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Existence of Optimal Stochastic Controls under Partial Observation

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Summary. This paper treats the problem of existence of optimal controls in partially observable systems whose dynamics are described by a nonlinear stochastic differential equation. The technique applied is based on weak convergence of probability measures and on the construction of stochastically equivalent processes.

1. Introduction

This paper concerns the control of a system whose dynamics are governed by the nonlinear stochastic differential equation

$$dx = f(t, x, u(t, x))\,dt + \sigma(t, x)\,dw, \qquad 0 \leqq t \leqq T, \tag{1.1}$$

with initial condition

$$x(0) = x_0. \tag{1.2}$$

Here w is an r-dimensional Brownian motion, f and σ are r-vector valued and $r \times r$-matrix valued nonanticipating functions, respectively, and the control u is a function whose value at time t may depend at most on specified information about the past of $x(\cdot)$ up to time t. The control is to be chosen so as to minimize the expected cost

$$J(x, u) = E\left\{\int_0^T l(t, x, u(t, x))\,dt\right\} + E\{g(x)\}. \tag{1.3}$$

Existence results for this type of problems involve heavily the information pattern available to the controller. The techniques based on the Girsanov measure transformation method (cf. [1, 5]) seem to work only in the case of complete information about the past.

Here we use a method applied by Kushner (cf. [6]), which is based on weak convergence of probability measures, and extend it to the case of partial

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observation. The underlying concept of solution to (1.1), (1.2) is that of weak solutions (cf. [7]), which takes into account only the distributions of the processes involved.

The results that will be obtained in this paper require only mild regularity assumptions about the drift $f(t,x,u)$. In particular, Roxin's condition (which requires the convexity of the velocity sets $f(t,x,\mathscr{U})$, $\mathscr{U}$ being the space of control points) is not needed here. Moreover, a fairly large class of information patterns is covered. However, this is at the price of a more restrictive class of control laws to be admitted.

2. Assumptions and Formulation of the Problem

The following notations and assumptions will be used throughout.

C_T^k = space of $\mathbb{R}^k$-valued continuous functions on $[0,T]$ with the sup-norm topology.

$\mathscr{C}_t^k$ = σ-algebra on C_T^k induced by the continuous functions on $(0,t]$ for $0 \leqq t \leqq T$; i.e. $\mathscr{C}_t^k$ is the σ-algebra generated by all sets of the form $\{\xi\colon \xi(s) \in \Gamma\}$, where $0 \leqq s \leqq t$, Γ is an arbitrary Borel set in $\mathbb{R}^k$ and ξ denotes the generic element of C_T^k.

In the sequel, r, l and m are positive integers, and ξ, η and ζ denote the generic elements of C_T^r, C_T^l and C_T^m, respectively.

(A1) Let $\varphi\colon C_T^r \to C_T^l$ be a continuous function satisfying $\varphi^{-1}(\mathscr{C}_t^l) \subset \mathscr{C}_t^r$ for all $t \in [0,T]$.

φ is the 'observation function', which means that $\varphi(x)(s)$, $0 \leqq s \leqq t$, represents the information available on the process x at time t. The measurability condition expresses that this information comprises at most the past of $x(\cdot)$ prior to t. It is equivalent to requiring that the process $\varphi(\xi)(t)$, $0 \leqq t \leqq T$, in C_T^r be adapted to $(\mathscr{C}_t^r)$. Let us give some examples of information patterns fitting our assumption.

1) Only certain components of the system process x can be observed:

$$\varphi(\xi) = \xi_2, \qquad \text{where } \xi = (\xi_1', \xi_2')'.$$

2) Delayed observation:

$\varphi(\xi)(t) = \xi(t-\tau)$, where τ is some fixed delay time and the convention $\xi(s) = \xi(0)$ for negative values of s is adapted.

3) $\varphi(\xi)(t) = \int_0^t \xi(s)\,ds.$

The choice of C_T^l as sample space for the observed process is not crucial. For instance, one might alternatively take the space of piecewise constant functions with jumps occuring at specified times $t_1, t_2, \ldots$. This would allow for observation taking place only at discrete times.

(A2) Let $\mathscr{U}\colon [0,T] \times C_T^l \to \mathbb{R}^m$ be a continuous point-to-set mapping (with respect to the Hausdorff metric) assigning to each pair (t,η) a closed set $\mathscr{U}(t,\eta)$ in $\mathbb{R}^m$.

Denote by $\mathscr{U}_0$ some measurable set containing all the sets $\mathscr{U}(t,\eta)$, and by $\mathscr{B}_T$ and $\mathscr{B}_{\mathscr{U}_0}$ the Borel σ-algebras on $[0,T]$ and $\mathscr{U}_0$, respectively.

(A3) Let $f: [0,T]\times C_T^r\times\mathscr{U}_0\to\mathbb{R}^r$ be measurable with respect to the σ-algebra $\mathscr{B}_T\otimes\mathscr{C}_T^r\otimes\mathscr{B}_{\mathscr{U}_0}$. For each pair $(t,u)\in[0,T]\times\mathscr{U}_0$ let the function $f(t,\cdot,u)$ be $\mathscr{C}_t^r$-measurable, and for each $t\in[0,T]$ let $f(t,\cdot,\cdot)$ be continuous.

(A4) Let $\sigma: [0,T]\times C_T^r\to r\times r$-matrices be measurable with respect to $\mathscr{B}_T\otimes\mathscr{C}_T^r$. For each $t\in[0,T]$, let $\sigma(t,\cdot)$ be continuous and $\mathscr{C}_t^r$-measurable.

(A5) Let $l: [0,T]\times C_T^r\times\mathscr{U}_0\to\mathbb{R}$ be a nonnegative function measurable with respect to $\mathscr{B}_T\otimes\mathscr{C}_T^r\otimes\mathscr{B}_{\mathscr{U}_0}$. For each pair $(t,u)\in[0,T]\times\mathscr{U}_0$ let $l(t,\cdot,u)$ be $\mathscr{C}_t^r$-measurable, and for each $t\in[0,T]$ let $l(t,\cdot,\cdot)$ be continuous.

(A6) $g: C_t^r\to\mathbb{R}$ is continuous and bounded.

Let us now make precise what we understand by an admissible control law. To make clear the principal ideas, we shall first deal with the case where only controls with continuous trajectories are admitted. The general case of measurable controls is postponed to Sect. 4.

A function $u: [0,T]\times C_T^l\to\mathbb{R}^m$ is called an admissible control if it satisfies the following set of assumptions (i)–(vi).

(i) $u(\cdot,\eta)\in C_T^m$ for all $\eta\in C_T^l$.

There exists a process $x(t)$, $0\leqq t\leqq T$, defined on some probability space $(\Omega,\mathscr{F},P)$, with continuous trajectories, such that the following conditions are fulfilled.

(ii) $x(0)$ has the prescribed distribution F_0.

(iii) For every $\rho>0$ there is a continuous function $r_\rho: [0,\rho]\to R^+$, $r_\rho(0)=0$, such that for all $0\leqq t\leqq T$

$$\|u(\cdot,\eta)-u(\cdot,\eta')\|_t\leqq r_\rho(\|\eta-\eta'\|_t)$$

for all $\eta,\eta'\in\mathscr{R}=\mathrm{cl}(\varphi(x(\Omega))$ satisfying $\|\eta-\eta'\|_t<\rho$. Here $\|\cdot\|_t$ denotes the sup-norm on $[0,t]$.

Example. $r_\rho(z)=K_\rho|z|^\alpha$, $\alpha>0$ (local Hölder condition). In particular, (iii) means that $u(s,\eta)=u(s,\eta')$ for all $0\leqq s\leqq t$ if $\eta_t=\eta'_t$ (with η_t denoting the restriction of η to $[0,t]$). As a consequence, the process $u(t,\eta)$, $0\leqq t\leqq T$, defined on $\mathscr{R}$ is adapted to $(\tilde{\mathscr{C}}_t^l)$, the σ-algebra induced on $\mathscr{R}$ by $(\mathscr{C}_t^l)$. Since the trajectories are continuous, it follows that $u(\cdot,\cdot): [0,T]\times\mathscr{R}\to\mathbb{R}^m$ is measurable with respect to $\mathscr{B}_T\otimes\tilde{\mathscr{C}}_T^l$.

(iv) $u(t,\eta)\in\mathscr{U}(t,\eta)$ for all $\eta\in\mathscr{R}$, $t\in[0,T]$.

(v) There exists an r-dimensional Brownian motion $(w(t),\mathscr{F}_t)$, $0\leqq t\leqq T$, on $(\Omega,\mathscr{F},P)$ such that $x(t)$ is nonanticipative with respect to $(\mathscr{F}_t)$ and the Ito equation

$$x(t)=x_0+\int_0^t f(s,x,u(s,\varphi(x)))\,ds+\int_0^t \sigma(s,x)\,dw(s) \tag{2.1}$$

holds with probability one for all $0\leqq t\leqq T$.

Remark. It is easily seen that the process $u(t, \varphi(x))$, $0 \leqq t \leqq T$, is adapted to $(\mathscr{F}_t)$.

The process x will be called a solution of (2.1) corresponding to the control u.

(vi) There is a constant K such that, uniformly in $0 \leqq t \leqq t+\Delta \leqq T$,

a) $$E \int_t^{t+\Delta} |f(s, x, u(s, \varphi(x)))| \, ds^2 \leqq K\Delta^2,$$

$$E \int_0^T |f(s, x, u(s, \varphi(x)))|^2 \, ds \leqq K;$$

b) $$E \int_0^T |\Sigma(s, x)|^4 \, ds \leqq K,$$

where $\Sigma = \sigma\sigma'$;

c) $$E \int_0^T |l(s, x, u(s, \varphi(x)))|^2 \, ds \leqq K;$$

d) $$E|u(0, \varphi(x))|^2 \leqq K, \qquad E|u(t+\Delta, \varphi(x)) - u(t, \varphi(x))|^2 \leqq K\Delta^2.$$

Note that the functions r_ρ in (iii) as well as the constant K in (vi) are assumed to be the same for all admissible u.

Let $\mathscr{A}$ denote the class of admissible controls. A rigorous formulation of the control problem can now be given:

$$\text{(P)} \quad \begin{cases} \text{minimize} \\ J(x, u) = E\left\{\int_0^T l(t, x, u(t, \varphi(x))) \, dt\right\} + E\{g(x)\} \\ \text{in the class } \mathscr{A} \text{ of admissible controls } u \text{ and} \\ \text{corresponding solutions } x. \end{cases}$$

By virtue of (A5) and (A6) $\hat{J} = \inf\{J(x, u)\colon u \in \mathscr{A},\ x \text{ solution corresponding to } u\}$ is finite.

3. Existence of Optimal Controls

Every $u \in \mathscr{A}$ defines a function $U\colon C_T^l \to C_T^m$ by setting $U(\eta)(t) = u(t, \eta)$. If x is a solution corresponding to u, then the mapping $V = U(\varphi(x(\cdot)))\colon \Omega \to C_T^m$ is measurable with respect to $\mathscr{F}_T$ by virtue of (iii).

Define functions

$$F(t) = \int_0^t f(s, x, u(s, \varphi(x))) \, ds, \qquad B(t) = \int_0^t \sigma(s, x) \, dw(s), \tag{3.1}$$

and

$$\Phi(t) = (x(t), F(t), B(t), V(t)). \tag{3.2}$$

The process Φ is adapted to $(\mathscr{F}_t)$ with paths in $S=C_T^{3r}\times C_T^m$. It induces on the Borel field of S a probability measure Q by

$$Q(A)=P_\Phi(A)=P[\Phi\in A].$$

Let $\mathscr{P}$ denote the class of all probability measures generated in this way with u ranging in $\mathscr{A}$. Then condition (vi) implies that $\mathscr{P}$ is tight (cf. [2], p. 95), hence every sequence in $\mathscr{P}$ contains a weakly convergent subsequence. The verification of the moment condition (12.51) in [2] is non-trivial only for the B-component of Φ. By Satz I.3.6 in [5]

$$\begin{aligned} E|B(t+\Delta)-B(t)|^4 &= E\left|\int_t^{t+\Delta}\sigma(s,x)\,dw(s)\right|^4 \\ &\leqq 6r^2\Delta\cdot\sum_{i,k}E\int_t^{t+\Delta}\sigma_{ik}^4(s,x)\,ds, \end{aligned}$$

and by Hölder's inequality

$$E\int_t^{t+\Delta}\sigma_{ik}^4\,ds\leqq\Delta^{1/2}\left[E\int_0^T\sigma_{ik}^8\,ds\right]^{1/2}\leqq\Delta^{1/2}\left[E\int_0^T|\Sigma|^4\,ds\right]^{1/2},$$

whence

$$E|B(t+\Delta)-B(t)|^4\leqq 6r^4K^{1/2}\Delta^{3/2}. \tag{3.3}$$

The following lemma shows that $\mathscr{P}$ is weakly closed.

Lemma 1. *Let u_n, $n=1,2,\ldots$, be a sequence of admissible controls with corresponding solutions x_n defined on probability spaces $(\Omega_n,\mathscr{F}_n,P_n)$. Let Q_n denote the probability measures induced on S, i.e. $Q_n=(P_n)_{\Phi_n}$, where Φ_n is defined by* (3.1), (3.2) *with $x=x_n$, $u=u_n$, and suppose that $Q_n\to Q_0$ weakly, where Q_0 is a probability measure on S. Then there exists a sequence $\tilde u_n$, $n=0,1,2,\ldots$, of admissible controls with corresponding solutions $\tilde x_n$, all defined on the same probability space $(\tilde\Omega,\tilde{\mathscr{F}},\tilde P)$, such that*

$$Q_n=\tilde P_{\tilde\Phi_n}=\tilde Q_n,\qquad n=0,1,2,\ldots,$$

and

$$\tilde\Phi_n\to\tilde\Phi_0\qquad \tilde P\text{-a.e.}$$

in the topology of S ($\tilde\Phi_n$ is defined analogously to Φ_n).

Proof. In the first step, we shall use a modification of Skorokod's proof of his lemma ([9], p. 10) to construct processes $\tilde\Phi_n$ in such a way that the fourth (=control-) component can be factorized via the first (=state-) component (cf. (3.9)).

Since S is a complete separable metric space, we can find Borel sets $S_{i_1,i_2,\ldots,i_k}$ in S as well as subintervals $\Delta^{(n)}_{i_1,i_2,\ldots,i_k}$ if [0, 1], left open and right closed, for all natural numbers $k,i_1,i_2,\ldots,i_k$, $n=1,2,\ldots$, with the properties required in [9]. For the reader's convenience, we repeat them here.

Assumptions about $S_{i_1, i_2, \ldots, i_k}$:

1. $S_{i_1, i_2, \ldots, i_k}$ and $S_{i'_1, i'_2, \ldots, i'_k}$ are disjoint for $i_k \neq i'_k$.

2. $\bigcup_{i_k=1}^{\infty} S_{i_1, i_2, \ldots, i_{k-1}, i_k} = S_{i_1, i_2, \ldots, i_{k-1}}$; $\bigcup_{i=1}^{\infty} S_i = S$.

3. The diameter of $S_{i_1, i_2, \ldots, i_k}$ does not exceed 2^{-k}.

4. For every $i_1, i_2, \ldots, i_k$,

$Q_0(\partial S_{i_1, i_2, \ldots, i_k}) = 0,$

where ∂A denotes the topological boundary of A.

Assumptions about $\Delta^{(n)}_{i_1, i_2, \ldots, i_k}$, $n = 1, 2, \ldots,$:

1. $\Delta^{(n)}_{i_1, i_2, \ldots, i_k}$ and $\Delta^{(n)}_{i'_1, i'_2, \ldots, i'_k}$ are disjoint for $i_k \neq i'_k$.

2. The interval $\Delta^{(n)}_{i_1, i_2, \ldots, i_k}$ is to the left of the interval $\Delta^{(n)}_{i'_1, i'_2, \ldots, i'_k}$ if $i_j = i'_j$ for $j < r$ and $i_r < i'_r$ for some r.

3. The length of the interval $\Delta^{(n)}_{i_1, i_2, \ldots, i_k}$ is equal to $Q_n(S_{i_1, i_2, \ldots, i_k})$.

Note that the intervals $\Delta^{(n)}_{i_1, i_2, \ldots, i_k}$ satisfy the same relation as is required form the $S_{i_1, i_2, \ldots, i_k}$ in condition 2.

Next, choose points

$$z^{(n)}_{i_1, i_2, \ldots, i_k} = \Phi_n(\omega^{(n)}_{i_1, i_2, \ldots, i_k}) \in S_{i_1, i_2, \ldots, i_k} \tag{3.4}$$

(with $\omega^{(n)}_{i_1, i_2, \ldots, i_k} \in \Omega_n$), if possible, and set

$$\tilde{\Phi}^k_n(\tilde{\omega}) = z^{(n)}_{i_1, i_2, \ldots, i_k} \quad \text{for } \tilde{\omega} \in \Delta^{(n)}_{i_1, i_2, \ldots, i_k}. \tag{3.5}$$

Since the choice (3.4) is possible whenever $Q_n(S_{i_1, i_2, \ldots, i_k})$ is positive, and $\Delta^{(n)}_{i_1, i_2, \ldots, i_k} = \emptyset$ when $Q_n(S_{i_1, i_2, \ldots, i_k}) = 0$, $\tilde{\Phi}^k_n$ is defined on the whole interval $[0, 1] = \tilde{\Omega}$ and is obviously Borel measurable.

Now consider the random functions $\hat{\Phi}^k_n$ obtained by direct application of Skorokhod's technique, which differ from the above functions by the fact that in (3.4) points $z_{i_1, i_2, \ldots, i_k} \in S_{i_1, i_2, \ldots, i_k}$ are chosen independently of n:

$$\hat{\Phi}^k_n(\tilde{\omega}) = z_{i_1, i_2, \ldots, i_k} \quad \text{for } \tilde{\omega} \in \Delta^{(n)}_{i_1, i_2, \ldots, i_k}.$$

Then, as is shown in [9], the limits

$$\tilde{\Phi}_n(\tilde{\omega}) = \lim_{n \to \infty} \hat{\Phi}^k_n(\tilde{\omega}), \qquad n = 1, 2, \ldots,$$

and

$$\tilde{\Phi}_0(\tilde{\omega}) = \lim_{n \to \infty} \tilde{\Phi}_n(\tilde{\omega}) \tag{3.6}$$

exist a.e.; moreover,

$$Q_n = \tilde{P}_{\tilde{\Phi}_n}, \qquad n = 0, 1, 2, \ldots. \tag{3.7}$$

But since, for $\tilde{\omega} \in \Delta^{(n)}_{i_1, i_2, \ldots, i_k}$, both $\tilde{\Phi}^k_n(\tilde{\omega})$ and $\hat{\Phi}^k_n(\tilde{\omega})$ lie in $S_{i_1, i_2, \ldots, i_k}$, it is easily established that

$$\tilde{\Phi}_n(\tilde{\omega}) = \lim_{k \to \infty} \tilde{\Phi}^k_n(\tilde{\omega}) \text{ a.e.}, \qquad n = 1, 2, \ldots. \tag{3.8}$$

To proceed, write

$$\tilde{\Phi}_n^k = (\tilde{x}_n^k, \tilde{F}_n^k, \tilde{B}_n^k, \tilde{V}_n^k), \qquad \tilde{\Phi}_n = (\tilde{x}_n, \tilde{F}_n, \tilde{B}_n, \tilde{V}_n).$$

By construction ((3.4) and (3.5)) and by the continuity properties of U_n and φ

$$\begin{aligned}\tilde{V}_n(\tilde{\omega}) &= \lim_{k\to\infty} \tilde{V}_n^k(\tilde{\omega}) \\ &= \lim_{k\to\infty} U_n(\varphi(\tilde{x}_n^k(\tilde{\omega}))) \\ &= U_n(\varphi(\tilde{x}_n(\tilde{\omega})))\end{aligned} \tag{3.9}$$

for almost every $\tilde{\omega}$. Morevoer,

$$\varphi(\tilde{x}_n(\tilde{\Omega})) \subset \mathrm{cl}(\varphi(x_n(\Omega_n))).$$

Now proceed in the same way as in [6]. The mapping

$$G_t(\xi, \zeta) = \int_0^t f(s, \xi, \zeta(s))\, ds$$

being measurable on its domain in $C_T^r \times C_T^m$ for each t, $F_n(t) = G_t(x_n, U_n(\varphi(x_n)))$, $n = 1, 2, \ldots$, together with (3.7) implies

$$\tilde{F}_n(t) = \int_0^t f(s, \tilde{x}_n, u_n(s, \varphi(\tilde{x}_n)))\, ds \tag{3.10}$$

a.e. for each t, and by continuity this holds for all t with probability one. Further, $\tilde{B}_n(t)$, $0 \leqq t \leqq T$, is a continuous vector-valued martingal with respect to $(\tilde{\mathscr{F}}_n(t)) = (\sigma\{\tilde{x}_n(s), \tilde{B}_n(s), 0 \leqq s \leqq t\})$ with quadratic variation

$$\langle \tilde{B}_n \rangle (t) = \int_0^t \Sigma(s, \tilde{x}_n)\, ds$$

for all $n = 0, 1, 2, \ldots$. Hence there exist Brownian motions $(\tilde{w}_n(t), \tilde{\mathscr{F}}_n(t))$ on the (possibly augmented) probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P})$ such that both $\tilde{x}_n(t)$ and $\tilde{B}_n(t)$, $0 \leqq t \leqq T$, are adapted to $(\tilde{\mathscr{F}}_n(t))$ and

$$\tilde{B}_n(t) = \int_0^t \sigma(s, \tilde{x}_n)\, d\tilde{w}_n(s). \tag{3.11}$$

Now it is easy to see that for $n = 1, 2, \ldots$

$$\begin{aligned}\tilde{x}_n(t) &= \tilde{x}_n(0) + \tilde{F}_n(t) + \tilde{B}_n(t) \\ &= \tilde{x}_n(0) + \int_0^t f(s, \tilde{x}_n, u_n(s, \varphi(\tilde{x}_n)))\, ds + \int_0^t \sigma(s, \tilde{x}_n)\, d\tilde{w}_n(s)\end{aligned} \tag{3.12}$$

holds for all t with probability one. This means that $\tilde{x}_n$ is a solution of (2.1) corresponding to the admissible control $\tilde{u}_n = u_n$, $n = 1, 2, \ldots$ (the tightness condition (vi) is easily verified by measurability arguments in connection with (3.7)).

It remains to show that we can find a function $\tilde{u}_0$: $[0, T] \times C_T^l \to R^m$ such that

$$\tilde{V}_0(\tilde{\omega}) = \tilde{U}_0(\varphi(\tilde{x}_0(\tilde{\omega}))) = \tilde{u}_0(\cdot, \varphi(\tilde{x}_0(\tilde{\omega}))) \tag{3.13}$$

holds for almost every $\tilde{\omega}$ and $\tilde{u}_0$ is an admissible control with $\tilde{x}_0$ as a corresponding solution.

To this end, define a function $\tilde{U}_0$ on $\varphi(\tilde{x}_0(\tilde{\Omega}))$ by setting

$$\tilde{U}_0(\eta) = \tilde{V}_0(\tilde{\omega}) \quad \text{for } \varphi(\tilde{x}_0(\tilde{\omega})) = \eta. \tag{3.14}$$

We have to make sure that the definition is unambiguous. This will follow from a more general result. For $\eta = \varphi(\tilde{x}_0(\omega))$, $\eta' = \varphi(\tilde{x}_0(\tilde{\omega}'))$ and arbitrary $t \in [0, T]$ we obtain the estimate

$$\begin{aligned} \|\tilde{U}_0(\eta) - \tilde{U}_0(\eta')\|_t &= \|\tilde{V}_0(\tilde{\omega}) - \tilde{V}_0(\tilde{\omega}')\|_t \\ &\leqq \|\tilde{V}_0(\tilde{\omega}) - \tilde{V}_n(\tilde{\omega})\|_t + \|U_n(\eta_n) - U_n(\eta_n')\|_t \\ &\quad + \|\tilde{V}_n(\tilde{\omega}') - \tilde{V}_0(\tilde{\omega}')\|_t \\ &\leqq 2\varepsilon + r_\rho(\|\eta_n - \eta_n'\|_t), \end{aligned}$$

where ρ is any positive number such that $\|\eta - \eta'\|_t < \rho$ as well as $\|\eta_n - \eta_n'\|_t < \rho$ for all n large enough, with $\eta_n = \varphi(\tilde{x}_n(\tilde{\omega}))$, $\eta_n' = \varphi(\tilde{x}_n(\tilde{\omega}'))$. (To account for those $\tilde{\omega}$ for which $\tilde{\Phi}_n(\tilde{\omega}) \not\to \tilde{\Phi}_0(\tilde{\omega})$, suppose that the $\tilde{\Phi}_n$, $n = 0, 1, \ldots$, have been redefined on this nullset to ensure convergence.) Now, since $\|\eta_n - \eta_n'\|_t \to \|\eta - \eta'\|_t$, it follows from the continuity of r_ρ that

$$\|U_0(\eta) - U_0(\eta')\|_t \leqq 2\varepsilon + r_\rho(\|\eta - \eta'\|_t);$$

since this estimate holds for arbitrary $\varepsilon > 0$, we find that

$$\|U_0(\eta) - U_0(\eta')\|_t \leqq r_\rho(\|\eta - \eta'\|_t)$$

for all $\eta, \eta' \in \varphi(\tilde{x}_0(\tilde{\Omega}))$ such that $\|\eta - \eta'\|_t < \rho$. In particular this means that $\tilde{U}_0$ is uniquely defined by (3.14) and $\tilde{u}_0(t, \eta) = \tilde{U}_0(\eta)(t)$ satisfies the admissibility condition (iii) if extended to $\mathrm{cl}(\varphi(\tilde{x}_0(\tilde{\Omega})))$ by continuity. Furthermore, with η and η_n as above,

$$\begin{aligned} \tilde{u}_0(t, \eta) &= \tilde{V}_0(\tilde{\omega})(t) \\ &= \lim_{n \to \infty} \tilde{V}_n(\tilde{\omega})(t) \\ &= \lim_{n \to \infty} U_n(\eta_n)(t). \end{aligned}$$

Since $U_n(\eta_n)(t) \in \mathscr{U}(t, \eta_n)$ and $\eta_n \to \eta$, $\tilde{u}_0(t, \eta) \in \mathscr{U}(t, \eta)$ follows from (A2).

Next let us show that (vi) is satisfied. Writing $f_n(s, \tilde{\omega}) = f(s, \tilde{x}_n(\tilde{\omega}), \tilde{u}_n(s, \varphi(\tilde{x}_n(\tilde{\omega}))))$, $n = 0, 1, 2, \ldots$, it follows from (3.6) and (A3) that

$$f_n \to f_0, \quad |f_n|^2 \to |f_0|^2 \qquad \tilde{P} \times \lambda_T\text{-a.e.} \tag{3.15}$$

(λ_T=Lebesgue measure on $[0, T]$), hence the second relation in (vi) a) follows from Fatou's lemma. To show the first relation, observe that the f_n are uniformly integrable and thus

$$\int_t^{t+\Delta} |f_n|\,ds \to \int_t^{t+\Delta} |f_0|\,ds \quad \text{in } L_1(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P}).$$

Then, for a subsequence $(n') \subset (n)$,

$$\left(\int_t^{t+\Delta} |f_{n'}|\,ds\right)^2 \to \left(\int_t^{t+\Delta} |f_0|\,ds\right)^2 \quad \tilde{P}\text{-a.e.}$$

Fatou's lemma completes the proof. The proof of the other conditions b)–d) follows the same arguments. Furthermore, it follows from

$$\tilde{F}_n(t) = \int_0^t f_n\,ds \to \int_0^t f_0\,ds \quad \text{in } L_1(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P})$$

for all t and (3.6) that

$$\tilde{F}_0(t) = \int_0^t f_0\,ds \quad \tilde{P}\text{-a.e.} \tag{3.16}$$

for all t, and by continuity (3.16) holds for all t with probability one. Hence, by (3.12), (3.6), (3.11) and (3.16),

$$\begin{aligned}\tilde{x}_0(t) &= \tilde{x}_0(0) + \tilde{F}_0(t) + \tilde{B}_0(t)\\ &= \tilde{x}_0(0) + \int_0^t f(s, \tilde{x}_0, \tilde{u}_0(s, \varphi(\tilde{x}_0)))\,ds + \int_0^t \sigma(s, \tilde{x}_0)\,d\tilde{w}_0(s)\end{aligned}$$

holds for all $0 \leqq t \leqq T$ with probability one.

This completes the proof of the Lemma.

The following theorem gives the existence result for the case of continuous controls.

Theorem 1. *Assume that conditions* (A1)–(A6) *are satisfied and that the class* $\mathscr{A}$ *of admissible controls is nonvoid. Then there exists an optimal admissible control in* $\mathscr{A}$.

Proof. Consider a minimizing sequence (u_n, x_n) of admissible controls u_n and corresponding solutions x_n, i.e. $J(x_n, u_n) \to \hat{J}$. Then, in the notation of Lemma 1, the sequence $Q_n = (P_n)_{\Phi_n}$ of measures induced on S is tight, and we can extract a subsequence – indexed equally by n in the sequel – converging to some $Q_0 \in \mathscr{P}$. Passing to the equivalent processes $\tilde{\Phi}_n$,

$$E_n \int_0^T l(s, x_n, u_n(s, \varphi(x_n)))\,ds = \tilde{E} \int_0^T l(s, \tilde{x}_n, \tilde{u}_n(s, \varphi(\tilde{x}_n)))\,ds$$

since the mapping $(\xi, \zeta) \to \int_0^T l(s, \xi, \zeta(s))\,ds$ from $C^r \times C^m$ to $\mathbb{R}$ is measurable. Since the sequence $l(\cdot, \tilde{x}_n, \tilde{u}_n(\cdot, \varphi(\tilde{x}_n)))$ is uniformly integrable and converges $\tilde{P} \times \lambda_T$-

a.e. to $l(\cdot, \tilde{x}_0, \tilde{u}_0(\cdot, \varphi(\tilde{x}_0))$,

$$\tilde{E} \int_0^T l(s, \tilde{x}_n, \tilde{u}_n(s, \varphi(\tilde{x}_n)))\, ds \to \tilde{E} \int_0^T l(s, \tilde{x}_0, \tilde{u}_0(s, \varphi(\tilde{x}_0)))\, ds.$$

Finally, since g is continuous and bounded,

$$E_n\{g(x_n)\} = \tilde{E}\{g(\tilde{x}_n)\} \to \tilde{E}\{g(\tilde{x}_0)\}.$$

Hence,

$$J(x_n, u_n) = J(\tilde{x}_n, \tilde{u}_n) \to J(\tilde{x}_0, \tilde{u}_0) = \hat{J}.$$

4. The Case of Measurable Controls

The situation becomes more complex when the admissible controls are allowed to have paths in $L_1^m[0, T]$ instead of C_T^m. Let us first look at the obvious changes to be made in the definition of admissibility. To begin with, (i) has to be replaced by

(i') $u(\cdot, \eta) \in L_1^m[0, T]$ for all $\eta \in C_T^l$.

(iii) can be overtaken in the above form if the symbol $\|\cdot\|_t$ on the left hand side of the inequality is interpreted as the restriction of the $L_1^m[0, T]$-norm to $[0, t]$, i.e.

$$\|v\|_t = \int_0^t |v(s)|\, ds.$$

Let us refer to this modification of (iii) as condition (iii'). To see what measurability properties are implied by (iii') consider the functions

$$u_h(t, \eta) = \frac{1}{h} \int_{t-h}^{t} u(s, \eta)\, ds \tag{4.1}$$

defined for all $\eta \in C_T^l$, $t \in [0, T]$ and $h > 0$ (put $u(s, \eta) = 0$ for $s < 0$). Then $u_h(\cdot, \eta) \in C_T^m$.

Lemma 2. *If u satisfies* (iii'), *then u_h satisfies* (iii) (*with* r_ρ/h).

Proof. For $0 \leqq s \leqq t$, $\eta, \eta' \in \mathscr{R}$, $\|\eta - \eta'\|_t < \rho$,

$$\begin{aligned} |u_h(s, \eta) - u_h(s, \eta')| &\leqq \frac{1}{h} \int_{s-h}^{s} |u(r, \eta) - u(r, \eta')|\, dr \\ &\leqq \frac{1}{h} \|u(\cdot, \eta) - u(\cdot, \eta')\|_t \\ &\leqq \frac{1}{h} r_\rho(\|\eta - \eta'\|_t). \end{aligned}$$

It follows that the restriction of the process $u_h(t,\cdot)$, $0 \leqq t \leqq T$, to $\mathscr{R}$ is adapted to $(\tilde{\mathscr{C}}_t^l)$ and measurable for every $h>0$. Take a sequence $h_n \searrow 0$. Then

$$u(t,\eta)=\lim_{n\to\infty} u_{h_n}(t,\eta)$$

for every η and for almost every t (cf. [8]). The exceptional set $E=\{(t,\eta)\in[0,T]\times\mathscr{R}: u_{h_n}(t,\eta)\nrightarrow u(t,\eta)\}$ is measurable with respect to $\mathscr{B}_T\otimes\tilde{\mathscr{C}}_T^l$, and its t-sections are measurable with respect to $\tilde{\mathscr{C}}_t^l$, since they are just those sets where $u_{h_n}(\cdot,\cdot)$ and $u_{h_n}(t,\cdot)$, respectively, do not converge. Hence, by redefining u on E, we can find a modification of u which is $\mathscr{B}_T\otimes\tilde{\mathscr{C}}_T^l$-measurable and adapted to $(\tilde{\mathscr{C}}_t^l)$.

(iv) has to be changed in that the relation $u(t,\eta)\in U(t,\eta)$ is required to hold only a.e. in t.

Finally, we have to replace (vi) by a suitable tightness criterion in L_1. To this end, define the Steklow functions

$$v_h(t)=\frac{1}{2h}\int_{t-h}^{t+h} v(s)\,ds$$

for $v\in L_1^m$, $h>0$ (put $v(s)=0$ for $s<0$ and $s>T$). Then a sufficient condition for a subset A of $L_1^m[0,T]$ to be relatively compact is the following:

a) $\sup_{v\in A}\|v\|<\infty$;

b) $\lim_{h\searrow 0}\sup_{v\in A}\|v_h-v\|=0$,

where $\|\cdot\|=\|\cdot\|_T$ denotes the $L_1^m[0,T]$-norm (cf. [10]). This yields the following condition for tightness in L_1^m:

Lemma 3. *A sequence (P_n) of probability measures on (the Borel field of) $L_1^m[0,T]$ is tight if the following conditions are fulfilled:*

a) *For every $\varepsilon>0$ there exists a number M such that*

$$P_n\{v: \|v\|>M\}\leqq\varepsilon \quad \textit{for all } n;$$

b) *For every $\varepsilon>0$, $\delta>0$, there exists an $h>0$ such that*

$$P_n\{v: \|v_{h'}-v\|\geqq\delta\}\leqq\varepsilon$$

for all n and all $0<h'<h$.

Lemma 3 simply transforms the concept of tightness in L_1^m by substituting for relative compactness the above characterization. Its proof is analogous to the if-part of the proof of Theorem 8.2 in [2].

For a sequence of random functions $V_n: \Omega\to L_1^m$ [1] tightness means tightness of the measures induced on L_1^m; hence a) and b) take on the form

a) $P[\|V_n\|>M]\leqq\varepsilon$ (i.e. the sequence $(\|V_n\|)$ is tight on the line);

b) $P[\|(V_n)_h-V_n\|\geqq\delta]\leqq\varepsilon$.

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[1] The V_n may be defined on different probability spaces. For simplicity we omitt the index n

By Čebyšev's inequality we obtain the following tightness condition to be required from an admissible L_1^m-control u and its corresponding solution x:

(vi′) d) There exist a constant K and a positive function $r(h)$, decreasing monotonically to 0 as $h \searrow 0$, such that, uniformly in u and x,

$$E\|u(\cdot,\varphi(x))\| \leqq K \quad \text{and}$$
$$E\|u_h(\cdot,\varphi(x)) - u(\cdot,\varphi(x))\| \leqq r(h) \qquad \text{for } h>0.$$

This is to be substituted for (vi) d), a)–c) remaining unchanged. The class $\mathscr{A}'$ of admissible controls now consists of all functions u satisfying the modified set of assumptions (i′)–(vi′).

Now, proceeding as in Chap. 3, we can define functions $U\colon C_T^l \to L_1^m[0,T]$ by $U(\eta)(\cdot) = [u(\cdot,\eta)]$ (the equivalence class of $u(\cdot,\eta)$), $V = U(\varphi(x(\cdot)))\colon \Omega \to L_1^m[0,T]$ and a measurable function $\Phi\colon \Omega \to S = C_T^{3r}\ L_1^m[0,T]$ as in (3.2). Then the proof of Lemma 1 carries over to the L_1-case in its essential parts. Only the verification of conditions (vi′) and (v) for the equivalent processes $(\tilde{u}_n, \tilde{x}_n)$, $n=0,1,2,\ldots$, turns out to be somewhat more involved. In principle, however, it runs along the same lines as the corresponding part in the proof of Lemma 1, making use of certain measurability and continuity properties of mappings of the type

$$(x,v) \to \int_s^t h(\tau, x, v(\tau))\, d\tau$$

defined on some domain in $C_T^r \times L_1^m$, and whose application to the functions f and l require the additional assumption of joint continuity. For the technical details cf. [3].

With the help of the properties just mentioned, the proof of Theorem 1, too, carries over to the case of L_1-controls.

Theorem 2. *Suppose that conditions* (A1)–(A6) *are satisfied and that, in addition,* $f(t,x,u)$ *and* $l(t,x,u)$ *are jointly continuous in* (t,x,u). *If the class* $\mathscr{A}'$ *of admissible controls is nonvoid, then there exists an optimal control in* $\mathscr{A}'$.

Remark. It should be clear that the proof of the weak compactness of $\mathscr{P}'$ (or $\mathscr{P}$) remains valid if the tightness criterion (vi′) d) (or (vi) d)) is replaced by some other condition (C) which is sufficient for tightness, provided the resulting class $\mathscr{A}_C$ of admissible controls has the following closedness properties:

1) If $u \in \mathscr{A}_C$ with corresponding solution x, then, for any distributionally equivalent pair $(\tilde{x}, \tilde{u})$, $\tilde{u}(\cdot, \varphi(\tilde{x}))$ satisfies (C).

2) If $u_n \in \mathscr{A}_C$ with corresponding solutions x_n and $x_n \to x$ in C_T, $u_n(\cdot, \varphi(x_n)) \to u(\cdot, \varphi(x))$ in L_1 a.e., then $u(\cdot, \varphi(x))$ satisfies (C).

Finally, let us remark that the methods used here are easily extended to include control of the diffusion term (cf. [3]).

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