\sqrt{s_2}=\text{constant}$ which cuts off the same area

Remark 2. Maybe a more natural way (which is closer to hydrodynamics) to look at the result of Theorem 1 (density profile and law of large numbers) is the following:

for $\varepsilon>0$ define the process $M^\varepsilon(t)$, $t\geqq 0$, whose values are measures on the real line, by

$$M^\varepsilon(t,[u,v]):=\varepsilon \sum_{u\leqq k\varepsilon\leqq v} X(k,t/\varepsilon), \qquad u,v\in R.$$

(That means, each wandering particle contributes a mass ε to the random measure; space and time are suitably rescaled.)

Then Theorem 1 states that for all $t>0$ the random measure $M^\varepsilon(t)$ converges to the deterministic measure $r(t,x)\,dx$ as ε tends to 0, in the sense that

$M^{\varepsilon}(t,[u,v])$ converges in L^1 (and a.s.) to $\int_u^v r(t,x)\,dx$ for all u, v, where r is defined by $r(t,x)=f(x/t)$ (f is the density profile and has been defined in (4)).

The remarkable fact is that r solves the evolution equation

$$\dot{r} = -\frac{\partial}{\partial x}(r(1-r)) \tag{6}$$

to the initial value $r(0,x)=1_{\{x\leqq 0\}}$. One immediately recognizes that (6) is a continuous version of (5); in a certain sense, (6) is the limit dynamics of the originally given simple exclusion process in the "hydrodynamical scaling".

2. Monotonicity Properties

Let X be the original particle process with state space E. On E, the coordinatewise order between points is defined by

$$x \leqq y \quad \text{iff} \quad x_i \leqq y_i \qquad \text{for all } i \in Z, \tag{1}$$

which induces the *stochastic order* for probability measures on E: we say that ν is *stochastically larger* than μ (in symbols $\mu \leqq \nu$) if there exists a measure σ on E^2 with

$$\mu=\pi^1(\sigma), \qquad \nu=\pi^2(\sigma), \qquad \sigma(x^1 \leqq x^2)=1. \tag{2}$$

(π^i: projection on coordinate i, $i=1,2$).

We will also have to look at the process S,

$$S(k,t)=\sum_{i>k} X(i,t), \tag{3}$$

whose appropriate state space E' is the set of all decreasing sequences S_k, $k \in Z$, of natural numbers. S is again Markov and a stochastic order is introduced on E', literally in the same way as in (1), (2).

Finally, we denote by $\mathscr{L}(Y)$ the distribution of a random variable Y; $*$ denotes convolution of measures on R.

Proposition 1. *For all $r, t \geqq 0$, $k, l \in N$ one has*

$$\mathscr{L}(S(k,r)) * \mathscr{L}(S(l,t)) \geqq \mathscr{L}(S(k+l, r+t)). \tag{5}$$

Proof. By a simple coupling argument one sees that the transition kernel of the S-process preserves the stochastic order of measures on E'. (In the picture of the growth process defined above: $s \leqq s'$ means that the black set corresponding to s is contained in the set corresponding to s'.) Hence, if one compares the process $S(u)$, $u \leqq r+t$, to the process $\tilde{S}$, defined in the following way:

$\tilde{S}$ evolves like S before time r; at the instant r, it is replaced by (6)

$$\tilde{S}(j,r)=\begin{cases} S(k,r) & \text{for } j \geqq k \\ S(k,r)+(k-j) & \text{for } j<k, \end{cases}$$

after time r it evolves again according to the dynamics of the S-process, one sees that $\mathscr{L}[\tilde{S}(r+t))\geqq\mathscr{L}(S(r+t))$ holds. But, conditioned on $S(r,k)$, the law of $\tilde{S}(k+l,r+t)-S(r,k)$, $l\in Z$, $t\geqq 0$ is identical to that of $S(l,t)$, $l\in Z$, $t\geqq 0$ and independent of $S(r,k)$. This proves the proposition. □

Proposition 2. *For all $u\in R$ the random variables $\frac{1}{t}S([u\,t],t)$ converge a.s. and in L^1 to a constant $h(u)$, as t goes to infinity. The function h is decreasing, convex; one has $h(u)=0$ for $u>1$ and $h(u)=-u$ for $u<-1$.*

Proof. By Proposition 1 we have

$$\mathscr{L}(S([u\,r],r)*\mathscr{L}(S([u\,t],t))\geqq\mathscr{L}(S[u(r+t)],r+t)). \tag{7}$$

The convergence statement follows from a lemma in subadditive ergodic theory. (See [6], the Kesten-Hammersley theorem.) To prove convexity of h, we deduce from Proposition 1, for $\alpha,\beta>0$, $\alpha+\beta=1$,

$$\mathscr{E}S([\alpha\,u\,t],\alpha\,t)+\mathscr{E}S([\beta\,v\,t],\beta\,t)\geqq\mathscr{E}S([(\alpha\,u+\beta\,v)\,t],t). \tag{8}$$

If one divides by t on either side one obtains

$$\alpha\,h(u)+\beta\,h(v)\geqq h(\alpha\,u+\beta\,v). \tag{9}$$

The other statements of Proposition 2 are obvious. □

Proposition 3. *If h is differentiable at u, one has*

$$\lim_{t\to\infty}\mathscr{E}X(k,t)=-h'(u)$$

whenever k/t tends to u.

Proof. We consider the functions h_t, defined by

$$h_t(v)=\int_v^\infty\mathscr{E}X([w\,t],t)\,dw; \tag{10}$$

they are convex (see Prop. 4, (12) below) and tend, as t goes to infinity, to $h=\lim\frac{1}{t}\mathscr{E}S([\cdot\,t],t)$. Hence, by an elementary lemma on convex functions of a real argument, one has even

$$\lim_t h_t'(v)=h'(u),\qquad \text{if } v\to u \text{ and } h'(u) \text{ exists},$$

where h_t' may be any (right or left) derivative of h_t. □

We define the (unknown) *density profile* f by

$$f(u)=-h'(u+0). \tag{11}$$

Theorem 1 is almost proved, except for the continuity of h' and the identification of f.

We recall the notation $\mu(k,t)=\mathscr{L}(X(k+l,t),\,l\in Z)$.

Proposition 4. *For all* $k \in Z$, $s, t \geqq 0$ *the following stochastic order properties hold:*

$$\mu(k,t) \geqq \mu(k+1,t) \tag{12}$$

$$\mu(k,t+s) \leqq \sum_l \pi(s,l)\,\mu(k-l,t) \tag{13a}$$

$$\mu(k,t+s) \geqq \sum_l \pi(s,l)\,\mu(k+l,t) \tag{13b}$$

where $\pi(s,\cdot)$ *is the Poisson distribution with mean* s.

Proof. (12) is obvious: $\mu(k+1,t)$ is the law of $X(k+l,t)$, $l \in Z$ under the initial condition $\tilde{x}$, where $\tilde{x}_i = 1_{\{i \leqq -1\}}$; since $\tilde{x} \leqq \bar{x}$, (12) follows from the monotonicity of T_t.

To prove (13a) one uses again this monotonicity. The position of the first particle at time s is Poisson distributed with mean s; conditioned on that position, one compares the original process with the process, where all sites behind the first particle are occupied at time s, and which evolves according to (T_t) after time s. (13b) follows from (13a), by symmetry between migration of particles and migration of holes. □

Proposition 5. *If* $h'(u)$ *exists, any weak limit* μ^* *of the measures* $\mu([ut],t)$, $t \to \infty$, *is an exchangeable measure, i.e. is of the form*

$$\mu^* = \int_0^1 \beta_a \rho(da) \tag{14}$$

with some probability ρ *on* $[0,1]$.

(β_a: *Bernoulli measure of density* a, *i.e.*

$$\beta_a(x_{i_j}=1, j=1,\ldots,n) = a^n, \textit{ for all } n,\ i_1 \ldots i_n).$$

The one point correlation of μ^* *is equal to* $\int a\,\rho(da) = f(u)$.

Proof. μ^* is stochastically larger than its image under shift, by Prop. 4. Both measures have the same one point correlations (Prop. 3), hence they are identical. (This follows immediately from the definition of stochastic order via coupling.) But shift invariance of μ^* implies also its invariance under the semigroup (T_t) if one uses the inequalities (13) in Prop. 4. By [4] one knows that shift invariance and invariance under (T_t) imply exchangeability. The rest is de Finetti's theorem. □

We have to exploit further the principle underlying the proof of Prop. 5, as preparation for the proof of Theorem 2. That principle, roughly speaking says, that $\mu(k,t)$ is "slowly varying" in space and time; more precisely: in order that $\mu(k,t)$ changes by an amount of $O(1)$, k or t has to change by an amount of $O(t)$, at least.

Denote by $\rho^n(k,F;t)$ the *n-point correlations* of $\mu(k,t)$, where F is a set of n sites; i.e.

$$\rho^n(k;F;t) = P(X(k+i,t)=1,\ i \in F). \tag{15}$$

Proposition 6. *Assume that $h'(\bar{u})$ exists. For any n and any finite set F of cardinality n and any $\varepsilon>0$ there exists a $\delta>0$, $t_0 \in R_+$ such that*

$$|\rho^n([u\,t], F; t) - \rho^n([\bar{u}\,t], F; t)| \leqq \varepsilon \tag{16}$$

for $|u-\bar{u}| \leqq \delta$ and $t \geqq t_0$,

$$|\rho^n([\bar{u}\,t], F; t+s) - \rho^n([\bar{u}\,t], F; t| \leqq \varepsilon \tag{17}$$

for $0 \leqq s \leqq \delta \cdot t$, $t \geqq t_0$.

Proof. We only show (16). The other statement then follows in connection with Prop. 4, (13), and the fact that $\pi(s, \cdot)$ is essentially carried by a set of the form $\{l: l \leqq c\,s\}$ in the limit $s \to \infty$. Also, by symmetry, we have only to consider the case $u > \bar{u}$.

Since $\mu(k, t)$ is stochastically decreasing in k we need an upper estimate only for

$$\rho^n([\bar{u}\,t], F; t) - \rho^n([u\,t], F; t).$$

But such an estimate, by the definition of stochastic order via coupling, is provided by

$$\sum_{i \in F} (\rho^1([\bar{u}\,t], i; t) - \rho^1([u\,t], i, t)) = \sum_{i \in F} \mathscr{E}(X([\bar{u}\,t] + i, t) - X([u\,t] + i, t)). \tag{18}$$

Take $\delta > 0$ such that $h'(\bar{u}+\delta)$ exists and satisfies $-h'(\bar{u}+\delta) = f(\bar{u}+\delta) \geqq f(\bar{u}) - \frac{\varepsilon}{2n}$ (f is continuous at $\bar{u}$). Proposition 3 gives

$$\liminf_{t \to \infty} \mathscr{E}(X([\bar{u}+\delta)\,t] + i, t) \geqq f(\bar{u}) - \frac{\varepsilon}{2n}, \tag{19}$$

hence, for any $u \leqq \bar{u} + \delta$, uniformly in u,

$$\limsup_{t \to \infty} \sum_{i \in F} [\rho^1([\bar{u}\,t], i; t) - \rho^1([u\,t], i; t)] \leqq \varepsilon/2 \tag{20}$$

which proves that

$$\rho^n([\bar{u}\,t], F; t) - \rho^n([u\,t], F; t)$$

becomes eventually smaller than ε, uniformly in $u \leqq \bar{u} + \delta$. □

3. Identification of the Profile f

We start by giving a lower estimate for the function $h(u) = \int_u^\infty f(w)\,dw = \lim_t \frac{1}{t} \mathscr{E} S([u\,t], t)$. To this end one has to look at the *interparticle distances* $Z(0,t) - Z(1,t)$, $Z(1,t) - Z(2,t)$, ... where $Z(i,t)$ is the position at time t of the particle originally located at $-i$. We write $Y(k,t)$ for $Z(k-1,t)-Z(k,t)$, $k=1,2,\ldots$.

Proposition 7. $h(u) \geqq \frac{1}{4}(1-u)^2$ *for* $|u| \leqq 1$.

Proof. We modify the dynamics of the system in the following way: the first particle is allowed to jump only at a rate b, where $b<1$; the jump rates of the other particles are unchanged, i.e. equal to one if they can jump at all. Expectations with respect to this process will be denoted by $\mathscr{E}^b$, its probability law by P^b.

These dynamics, in terms of the Y-process, may be described as follows: the state space is the set of all sequences of integers greater or equal to 1; from the point $(y_1, y_2, \ldots)$ the following jumps are allowed:

to $(y_1, y_2, \ldots, y_i-1, y_{i+1}+1, \ldots)$ at rate 1 if $y_i>1$,
to $(y_1+1, y_2, \ldots)$ at rate b.

One checks immediately the following two statements about this process, resp. its semigroup: first, it preserves stochastic order; second, an invariant measure is γ^b, described by

all coordinates are independent, identically distributed under γ^b;
$\gamma^b(y_i>m)=b^m$ for all $m \geqq 0$.

(See for example [7], the computations in Chap. 12-4.)

We now compare two Y-processes, one with the initial value we are interested in, viz. $y_1=y_2=\ldots=1$, and the stationary process with initial measure γ^b. Since the law of the first process is stochastically smaller than that of the second at time $t=0$, this holds for all times. We thus get for all $t \geqq 0$, $k \in N$

$$\mathscr{E}^b\left(\sum_1^k Y(j,t)\right) \leqq \int \left(\sum_1^k y_j\right) \cdot \gamma^b(dy) = k \cdot \sum_{m \geqq 0} \gamma^b(y_1>m) = k \cdot (1-b)^{-1}.$$

If we choose $k=[a\,t]$, $0<a<1$ fixed, and let t tend to infinity we arrive at

$$\limsup_{t\to\infty} \frac{1}{t}\, \mathscr{E}^b\Big(\sum_{j \leqq at} Y(j,t)\Big) \leqq a(1-b)^{-1}. \tag{1}$$

This result can be sharpened to provide us with a stochastic upper bound for $\sum_{j \leqq at} Y(j,t)$; the weak law of large numbers for γ^b gives

$$\text{for any } \varepsilon>0, \qquad \lim_{t\to\infty} P^b\left(\frac{1}{t} \cdot \sum_{j \leqq at} Y(j,t) > a(1-b)^{-1}+\varepsilon\right)=0. \tag{2}$$

About the displacement process Z we know that $Z(0,t)$ is Poisson distributed with mean $b\,t$, hence by a law of large numbers for Poisson distributions we obtain from (2)

$$\lim_{t\to\infty} P^b\left(\frac{1}{t} \cdot Z([a\,t],t) < b-a(1-b)^{-1}-\varepsilon\right)=0 \qquad \text{for any } \varepsilon>0. \tag{3}$$

The Z-process gets stochastically larger if b increases to 1; hence (3) remains valid if we replace P^b by P, the original dynamics of the Z-process. This holds for every b. We choose b (depending on a) in such a way that the right hand side within the brackets becomes maximal. Since

$$\max_{0<b<1} (b-a(1-b)^{-1})=1-2\sqrt{a} \quad \text{for } 0<a<1 \tag{4}$$

we get finally

$$\lim_{t\to\infty} P\left(\frac{1}{t}\cdot Z([a\,t],t)<1-2\sqrt{a}-\varepsilon\right)=0 \quad \text{for all } \varepsilon>0,\ 0<a<1. \tag{5}$$

Translating (5) into terms of the S-process one obtains for all $\varepsilon>0$, $0<a<1$

$$\lim_{t\to\infty} P\left(\frac{1}{t} S([(1-2\sqrt{a}-\varepsilon)\,t],t)>a\right)=1, \tag{6}$$

hence

$$h(1-2\sqrt{a})\geqq a \quad \text{or} \quad h(u)\geqq\tfrac{1}{4}(1-u)^2 \quad \text{for } |u|\leqq 1. \quad \square \tag{7}$$

The final step in order to prove Theorem 1 is the following upper estimate on h.

Proposition 8. $h(u)\leqq\frac{1}{4}(1-u)^2$ *for* $|u|\leqq 1$.

Proof. Consider the case of $u>0$, put $u^{-1}=w$; assume $h'(u)$ exists. We compute the expectation of $S([u\,t],t)$ which is the number of particles which have passed an observer travelling at speed u, minus the number of particles passed by the observer:

$$\begin{aligned}\mathscr{E}S(k,k\,w)&=\sum_{l=1}^{k}\mathscr{E}(S(l,l\,w)-S(l-1,l\,w))\\&\quad+\sum_{l=1}^{k}\mathscr{E}(S(l-1,l\,w)-S(l-1,(l-1)\,w))\\&=-\sum_{l=1}^{k}\mathscr{E}X(l,l\,w)+\sum_{l=1}^{k}\int_{(l-1)w}^{lw}P(X(l-1,t)=1,X(l,t)=0)\,dt.\end{aligned} \tag{8}$$

(The integral is the expected number of jumps from $l-1$ to l in the time interval $((l-1)\,w,l\,w)$.)

Multiplying both sides by $u\,k^{-1}$ one gets, if k tends to infinity,

$$h(u)=u\cdot h'(u)+\lim_{T\to\infty}\frac{1}{T}\int_0^T \mu([u\,t],t)\,(x_0=1,x_1=0). \tag{9}$$

The same way one shows that (9) holds for negative u, too. We need this relation for a dense set of u's in $[-1,1]$.

Now, by Proposition 5, any limit of

$$\mu([u\,t],t)\,(x_0=1,x_1=0), \qquad t\to\infty,$$

is of the form

$$\int (a-a^2)\,\rho(da) \quad \text{with} \quad \int a\,\rho(da) = f(u). \tag{10}$$

Hence, by Jensen's inequality we get

$$\limsup_{t\to\infty} \mu([u\,t], t)\,(x_0=1, x_1=0) \leqq f(u) - (f(u))^2. \tag{11}$$

Combining (9) and (11), we obtain, since $0 \leqq f(u) \leqq 1$,

$$h(u) \leqq \sup_{0<b<1} \{b\,u + b - b^2\} = \tfrac{1}{4}(1-u)^2. \tag{12}$$

This proves Proposition 8. □

From Proposition 2, 3, 7, 8 together we obtain

Theorem 1. (Density profile and law of large numbers.) *For any $u \in R$ the limit $\lim_{t\to\infty} \mathscr{E} X(k,t)$ exists and is equal to $f(u)$, whenever k/t tends to u. The function f is given by*

$$f(u) = \begin{cases} \frac{1}{2}(1-u) & \textit{for } |u| \leqq 1 \\ 1\,(0) & \textit{for } u < -1\,(u > 1). \end{cases} \tag{13}$$

The quantities $\frac{1}{t} \sum_{ut<k<vt} X(k,t)$ converge a.s. to the constant value $\int_u^v f(w)\,dw$, for $u < v$.

Remark. Returning to the picture of the growth process in R_+^2 (Remark 1 in Section 1) we can now derive an explicit expression for the asymptotic shape of B_t. Consider $|u| \leqq 1$. Stochastic convergence of $\frac{1}{t} S([ut], t)$ to $h(u)$ means that particle number $[h(u)\,t]$ at time t is at site $ut + o(t)$, what is equivalent to saying that this particle at time t has travelled a distance $(h(u)+u)\,t + o(t)$ from its initial position. So the boundary of B_t is close (up to terms of order $o(t)$) to the point

$$((h(u)+u)\,t,\, h(u)\,t).$$

This gives a parametrisation of the boundary arc of B

$$s_1 = h(u) + u, \quad s_2 = h(u), \quad -1 < u \leqq 1 \tag{14}$$

which because of $h(u) = \frac{1}{4}(1-u)^2$ leads to

$$\sqrt{s_1} + \sqrt{s_2} = 1. \tag{15}$$

4. Propagation of Chaos

We want to show weak convergence of $\mu([ut], t)$ to the Bernoulli measure $\beta_{f(u)}$ for any $u \in R$. The statement is non-trivial only for $|u| \leqq 1$, and only this case will be discussed below. Also, by monotonicity, if this result can be proven, we have automatically convergence of $\mu(k,t)$ to $\beta_{f(u)}$ for all $k=k(t)$ with $k/t \to u$.

We recollect the results obtained in the last section about the asymptotic behaviour of $\mu([ut],t)(x_0=1, x_1=0)$. In (9) we have stated Cesàro convergence towards $h(u)+uf(u)$, which turns out to be equal to $\frac{1}{4}(1-u^2)$ or $f(u)-f^2(u)$; on the other hand, (11) states that the lim sup is less or equal to the same quantity $f(u)-f^2(u)$.

We formulate this result in terms of correlation functions (notations as in Section 2, Proposition 6); since $\lim_t \rho^1([ut];k;t)=f(u)$ for $k=0$ or 1, we get

$$\liminf_{t\to\infty} \rho^2([ut];0,1;t) \geqq f(u)^2 \tag{1}$$

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \rho^2([ut];0,1;t)=f(u)^2. \tag{2}$$

We want to show that

$$\limsup_{t\to\infty} \rho^2([ut];0,1;t) \leqq f(u)^2. \tag{3}$$

If (3) holds the only weak limit of any subsequence of $\mu([ut],t)$ is the Bernoulli measure $\beta_{f(u)}$, because any limit is of the form $\int \rho(da)\,\beta_a$, where ρ satisfies (in virtue of (1) and (3))

$$\int a\,\rho(da)=f(u), \qquad \int a^2\rho(da)=f(u)^2. \tag{4}$$

But (4) implies that ρ is the unit mass at $f(u)$.

To show (3), we conclude from (1) and (2) that

$$\text{for any } \varepsilon>0 \text{ the set } A_\varepsilon=\{t:\rho^2([ut];0,1;t)>f^2(u)+\varepsilon\} \tag{5}$$

is of Cesàro density zero.

We have to show that the sets A_ε are actually bounded. To this purpose one uses Proposition 6, relation (17):

$$\begin{gathered}\text{there is a } \delta>0,\ t_0\in R_+ \text{ such that } t\geqq t_0,\ t\in A_\varepsilon\\ \text{implies } s\in A_{\varepsilon/2} \text{ for all } s\in(t,t+\delta t).\end{gathered} \tag{6}$$

Hence if A_ε is not bounded, $A_{\varepsilon/2}$ is not of Cesàro density zero, in contradiction to (5). So we have proven

Theorem 2. *For any $u\in R$, the measures $\mu(k,t)$, as t tends to infinity, tend weakly to the Bernoulli measure with density $f(u)$, whenever k/t tends to u.*

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