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On Some Selection Procedures in Two-Way Layouts*

P.K. Sen and M.L. Puri

1. Introduction

For a two-factor complete block design with one observation per cell, we express the observable random variables $X_{i\alpha}$ $(i=1, \ldots, c;\ \alpha=1, \ldots, n)$ as

$$X_{i\alpha}=\mu+\beta_\alpha+\tau_i+\varepsilon_{i\alpha}, \qquad \sum_{i=1}^{c} \tau_i=0, \tag{1.1}$$

where μ is the *mean-effect*, $\beta_1, \ldots, \beta_n$ are the *block effects* (nuisance parameters for the *fixed effects* model or random variables for the *mixed effects* model), $\tau_1, \ldots, \tau_c$ are the *treatments effects*, and the $\varepsilon_{i\alpha}$ are the *error* components. It is assumed that $\boldsymbol{\varepsilon}_\alpha=(\varepsilon_{1\alpha}, \ldots, \varepsilon_{c\alpha})$, $\alpha=1, \ldots, n$ are independent and identically distributed stochastic vectors with a continuous cumulative distribution function (cdf) $F(\boldsymbol{\varepsilon})$, $\boldsymbol{\varepsilon} \in R^c$ (the real c space), where $F(\boldsymbol{\varepsilon})$ is symmetric in its c arguments, that is, for any $\boldsymbol{\varepsilon} \in R^c$ and any permutation $(i_1, \ldots, i_c)$ of $(1, \ldots, c)$

$$F(\varepsilon_1, \ldots, \varepsilon_c)=F(\varepsilon_{i_1}, \ldots, \varepsilon_{i_c}). \tag{1.2}$$

[Note that if all the nc errors are independent and identically distributed, (1.2) holds, but the converse is not necessarily true.] Our purpose is to study some (parametric as well as nonparametric) multiple decision procedures for the following three problems: (i) selection of the best t treatments *without* regard to order, (ii) selection of the *best t* treatments *with* regard to order, and (iii) selection of all the treatments which are *as good as or better* than a standard treatment. The *quality* of the treatments is judged by the *largeness* of the τ_i.

In the parametric case, Bechhofer [1] has studied the first problem under the assumption that the errors are independent and normally distributed. It is shown here that if the errors are jointly (within each block) normally distributed and are *equally correlated* then his procedure remains valid. This covers the situation (1.2) which may arise often in *mixed effects* model. It is also shown that if (1.2) holds and F admits of the existence of second order moments, then the Bechhofer procedure remains valid for large samples, even if F is not normal.

In the nonparametric setup, Puri and Puri [4] have considered these selection procedures for the one-way layout problems, and their procedures are based on a class of rank order statistics. Here, instead of these statistics, we consider the rank order estimators of $\{\tau_i\}$ by Puri and Sen [5] to provide asymptotically

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distribution-free selection procedures for the two-way layout problems. The asymptotic relative performances and efficiencies of the parametric as well as nonparametric procedures are studied. The cases of paired comparisons designs as well as one-way layout problems are also briefly presented.

2. Parametric Solution to Problem 1

In the case of independent and normally distributed errors, Bechhofer [1] considered the solution based on the order statistics associated with the treatment means. As the errors in (1.2) are not independent, his technique is not directly applicable. For this reason, we consider the following modification of his procedure, to be termed the *Extended Bechhofer* (B^*-) *procedure.*

Let $\tau_{[1]} \leqq \cdots \leqq \tau_{[c]}$ be the actual ranked τ's (which are unknown), and let

$$Z_{ni} = \bar{X}_{ni} - \bar{\bar{X}}_n, \qquad \bar{X}_{ni} = n^{-1} \sum_{\alpha=1}^{n} X_{i\alpha}, \quad i=1, \ldots, n; \qquad \bar{\bar{X}}_n = c^{-1} \sum_{i=1}^{c} \bar{X}_{ni}. \tag{2.1}$$

We denote the ordered values of the Z_{ni} by $Z_{n[1]} \leqq \cdots \leqq Z_{n[c]}$, and let $Z_{n(i)}$ be the statistic associated with $\tau_{[i]}$, $i=1, \ldots, c$. Then, we select the t best treatments which are associated with

$$Z_{n[c-t+1]}, \ldots, Z_{n[c]}. \tag{2.2}$$

Our basic problem is to determine the sample size n in such a way that for any preassigned γ $(0<\gamma<1)$, the probability of correct selection of the t best population is $\geqq \gamma$, where $\tau_{[c-t]}$ and $\tau_{[c-t+1]}$ are subject to the condition that

$$\tau_{[c-t+1]} - \tau_{[c-t]} \geqq \zeta, \tag{2.3}$$

ζ being the smallest *worth detecting* difference. Note that the choice of ζ is left to tha practical considerations. We denote the condition in (2.3) by a sequence $\{\zeta_m\}$ where m is a positive integer and $\zeta_m \to 0$ as $m \to \infty$. We denote the corresponding sequence of conditions by $\{(A_m)\}$. In the context of one-way layout (which also extends to two-way layouts for independent errors), Bechhofer [1] has shown that for the parent distribution being normal,

$$P\{\text{correct selection of the } t \text{ best treatments}\} = \gamma \tag{2.4}$$

when the following least favorable configuration holds:

$$\begin{aligned} \tau_{[1]} = \cdots = \tau_{[c-t]} = \tau_{[c-t+1]} - \zeta, \\ \tau_{[c-t+1]} = \cdots = \tau_{[c]}. \end{aligned} \tag{2.5}$$

We denote by $L(c, t; \zeta)$ the configuration in (2.5), and note that if ζ be replaced by $\{\zeta_m\}$, the corresponding sequence will be denoted by $\{L(c, t; \zeta_m)\}$.

We first show that $L(c, t; \zeta)$ is the least favorable configuration for the entire class of symmetric dependent multinormal distributions. This basic result will be used throughout the paper.

Theorem 2.1. *Let* $(W_1, \ldots, W_c)$ *have jointly a multinormal distribution with mean vector* $(\tau_{[1]}, \ldots, \tau_{[c]})$ *and dispersion matrix* $\mathbf{\Sigma} = \sigma^2[(1-\rho)\mathbf{I} + \rho\mathbf{J}]$, $(\sigma^2 > 0, \ -1/(c-1) \leqq \rho < 1$, $\mathbf{I}$ *is the identity matrix of order c and* $\mathbf{J} = \mathbf{1}_c' \mathbf{1}_c)$. *Then for given* γ, *under* (A), (2.4) *holds when* (2.5) *holds.*

Proof. Denote by

$$U_{ij}=(W_i-W_j-[\tau_{[i]}-\tau_{[j]}])/[2\sigma^2(1-\rho)]^{\frac{1}{2}}, \quad 1\leqq i\leqq c-t,\ c-t+1\leqq j\leqq c, \tag{2.6}$$

and note that $\{U_{ij},\ 1\leqq i\leqq c-t,\ c-t+1\leqq j\leqq c\}$ have jointly a (singular) multinormal distribution with null mean vector and dispersion matrix with the elements $\mathrm{Cov}(U_{ij}, U_{i'j'})=1, \frac{1}{2}$ or 0 according as $i=i', j=j'$; $i=i', j\neq j'$ $(i\neq i', j=j')$ or $i\neq i', j\neq j'$. Now, by the procedure in (2.2), the probability of a correct decision is

$$\begin{aligned} &P\{\mathrm{Max}[W_1,\dots,W_{c-t}]<\mathrm{Min}[W_{c-t+1},\dots,W_c]\}\\ &\quad=P\{W_i-W_j<0,\ i=1,\dots,c-t,\ j=c-t+1,\dots,c\}\\ &\quad=P\{U_{ij}\leqq(\tau_{[j]}-\tau_{[i]})/[2\sigma^2(1-\rho)]^{\frac{1}{2}},\ i=1,\dots,c-t,\ j=c-t+1,\dots,c\}. \end{aligned} \tag{2.7}$$

Now, in view of (2.3) the $\tau_{[i]}$ satisfy

$$\tau_{[1]}\leqq\cdots\leqq\tau_{[c-t]}\leqq\tau_{[c-t+1]}-\zeta\leqq\tau_{[c-t+2]}\leqq\cdots\leqq\tau_{[c]}. \tag{2.8}$$

Since the right hand side of (2.7) agrees with the corresponding expression in [1, pp. 20–21], [with the only change that his σ^2 is replaced by our $\sigma^2(1-\rho)$], the least favorable configuration for which (2.4) holds can easily be shown (as in [1, pp. 20–21]) to be (2.5). Hence the theorem.

It may be noted that for the Z_{ni}, defined by (2.1), σ^2 in Theorem 2.1 has to be replaced by σ^2/n, and as in Sen [8], it follows that the mean square due to error consistently estimates $\sigma^2(1-\rho)$, for all parent distribution having finite second moments. While proceeding to the large sample solutions to the problem, we note that by virtue of the consistency of the estimates Z_{ni}, $i=1,\dots,c$, for any fixed ζ in (2.3), the probability of correct decision will tend to unity as $n\to\infty$. Hence, to avoid this limiting degeneracy, we conceive of a sequence of *worth-detecting* differences which also converge to zero as $n\to\infty$, in such a way that (2.4) holds with a γ, strictly less than 1. The justification of the use of such a sequence can be made in the same light as the use of the well-known *Pitman-efficiency* is justified in nonparametric inference procedures. Theoretically speaking, we replace the single sequence $\{X_{i\alpha},\ \alpha=1,\dots,n,\ i=1,\dots,c\}$ by a double sequence $\{[X_{i\alpha}^{(n)},\ \alpha=1,\dots,n,\ i=1,\dots,c],\ n\geqq 1\}$, where for the $X_{i\alpha}^{(n)}$, the corresponding $\tau_{[i]}$ (say $\tau_{[i]}^{(n)}$) satisfy (2.3) with ζ replaced by ζ_n, and $\zeta_n\to 0$ as $n\to\infty$. On the other hand, to practicing people, the solution can be made justified for small values of ζ; small enough to make the applied approximations to be valid and reasonable.

We now drop the assumption of multinormality of the errors, and, as in (1.2), we let F to be arbitrary. For the parametric procedures based on sample means, we require that F possesses second order moments.

Theorem 2.2. *Suppose in* (2.4) γ *is fixed, and in* (2.3), ζ *is replaced by* ζ_n. *Then for a* γ*-probability of correct selection, we have*

$$|n^{\frac{1}{2}}\zeta_n-\delta\sigma\sqrt{1-\rho}|\to 0 \quad \text{as} \quad n\to\infty, \tag{2.9}$$

where σ^2 *is the variance of* $e_{i\alpha}$, ρ *is the common correlation of* $e_{i\alpha}$, $e_{i\beta}$ $(\beta\neq\alpha)$, *and* δ *is determined by the condition:*

$$\gamma=tQ_{c-1}(\underbrace{\delta/\sqrt{2},\ \dots,\ \delta/\sqrt{2}}_{c-t\text{ times}},\underbrace{0,\ \dots,\ 0}_{t-1\text{ times}}), \tag{2.10}$$

Q_{c-1} being the cumulative distribution function of a normally distributed $(c-1)$-vector $(U_1, \ldots, U_{c-t}, W_{c-t+2}, \ldots, W_c)$, with $EU_i = EW_j = 0$, $\mathrm{Cov}[U_i, U_{i'}] = \frac{1}{2}(1+\delta_{ii'})$, $\mathrm{Cov}[W_j, W_{j'}] = \frac{1}{2}(1+\delta_{jj'})$ and $\mathrm{Cov}[U_i, W_j] = -\frac{1}{2}$ for $i, i' = 1, \ldots, c-t$, $j, j' = c-t+2, \ldots, c$; $\delta_{rs} = 0, 1$ according as $r \neq s$ or $r = s$.

The proof follows as a special case of (4.4) [with simplified (4.1)], and hence is omitted.

Suppose now we are given a small ζ^* and we wish to determine n such that (2.4) holds [subject to (2.2), with ζ replaced by ζ^*]. Then Theorem 2.2 provides the following large sample solution:

$$n \simeq \delta^2 \sigma^2 (1-\rho)/(\zeta^*)^2, \tag{2.11}$$

and as s^2, the mean square due to error estimates $\sigma^2(1-\rho)$ consistently, we have also asymptotically $n \simeq \delta^2 s^2/(\zeta^*)^2$.

3. Procedures Based on Rank Order Estimates

Let us define

$$X^*_{ij,\alpha} = X_{i\alpha} - X_{j\alpha}, \quad e_{ij,\alpha} = \varepsilon_{i\alpha} - \varepsilon_{j\alpha}, \quad \alpha = 1, \ldots, n; \qquad \Delta_{ij} = \tau_i - \tau_j, \quad 1 \leqq i < j \leqq c. \tag{3.1}$$

By (2.1), $e_{ij,\alpha}(X^*_{ij,\alpha})$ is symmetrically distributed about 0 (Δ_{ij}), and we denote its distribution by G. For $\mathbf{X}^*_{ij} = (X^*_{ij,1}, \ldots, X^*_{ij,n})$, consider the usual one-sample rank order statistic (viz., Puri and Sen [5, 6])

$$h_n(\mathbf{X}^*_{ij}) = n^{-1} \sum_{\alpha=1}^{n} J_n\big((n+1)^{-1} R_{ij,\alpha}\big) \operatorname{sgn} X^*_{ij,\alpha}, \tag{3.2}$$

where $R_{ij,\alpha}$ = Rank of $|X^*_{ij,\alpha}|$ among $|X^*_{ij,1}|, \ldots, |X^*_{ij,n}|$, sgn u is equal to 1, 0 or -1 according as u is $>$, $=$ or <0, and $J_n\big((n+1)^{-1} i\big)$ is the expected value of the ith order statistic of a sample of size n from the distribution $\Psi^*(x) = \Psi(x) - \Psi(-x)$, $x \geqq 0$, $i = 1, \ldots, n$, where $\Psi(x)$ is a symmetric (about 0) non-degenerate distribution satisfying the assumptions I, II and III in [5, 6]. Notable cases of h_n are the Wilcoxon signed rank and the normal scores statistics for which Ψ is the uniform over $(-1, 1)$ and the standard normal distribution respectively. As in [5], we let

$$\hat{\Delta}^{(n)}_{ij,1} = \sup\{t\colon h_n(\mathbf{X}^*_{ij} - t\,\mathbf{1}_n) > 0\}, \qquad \hat{\Delta}^{(n)}_{ij,2} = \inf\{t\colon h_n(\mathbf{X}^*_{ij} - t\,\mathbf{1}_n) < 0\}; \tag{3.3}$$

$$\hat{\Delta}^{(n)}_{ij} = [\Delta^{(n)}_{ij,1} + \Delta^{(n)}_{ij,2}]/2, \qquad 1 \leqq i < j \leqq c. \tag{3.4}$$

Then, the compatible estimates of $\{\Delta_{ij}\}$ are $\{Y^{(n)}_i - Y^{(n)}_j\}$, where

$$Y^{(n)}_i = c^{-1} \sum_{j=1}^{c} \hat{\Delta}^{(n)}_{ij}, \qquad \hat{\Delta}^{(n)}_{ii} = \Delta_{ii} = 0, \qquad i = 1, \ldots, c. \tag{3.5}$$

Let then $Y^{(n)}_{[1]} < \cdots < Y^{(n)}_{[c]}$ be the ordered values of the $Y^{(n)}_i$, and let $Y^{(n)}_{(i)}$ be the statistic associated with $\tau_{[i]}$, $i = 1, \ldots, c$. Our proposed procedure consists in selecting the t populations associated with

$$Y^{(n)}_{[c-t+1]}, \ldots, Y^{(n)}_{[c]}. \tag{3.6}$$

We consider now the basic theorem of this section which extends Theorem 3.2 of [5] along the multivariate set up in Sen and Puri [9] and Lemma 2.1 of [8]. For intended brevity, the proof is omitted.

Theorem 3.1. *Under the assumptions made before, $n^{\frac{1}{2}}[Y_{(i)}^{(n)} - Y_{(j)}^{(n)} - \Delta_{ij}; 1 \leqq i \leqq c-t, c-t+1 \leqq j \leqq c]$ has asymptotically a multivariate normal distribution with zero means and a dispersion matrix with elements $\{v_{ij,i'j'}\}$, where $v_{ij,i'j'} = \sigma_0^2, \sigma_0^2/2$ or 0 according as $i=i', j=j'$; $i=i', j \neq j'$ $(i \neq i', j=j')$ or $i \neq i', j \neq j'$, and*

$$\sigma_0^2 = [A^2 + (c-2)\lambda_J(F)]/(cB^2), \tag{3.7}$$

$$A^2 = \int_0^1 J^2(u)\,du, \quad B = \int_{-\infty}^{\infty} (d/dx) J[G(x)]\,dG(x); \quad J(u) = \Psi^{-1}(u), \tag{3.8}$$

$$\lambda_J(F) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} J[G(x)]\,J[G(y)]\,dG^*(x,y), \tag{3.9}$$

where $G^*(x,y)$ is the joint cdf of $(e_{ij,\alpha}, e_{ij',\alpha})$, $j \neq j'$.

Now, the probability of correct selection of t best populations is given by

$$\begin{aligned} &P\{\max[Y_{(1)}^{(n)}, \ldots, Y_{(c-t)}^{(n)}] < \min[Y_{(c-t+1)}^{(n)}, \ldots, Y_{(c)}^{(n)}]\} \\ &= P\left\{\left(\frac{n}{2\sigma_0^2}\right)^{\frac{1}{2}} [Y_{(i)}^{(n)} - Y_{(j)}^{(n)} - \Delta_{ij}] < \left(\frac{n}{2\sigma_0^2}\right)^{\frac{1}{2}} \Delta_{ji}, \right. \\ &\qquad \left. i=1,\ldots,c-t;\ j=c-t+1,\ldots,c\right\}. \end{aligned} \tag{3.10}$$

Thus, as in the asymptotic parametric case, we replace Δ_{ji} by a sequence $\{\Delta_{ji}^{(n)}\}$ such that $n^{\frac{1}{2}}\Delta_{ji}^{(n)} \to \lambda_{ji}$ (real and finite) as $n \to \infty$. Then, by using Theorem 3.1, we conclude that the right hand side of (3.10) is asymptotically equal to

$$P\{U_{ij} < (n/2\sigma_0^2)^{\frac{1}{2}} \Delta_{ij}^{(n)},\ i=1,\ldots,c-t;\ j=c-t+1,\ldots,c\}, \tag{3.11}$$

where the U_{ij} have a multinormal distribution with null mean vector and covariance matrix $((v_{ij,i'j'}))$. Since, this multinormal distribution satisfies the condition of Theorem 2.1, we can again check easily that the least favorable configuration turns out to be (2.5) with ζ replaced by ζ_n. Moreover, by the same technique as in Theorem 2.2, it follows that

$$|n^{\frac{1}{2}}\zeta_n - \delta\sigma_0| \to 0 \quad \text{as} \quad n \to \infty, \tag{3.12}$$

where σ_0 is defined by (3.7) and δ by (2.10). Thus, a large sample solution to the sample size needed for the probability of correct selection being equal to γ is given by

$$n \simeq \delta^2 \sigma_0^2/(\zeta^*)^2, \tag{3.13}$$

where ζ^* is a given small worth-detecting difference. Now, σ_0^2 involves the two unknown parameters B and $\lambda_J(F)$. B can be estimated (by $\hat{B}$) as in Sen [7], while $\lambda_J(F)$ can be estimated (by $L_J(F)$) as in [5]. Hence, we have

$$n \simeq \delta^2 [A^2 + (c-2) L_J(F)]/c\hat{B}^2. \tag{3.14}$$

Now, using the same notion of asymptotic relative efficiency (ARE) as in [4], it follows from (2.11) and (3.13) that the ARE of the rank order procedure (based on ψ-scores) with respect to the B^*-procedure is

$$\begin{aligned} e(\psi, B^*) &= \sigma^2(1-\rho)/\sigma_0^2 \\ &= \{[2\sigma^2(1-\rho)]B^2\}\{c/2[A^2+(c-2)\lambda_J(F)\}. \end{aligned} \tag{3.15}$$

Now, B relates to the cdf G [cf. (3.1)] whose variance is $2\sigma^2(1-\rho)$. Hence, the first factor on the right hand side of (3.15) is the ARE of the one-sample rank order tests (for location) with respect to the Student t-test when the parent distribution is $G(x)$ (cf. [6]), and we denote it by $e_{\psi, B^*}(G)$. Further, it has been shown in [5] that $\lambda_J(F) \leqq \frac{1}{2}A^2$, where the equality sign holds *iff* $J[F(x)]$ is linear in x, with probability one. As such, we have

$$e(\psi, B^*) \geqq e_{\psi, B^*}(G), \tag{3.16}$$

where the equality sign holds *iff* $J[F(x)] = a + bx$, with probability one. Now, various known bounds for $e_{\psi, B^*}(G)$ can be used to provide bounds to $e(\psi, B^*)$. For example, if we use Ψ as the standard normal distribution, $e_{\psi, B^*}(G)$ is bounded below by 1, where the lower bound is attained *iff* G is normal. *Thus, the procedure based on the normal Scores estimators is asymptotically at least as efficient as the extended Bechhofer procedure.* If we use the Wilcoxon-Scores estimator, it follows that $e(\psi, B^*)$ is bounded below by 0.864 (though not attainable) for all $F(G)$, while the same can be greater than unity for many non-normal F. For normal F, it is bounded below by $3/\pi$ for all $c(\geqq 2)$, while it can be as high as 0.98.

4. Relative Performance Characteristics when (2.5) Is not Necessarily True

To compare the Ψ-score procedure with the B^*-procedure, we consider any sequence of parameter points satisfying

$$\tau_{[c-t+1]}^{(n)} - \tau_{[i]}^{(n)} = \delta_i^{(n)} = n^{-\frac{1}{2}}\theta_i + o(n^{-\frac{1}{2}}), \qquad i = 1, \dots, c-t,\ c-t+2, \dots, c, \tag{4.1}$$

where not all the $\theta_1, \dots, \theta_{c-t}$ are equal to θ, and/or not all the $\theta_{c-t+2}, \dots, \theta_c$ are equal to zero, i.e. the least favorable configuration does not hold, but $\tau_{[i]}^{(n)} \leqq \tau_{[j]}^{(n)}$ whenever $1 \leqq i \leqq j \leqq c$. Then, for the B^*-procedure, we have

$$\begin{aligned} &\lim_{n\to\infty} P\{\text{correct selection of } t \text{ best treatments}\} \\ &= \lim_{n\to\infty} P\{\max[Z_{n(1)}, \dots, Z_{n(c-t)}] < \min[Z_{n(c-t+1)}, \dots, Z_{n(c)}]\} \\ &= \lim_{n\to\infty} \Big[\sum_{l=1}^{t} P\{\text{all the } Z_{n(1)}, \dots, Z_{n(c-t)} < Z_{(c-t+l)} \\ &\qquad\qquad < \min[Z_{n(i)}, i = c-t+1, \dots, c(\neq c-t+l)]\}\Big] \\ &= \sum_{l=1}^{t} \Big[\lim_{n\to\infty} P\Big\{\begin{matrix} Z_{n(s)} - Z_{n(l)} < 0, & s = 1, \dots, c-t, \\ Z_{n(l)} - Z_{n(r)} < 0, & r = c-t+1, \dots, c(\neq c-t+l) \end{matrix}\Big\}\Big] \\ &= \sum_{l=1}^{t} \Big[\lim_{n\to\infty} P\Big\{U_s < \xi_{l,s}^{(n)},\ W_r < \xi_{r,l}^{(n)},\ \begin{matrix} s = 1, \dots, c-t, \\ r = c-t+1, \dots, c(\neq c-t+l) \end{matrix}\Big\}\Big], \end{aligned} \tag{4.2}$$

where $[U_1, \ldots, U_{c-t}, W_{c-t+1}, \ldots, W_{i-1}, W_{i+1}, \ldots, W_c]$ has the multivariate normal distribution Q_{c-1}, defined in Theorem 2.2, and

$$\zeta_{r,s}^{(n)} = [n/2\sigma^2(1-\rho)]^{\frac{1}{2}}[\tau_{[r]}^{(n)} - \tau_{[s]}^{(n)}], \quad 1 \leq r \leq s \leq c; \tag{4.3}$$

the last identity in (4.2) is a direct consequence of the central limit theorem as applied to the $Z_{n(r)}$. Thus, from (4.1), (4.2) and (4.3), it follows that (4.2) is equal to

$$\sum_{l=c-t+1}^{c} Q_{c-1}\left(\frac{1}{\sqrt{2(1-\rho)}}[\theta_{1,l}, \ldots, \theta_{c-t,l}, \theta_{c-t+1,l}, \ldots, \theta_{l-1,l}, \theta_{l+1,l}, \ldots, \theta_{c,l}]\right), \tag{4.4}$$

where $\theta_{i,j} = \theta_i - \theta_j$.

Consider now the rank procedures, where the assumptions implicit in Section 3 hold and the sequence in (4.1) also holds. Then, by Theorem 3.1, it follows as in (4.2) that under (4.1), for the ψ-scores procedure

$$\lim_{n\to\infty} P\{\text{correct solution of } t \text{ best treatments}\}$$
$$= \sum_{=c-t+1}^{c} Q_{c-1}\left(\frac{\sigma}{\sqrt{2}\sigma_0}[\theta_{1,l}, \ldots, \theta_{c-t,l}, \theta_{c-t+1,l}, \ldots, \theta_{l-1,l}, \theta_{l+1,l}, \ldots, \theta_{c,l}]\right), \tag{4.5}$$

where σ_0^2 is defined by (3.7). Thus, comparing (4.4) and (4.5) (in the Pitman sense), we may conclude that the ARE remains the same for (4.1), even when the least favorable configuration may not hold.

5. Selection of Best Treatments with Regard to Order

Here the *Bechhofer procedure* consists in selecting the t best treatments associated with $Z_{n[c-t+1]}, \ldots, Z_{n[c]}$ respectively. By virtue of our Theorem 2.1, we can readily extend the original procedure by Bechhofer [1], and derive the following results. [The details are omitted for intended brevity.]

For a fixed $\gamma (0<\gamma<1)$ *and under the condition that*

$$\tau_{[i+1]} - \tau_{[i]} \geq \zeta_n \quad (\textit{worth detecting distance}), \quad i = c-t, \ldots, c-1, \tag{5.1}$$

let n be determined so that (a) *the following (least favorable) configuration holds:*

$$\tau_{[1]} = \cdots = \tau_{[c-t]} = \tau_{[c-t+1]} - \zeta_n, \quad \tau_{[i+1]} - \tau_{[i]} = \zeta_n, \quad i = c-t, \ldots, c-1, \tag{5.2}$$

and (b)

$$P\{\max_{1\leq i\leq c-t} Z_{n(i)} < Z_{n(c-t+1)} < \cdots < Z_{n(c)}\} = \gamma. \tag{5.3}$$

Then asymptotically,

$$|n^{\frac{1}{2}}\zeta_n - \delta\sigma\sqrt{1-\rho}| \to 0, \quad (\text{as } n\to\infty), \tag{5.4}$$

where σ *and* ρ *are defined as in Theorem* 2.2 *and* δ *is determined by the condition*

$$(c-t)\,Q_{c-1}(0, \underbrace{\ldots}_{c-t-1 \text{ times}}, 0, \delta/\sqrt{2}, \underbrace{\ldots}_{t \text{ times}}, \delta/\sqrt{2}) = \gamma, \tag{5.5}$$

and Q_{c-1} *is the cdf of a normally distributed vector* $(U_1, \ldots, U_{c-t-1}, W_{c-t}, \ldots, W_{c-1})$ *satisfying* $EU_i = EW_j = 0$, $i = 1, \ldots, c-t-1$, $j = c-t, \ldots, c-1$ *and* (i) $\mathrm{Cov}(U_i, U_{i'}) = \frac{1}{2}(1+\delta_{ii'})$, (ii) $\mathrm{Cov}(U_i, W_j) = -\frac{1}{2}$ *if* $j = c-t$ *and* 0, *otherwise, and* (iii) $\mathrm{Cov}(W_j, W_{j'}) = 1$,

$-\frac{1}{2}$ *or* 0 *according as* $j=j'$, $|j-j'|=1$ *or* $|j-j'|>1$, *for* $i, i'=1, \ldots, c-t-1$, $j, j'=c-t, \ldots, c-1$; $\delta_{ii'}$ *being the usual Kronecker delta.*

Again by virtue of our Theorems 2.1 and 3.1, it follows along the same line as in [4] that for the rank scores procedure based on (3.6) (with regard to the order), the least favorable configuration turns out to be (5.2) (for small ζ_n), and hence, for this procedure

$$|n^{\frac{1}{2}}\zeta_n - \delta\sigma_0| \to 0, \quad \text{as} \quad n \to \infty, \tag{5.6}$$

where σ_0 is defined by (3.7) and δ by (5.5).

Hence, the ARE is the same as in (3.15) and (3.16). Also, the results of Section 4 can readily be extended in this situation and similar conclusions be derived; for brevity the details are omitted.

6. Selection of a Subset of Treatments Better than a Standard One

Instead of (1.1), we consider the model

$$X_{i\alpha} = \mu + \beta_\alpha + \tau_i + \varepsilon_{i\alpha}, \quad i=0,1,\ldots,c;\ \alpha=1,\ldots,n, \tag{6.1}$$

where the notations are all explained after (1.1) and τ_0 is the standard treatment effect. We say the ith treatment is *better* than the standard if

$$\tau_i \geqq \tau_0 + \zeta; \quad \zeta = \text{worth detecting difference}. \tag{6.2}$$

For the one-way layout case with normally distributed errors, Gupta and Sobel [2] proposed the procedure: Select the subset of treatments for which $X_{ni} - X_{n0} > 0$, where

$$X_{ni} = n^{-1}\sum_{\alpha=1}^{n} X_{i\alpha}, \quad i=0,1,\ldots,c.$$

An elegant solution for the sample size (n) needed to achieve a $\gamma(0<\gamma<1)$ probability of correct selection has also been provided by them. Noting that for (within-block) symmetric dependent normally distributed errors, the distribution of $[n^{\frac{1}{2}}(\overline{X}_{ni} - \overline{X}_{n0} - \tau_i + \tau_0),\ i=1,\ldots,c]$ is also multinormal with null means and covariance matrix $\sigma^2(1-\rho)[I_c + J_c]$, (where σ^2 and ρ are defined in Theorem 2.2), it turns out that the only change needed in Gupta-Sobel solution for the two-way layout problem is to replace their σ^2 by $\sigma^2(1-\rho)$. By virtue of the central limit theorem, the same solution holds asymptotically for the entire class of cdf's with finite second moments.

For the rank scores procedure, we define the estimators $\hat{\Delta}_{i0}^{(n)}$, $i=1,\ldots,c$ as in (3.3) and (3.4). Then, we select the subset of treatments for which $\hat{\Delta}_{i0}^{(n)} > 0$. Along the same line as in Theorem 3.1, it follows that $[n^{\frac{1}{2}}(\hat{\Delta}_{i0}^{(n)} - \tau_i + \tau_0), i=1,\ldots,c]$ have asymptotically a multinormal distribution with null means and covariance matrix $\sigma_0^2[I_c + J_c]$, where σ_0^2 is defined by (3.7). Consequently, the Gupta-Sobel solution also asymptotically holds for the rank scores procedure provided we replace σ^2 by σ_0^2. Hence, the ARE of the rank scores procedures with respect to the Gupta-Sobel procedure can again be measured by $\sigma^2(1-\rho)/\rho_0^2$, and it agrees with (3.15). The details are therefore omitted. However, the results clearly indicate the superiority of the normal scores procedure over the normal theory procedure (for large samples).

7. Few Additional Comments

Unlike Puri and Puri [4], we have considered here the procedures based on the rank order estimates, not statistics. The same procedure can be suggested for the one-way layout problem as an alternative to the procedures in [4]. Also, we note that the procedure considered in this paper can readily be extended to incomplete block designs (such as the paired comparisons designs, etc.), whereas the original procedures by Puri and Puri face considerable difficulties.

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Pranab Kumar Sen
School of Public Health
Department of Biostatistics
University of North Carolina
Chapel Hill, N.C. 27514
USA

Madan Lal Puri
Department of Mathematics
Indiana University
Swain Hall-East
Bloomington, Indiana 47401
USA

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