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# On the Robustness of Rank Order Tests and Estimates in the Generalized Multivariate One-Sample Location Problem*

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## 1. Introduction

In an earlier paper [9], the authors developed the theory of rank order tests for location in the multivariate one-sample problem. A later paper [5], dealing with the univariate case, relaxes the basic regularity conditions to some extent. An alternative novel approach to the same problem by Pyke and Shorack [6] also deserves special mention. Further, one of the authors [8] has shown that for the univariate one-sample location problem, the homogeneity of the distributions of the sample observations is redundant. In the present paper, generalizing the approach of [8] to the multivariate case, the results of [9] are extended to nonidentically distributed (independent) stochastic vectors. It is shown here that the permutation distribution theory of multivariate rank order statistics (under the finite group of sign-invariant transformations), developed in [9], readily extends to the case of heterogeneous distributions. The asymptotic (multi-) normality of the rank order statistics, when the null hypothesis of sign-invariance is not necessarily true, is also extended to the heterogeneous case. An interesting feature is the role played by the average distribution function in this theory. In this context, a useful result on the asymptotic covariance matrix of the rank statistics is obtained. As in [4], these statistics are used to derive suitable estimates of location, and the asymptotic relative efficiencies of these tests and estimates are studied.

## 2. Preliminary Notions

Let $\mathbf{X}_\alpha=(X_{1\alpha},\dots,X_{p\alpha})'$, $\alpha=1,\dots,N$ be $N$ independent stochastic vectors having continuous $p(\geqq 1)$-variate cumulative distribution functions $F_1(\mathbf{x}),\dots,F_N(\mathbf{x})$, respectively, where $\mathbf{x}\in R^p$, the real $p$-space. We denote by $\mathscr{F}_p(\mathscr{F}_p^o)$ the class of all $p$-variate continuous cdf's (diagonally summetric about $\mathbf{0}$), so that $\mathscr{F}_p^o\subset\mathscr{F}_p$. (For the definition of diagonal symmetry, see [1, 9].) Let $\mathbf{F}_N=(F_1,\dots,F_N)$, and let

$$\begin{aligned}\mathscr{F}_{pN}&=\{\mathbf{F}_N\colon F_i\in\mathscr{F}_p,\ i=1,\dots,N\},\\ \mathscr{F}_{pN}^o&=\{\mathbf{F}_N\colon F_i\in\mathscr{F}_p^o,\ i=1,\dots,N\}.\end{aligned}\tag{2.1}$$

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Also, let $\mathscr{F}_{pN}^{*}(\mathscr{F}_{pN}^{o})$ be the subset of $\mathscr{F}_{pN}(\mathscr{F}_{pN}^{o})$ for which $F_1 = \cdots = F_N$. We desire to test the null hypothesis

$$H_o\colon \mathbf{F}_N \in \mathscr{F}_{pN}^{o} \quad \text{against} \quad K\colon \mathbf{F}_N \notin \mathscr{F}_{pN}^{o}, \tag{2.2}$$

without unnecessarily imposing the restriction that $\mathbf{F}_N \in \mathscr{F}_{pN}^{*}$. With the notable exception of the bivariate sign test by Chatterjee [1], all the other multivariate rank type tests available in the literature (cf. [9] for the references cited therein), are proposed and worked out only for the situation where $\mathbf{F}_N \in \mathscr{F}_{pN}^{*}$. We propose to show here that the general class of rank order tests, developed by the authors [9] for the situation $\mathbf{F}_N \in \mathscr{F}_{pN}^{*}$, are valid and robust for the situation $\mathbf{F}_N \in \mathscr{F}_{pN}$. For this, we introduce the following notations. Let $R_{j\alpha}$ be the rank of $|X_{j\alpha}|$ among $|X_{j1}|, \ldots, |X_{jn}|$ i.e.,

$$R_{j\alpha} = \tfrac{1}{2} + \sum_{\beta=1}^{N} c(|X_{j\alpha}| - |X_{j\beta}|), \qquad \alpha = 1, \ldots, N;\ j = 1, \ldots, p, \tag{2.3}$$

where $c(u)$ is 1, $\frac{1}{2}$, or 0 according as $u$ is $>$, $=$, or $<0$. Also, let

$$S_{j\alpha} = 2c(X_{j\alpha}) - 1, \qquad \alpha = 1, \ldots, N;\ j = 1, \ldots, p. \tag{2.4}$$

Then, define the rank order statistics

$$T_{N,j} = N^{-1} \sum_{\alpha=1}^{N} E_{N,R_{j\alpha}}^{(j)} S_{j\alpha}, \qquad j = 1, \ldots, p, \tag{2.5}$$

where the rank scores $E_{N,\alpha}^{(j)} = J_{N,j}\left(\frac{\alpha}{(N+1)}\right)$, $1 \leqq \alpha \leqq N$, $j = 1, \ldots, p$, satisfy the following conditions (cf. [5, 8, 9]):

(a) for each $j(=1, \ldots, p)$, $\lim_{N\to\infty} J_{N,j}(u) = J_j(u)$ exists for $0<u<1$ and is not a constant;

$$\text{(b)} \qquad N^{-1} \sum_{\alpha=1}^{N} \left| J_{N,j}\left(\frac{\alpha}{N+1}\right) - J_j\left(\frac{\alpha}{N+1}\right) \right| = o(N^{-\frac{1}{2}}), \qquad i = 1, \ldots, p; \tag{2.6}$$

and

(c) $J_j(u)$ is absolutely continuous with

$$\left| \left(\frac{d^r}{du^r}\right) J_j(u) \right| \leqq K\{u(1-u)\}^{-r-\frac{1}{2}+\delta}, \qquad \delta > 0, \tag{2.7}$$

for $r = 0, 1$; $i = 1, \ldots, p$, where $K(>0)$ is a finite constant.

Finally, we define

$$v_{N,ij} = N^{-1} \sum_{\alpha=1}^{N} S_{i\alpha} S_{j\alpha} E_{N,R_{i\alpha}}^{(i)} E_{N,R_{j\alpha}}^{(j)}, \qquad i, j = 1, \ldots, p; \tag{2.8}$$

$$\mathbf{V}_N = ((v_{N,ij})) \quad \text{and} \quad \mathbf{V}_N^{-1} = ((v_N^{ij})). \tag{2.9}$$

Then, our proposed test statistic is

$$\mathscr{L}_N = N \sum_{i=1}^{p} \sum_{j=1}^{p} v_N^{ij} T_{N,i} T_{N,j}. \tag{2.10}$$

(Note that the same statistic was proposed in [9] for testing the hypothesis that $\mathbf{F}_N \in \mathscr{F}_{pN}^{o*}$. The robustness will be studied here.) Further, when all the $F_i$ $(i = 1, \ldots, N)$ are diagonally symmetric around a common $\boldsymbol{\theta}$, we may use the statistics $T_{N,j}$, $j = 1, \ldots, p$ to estimate $\boldsymbol{\theta}$. The properties of such estimators are studied in Section 7.

## 3. The Basic Permutation Principle

Let $\mathbf{Z}_N=(\mathbf{X}_1, \ldots, \mathbf{X}_N)$ be the sample point and $\mathfrak{Z}_N$ be the sample space. Consider the finite group $\mathscr{G}_N$ of transformations $\{g_N\}$, where typically a $g_N$ is given by

$$g_N \mathbf{Z}_N=\mathbf{Z}_N^*=[(-1)^{j_1}\mathbf{X}_1, \ldots, (-1)^{j_N}\mathbf{X}_N], \quad j_\alpha=0,1;\ \alpha=1, \ldots, N, \tag{3.1}$$

and $(-1)\mathbf{X}=(-X_1, \ldots, -X_p)'$. Since, for $\mathbf{F}_N\in\mathscr{F}_{pN}^o$, $\mathbf{X}_\alpha$ and $(-1)\mathbf{X}_\alpha$ both have the common cdf $F_\alpha$, $\alpha=1, \ldots, N$, the distribution of $\mathbf{Z}_N^*$ is the same as that of $\mathbf{Z}_N$ for all $g_N\in\mathscr{G}_N$, when $\mathbf{F}_N\in\mathscr{F}_{pN}^o$. Also, $\mathscr{G}_N$ has $2^N$ distinct elements $\{g_N\}$. Thus, if $S(\mathbf{Z}_N)=\{\mathbf{Z}_N^*\colon \mathbf{Z}_N^*=g_N\mathbf{Z}_N,\ g_N\in\mathscr{G}_N\}$, we have

$$P\{\mathbf{Z}_N=\mathbf{Z}_N^* | S(\mathbf{Z}_N^*), H_o\}=2^{-N}, \tag{3.2}$$

for all $\mathbf{Z}_N^*\in S(\mathbf{Z}_N^*)$, whatever be $\mathbf{F}_N\in\mathscr{F}_{pN}^o$. Let us denote the permutational (conditional) probability measure in (3.2) by $\mathscr{P}_N$. Since $\mathscr{P}_N$ is completely specified, we can always select a test function $\varphi(\mathbf{Z}_N)[0\leqq\varphi\leqq1]$, chosen in such a way that

$$E_{\mathscr{P}_N}[\varphi(\mathbf{Z}_N)]=\varepsilon(0<\varepsilon<1), \quad \text{the desired level of significance.} \tag{3.3}$$

In particular, we shall let

$$\varphi(Z_N)=\begin{cases}1, & \mathscr{L}_N>\mathscr{L}_{N,\varepsilon}(\mathscr{P}_N),\\ \mathscr{A}_\varepsilon(\mathscr{P}_N), & \mathscr{L}_N=\mathscr{L}_{N,\varepsilon}(\mathscr{P}_N),\\ 0, & \mathscr{L}_N<\mathscr{L}_{N,\varepsilon}(\mathscr{P}_N),\end{cases} \tag{3.4}$$

where $\mathscr{L}_{N,\varepsilon}(\mathscr{P}_N)$ and $\mathscr{A}_\varepsilon(\mathscr{P}_N)$ $(0\leqq\mathscr{A}_\varepsilon<1)$ are so chosen that (3.3) holds. (3.3) and (3.4) characterize the existence of a conditionally distribution-free test for $H_o$: $\mathbf{F}_N\in\mathscr{F}_{pN}^o$ based on the statistic $\mathscr{L}_N$.

As in [9], for small values of $N$, $\mathscr{L}_{N,\varepsilon}(\mathscr{P}_N)$ and $\mathscr{A}_\varepsilon(\mathscr{P}_N)$ are to be determined from the exact permutation (conditional) distribution of $\mathscr{L}_N$ (under (3.2)). For large values of $N$, we have the simplifications to be considered in the next section.

## 4. Large Sample Permutation Distribution of $\mathscr{L}_N$

Let us denote the marginal cdf of $X_{j\alpha}$ by $F_{j,\alpha}(x)$ and the joint cdf of $(X_{j\alpha}, X_{k\alpha})$ by $F_{jk,\alpha}(x, y)$, for $j\neq k=1, \ldots, p$, $\alpha=1, \ldots, N$. Let then

$$F_{j(N)}^*(x)=N^{-1}\sum_{\alpha=1}^N F_{j,\alpha}(x); \quad F_{jk(N)}^*(x, y)=N^{-1}\sum_{\alpha=1}^N F_{jk,\alpha}(x, y), \quad j\neq k=1, \ldots, p; \tag{4.1}$$

$$\begin{aligned} H_{j,\alpha}(x)&=F_{j,\alpha}(x)-F_{j,\alpha}(-x) \quad \text{if } x\geqq0\\ &=0, \quad \text{otherwise,}\end{aligned} \tag{4.2}$$

$$H_{j(N)}^*=N^{-1}\sum_{\alpha=1}^N H_{j,\alpha}(x), \quad j=1, \ldots, p; \tag{4.3}$$

$$\begin{aligned} H_{jk,\alpha}(x, y)=F_{jk,\alpha}(x, y)+F_{jk,\alpha}(-x, y)+F_{jk,\alpha}(x, -y)+F_{jk,\alpha}(-x, -y),\\ \text{for } x\geqq0, y\geqq0, j=k=1, \ldots, p;\ \alpha=1, \ldots, N;\end{aligned} \tag{4.4}$$

$$H_{jk(N)}^*(x, y)=N^{-1}\sum_{\alpha=1}^N H_{jk,\alpha}(x, y), \quad j\neq k=1, \ldots, p. \tag{4.5}$$

Also, define the corresponding empirical cdf's as

$$F_{N,j}(x)=N^{-1}\sum_{\alpha=1}^{N} c(x-X_{j\alpha}), \quad j=1,\ldots,p. \tag{4.6}$$

$$F_{N,jk}(x,y)=N^{-1}\sum_{\alpha=1}^{N} c(x-X_{j\alpha})\,c(y-X_{k\alpha}), \quad j=k=1,\ldots,p; \tag{4.7}$$

$$H_{N,j}(x)=\begin{cases}F_{N,j}(x)-F_{N,j}(-x-), & x\geqq 0,\\ 0, & x<0,\ j=1,\ldots,p;\end{cases} \tag{4.8}$$

$$\begin{aligned}H_{N,jk}(x,y)=&F_{N,jk}(x,y)+F_{N,jk}(-x-,y)+F_{N,jk}(x,-y-)\\ &+F_{N,jk}(-x-,-y-), \quad \text{for } j\neq k=1,\ldots,p.\end{aligned} \tag{4.9}$$

Our first problem is to study the stochastic behavior of $\mathbf{V}_N$, which, in turn, governs the large sample distribution of $\mathscr{L}_N$. For this, let us define

$$v_{jk}^{(N)}=\int_0^\infty\int_0^\infty J_j(H_{j(N)}^*(x))\,J_k(H_{k(N)}^*(y))\,dH_{jk(N)}^*(x,y), \quad j,k=1,\ldots,p, \tag{4.10}$$

$$\boldsymbol{v}^{(N)}=((v_{jk}^{(N)})). \tag{4.11}$$

Then, we assume that (a)

$$\boldsymbol{v}^{(N)} \text{ is positive definite for all } N\geqq N_0 \text{ (i.e., the characteristic roots of } \boldsymbol{v}^{(N)} \text{ are all bounded away from zero as } N\to\infty), \tag{4.12}$$

and (b) for each $j(=1,\ldots,p)$

$$\lim_{N\to\infty}\int_0^1 [J_{N,j}(u)-J_j(u)]^2\,du=0. \tag{4.13}$$

**Theorem 4.1.** *Under the assumptions* (a), (b), (c) *of Section* 2 *and* (4.13), $\mathbf{V}_N$ *is stochastically equivalent to* $\boldsymbol{v}^{(N)}$, *that is,* $\mathbf{V}_N-\boldsymbol{v}^{(N)}\xrightarrow{p}\mathbf{0}^{p\times p}$ *as* $N\to\infty$. *Thus, under* (4.12), $\mathbf{V}^{(N)}$ *is positive definite, in probability as* $N\to\infty$.

*Proof.* Using (2.8) and (4.9), we can write

$$v_{N,jk}=\int_0^\infty\int_0^\infty J_{N,j}\left(\frac{N}{N+1}H_{N,j}(x)\right) J_{N,k}\left(\frac{N}{N+1}H_{N,k}(y)\right) dH_{N,jk}(x,y), \tag{4.14}$$

for $j,k=1,\ldots,p$. Note that for $j=k$,

$$v_{N,jj}=\int_0^1 [J_{N,j}(u)]^2\,du, \quad \text{and} \quad v_{jj}^{(N)}=\int_0^1 [J_j(u)]^2\,du=v_{jj}. \tag{4.15}$$

Thus, by (4.13), we have $v_{N,ij}-v_{ij}^{(N)}\to 0$, as $N\to\infty$. Consequently, it suffices to prove the result only for $j\neq k=1,\ldots,p$. Now, using (4.13) and the Schwarz inequality, we can write (4.14) as

$$v_{N,jk}=\int_0^\infty\int_0^\infty J_j\left(\frac{N}{N+1}H_{N,j}(x)\right) J_k\left(\frac{N}{N+1}H_{N,k}(y)\right) dH_{N,jk}(x,y)+o(1). \tag{4.16}$$

The first term on the right hand side of (4.16) can be written as

$$\begin{aligned}&\int_0^\infty\int_0^\infty J_j(H^*_{j(N)}(x))\,J_k(H^*_{j(N)}(y))\,dH_{N,jk}(x,y)\\&+\int_0^\infty\int_0^\infty\left\{J_j\left(\frac{N}{N+1}H_{N,j}(x)\right)-J_j(H^*_{j(N)}(x))\right\}J_k\left(\frac{N}{N+1}H_{N,k}(y)\right)dH_{N,jk}(x,y)\qquad(4.17)\\&+\int_0^\infty\int_0^\infty J_j(H^*_{j(N)}(x))\left\{J_k\left(\frac{N}{N+1}H_{N,k}(y)\right)-J_k(H^*_{k(N)}(y))\right\}dH_{N,jk}(x,y).\end{aligned}$$

Now, the first term of (4.17) is an average over the $N$ independent random variables $\{J_j(H^*_{j(N)}(X_{j\alpha}))\,J_k(H^*_{k(N)}(X_{k\alpha}))\}$, $\alpha=1,\ldots,N$. Hence, using (2.7) (with $r=0$) and a theorem by Loève ([2], p. 275) on the law of large numbers, it readily follows that it is asymptotically equivalent, in probability, to $v^{(N)}_{jk}$, defined by (4.10). Hence, we require only to show that the second and the third terms of (4.17) both converge, in probability, to zero, as $N\to\infty$. Now, applying Schwarz-inequality, we obtain that the absolute value of the second integral in (4.17) is bounded above by

$$\left\{\left[\int_0^\infty\left[J_j\left(\frac{N}{N+1}H_{N,j}(x)\right)-J_j(H^*_{j,N}(x))\right]^2dH_{N,j}(x)\right]\cdot\left[\int_0^\infty\left[J_k\left(\frac{N}{N+1}H_{N,k}(y)\right)\right]^2dH_{N,k}(y)\right]\right\}^{\frac{1}{2}},\qquad(4.18)$$

where, by (2.7), the second factor is finite. Let us now define two sequences of real numbers $\{a_{N,j}\}$ and $\{b_{N,j}\}$ by

$$H^*_{j,N}(a_{N,j})=1-H^*_{j,N}(b_{N,j})=\varepsilon_N=O(N^{-\frac{1}{2}}).\qquad(4.19)$$

Then,

$$\begin{aligned}&\int_0^{a_{N,j}}\left[J_j\left(\frac{N}{N+1}H_{N,j}(x)\right)-J_j(H^*_{j,N}(x))\right]^2dH_{N,j}(x)\\&\leqq 2\left[\int_0^{a_{N,j}}\left[J_j\left(\frac{N}{N+1}H_{N,j}(x)\right)\right]^2dH_{N,j}(x)+\int_0^{a_{N,j}}[J_j(H^*_{j,N}(x))]^2\,dH_{N,j}(x)\right].\end{aligned}\qquad(4.20)$$

Now, using (2.7) and the fact that $H_{N,j}(x)-H^*_{j,N}(x)\to 0$ [cf. (4.21)], it can be easily shown that the right hand side of (4.20) is bounded above (in probability) by $(4/\delta)(\varepsilon_N)^\delta$, and thus converges, in probability, to zero as $N\to\infty$. Similarly,

$$\int_{b_{N,j}}^\infty\left[J_j\left(\frac{N}{N+1}H_{N,j}(x)\right)-J_j(H^*_{N,j}(x))\right]^2dH_{N,j}(x)\xrightarrow{p}0\quad\text{as}\quad N\to\infty.$$

Now, it follows from Theorem 5.2 of Sen [8] that for every $\varepsilon>0$ there exists a finite $c(\varepsilon)$ $(>0)$ such that for $0<\delta'<\frac{1}{2}$ and $N\geqq N_0(\varepsilon)$,

$$P\left\{\sup_x\frac{N^{\frac{1}{2}}|H_{N,j}(x)-H^*_{j,N}(x)|}{\{H^*_{j,N}(x)[1-H^*_{j,N}(x)]\}^{\frac{1}{2}-\delta}}\geqq c(\varepsilon)\right\}<\varepsilon,\qquad\forall\,\mathbf{F}_N\in\mathscr{F}_{pN}.\qquad(4.21)$$

Consequently, writing

$$\begin{aligned}\left|J_j\left(\frac{N}{N+1}H_{N,j}\right)-J_j(H^*_{j,N})\right| \\ =\left|\frac{N}{N+1}H_{N,j}-H^*_{j,N}\right|\left|J_j'\left(\frac{\theta N}{N+1}H_{N,j}+(1-\theta)H^*_{j,N}\right)\right|,\end{aligned} \tag{4.22}$$

and making use of (2.7) (with $\delta'<\delta$) and (4.21), it can be easily shown that

$$\int_{a_{N,j}}^{b_{N,j}}\left[J_j\left(\frac{N}{N+1}H_{N,j}\right)-J_j(H^*_{j,N})\right]^2 dH_{N,j}\xrightarrow{p}0, \quad \text{as} \quad N\to\infty. \tag{4.23}$$

Consequently,

$$v_{N,jk}\overset{p}{\sim}v_{jk}^{(N)} \quad \text{for all } j,k=1,\dots,p. \tag{4.24}$$

The second part of the theorem directly follows from (4.12) and (4.24). Q.E.D.

*Remark.* The theorem generalizes the results of Theorem 4.2 of Puri and Sen [3] to non-identically (even groupwise) distributed random variables.

**Theorem 4.2.** *Under the conditions of Theorem* 4.1, $\mathscr{L}_N$ *has asymptotically a chi-square distribution with p d.f.*

The proof follows precisely on the same line as in Theorem 3.2 of Sen and Puri [9] (with their Theorem 3.1 replaced by the present Theorem 4.1), and hence is omitted.

Thus, for the test function $\varphi(Z_N)$ in (3.4), we have

$$\mathscr{L}_{N,\varepsilon}(\mathscr{P}_N)\xrightarrow{p}\chi^2_{p,\varepsilon} \quad \text{and} \quad \mathscr{A}_\varepsilon(\mathscr{P}_N)\xrightarrow{p}0, \tag{4.25}$$

where $\chi^2_{p,\varepsilon}$ is the upper $100\varepsilon\%$ point of the chi-square distribution with $p$ degrees of freedom (d.f.).

Thus, the rank order tests proposed in Sen and Puri [9] remain valid and robust for the entire class $\mathscr{F}^o_{pN}$ of $p$-variate distributions. Next, we intend to study the power properties of the test based on $\mathscr{L}_N$ in the general case of heterogeneous distributions. For this we require first the following.

## 5. Asymptotic Multinormality of $\mathbf{T}_N$ for Arbitrary $\mathbf{F}_N$

As in Sen and Puri [9], we shall, for the sake of convenience of presentation, consider the following statistics, linearly related to $\mathbf{T}_N$. Let

$$\mathbf{T}^*_N=(T^*_{N,1},\dots,T^*_{N,p})'; \quad T^*_{N,j}=N^{-1}\sum_{\alpha=1}^{N}E^{(j)}_{N,R_{j\alpha}}c(X_{j\alpha}), \quad j=1,\dots,p. \tag{5.1}$$

We introduce the following notations:

$$\mu_{N,j,\alpha}=\int_0^\infty J_j\big(H^*_{j(N)}(x)\big)\,dF_{j,\alpha}(x), \quad j=1,\dots,p,\ \alpha=1,\dots,N. \tag{5.2}$$

$$\mu^*_{Nj}=N^{-1}\sum_{\alpha=1}^{N}\mu_{Nj,\alpha}, \quad j=1,\dots,p;\ \boldsymbol{\mu}^*_N=(\mu^*_{N,1},\dots,\mu^*_{N,p})'. \tag{5.3}$$

It may be noted that

$$\mu_{Nj}^* = \int_0^\infty J_j\big(H_{j(N)}^*(x)\big)\, dF_{j(N)}^*(x), \quad j=1,\ldots,p. \tag{5.4}$$

In the sequel, we denote by $F_{jk,\alpha}(x,y)$ a bivariate cdf, which for $j=k$ means $F_j(\min[x,y])$. A similar notation is used for $H_{jk,\alpha}$ and $H_{jk(N)}^*$.

**Theorem 5.1.** *Under the conditions* (a), (b) *and* (c) *of Section* 2, $N^{\frac{1}{2}}(\mathbf{T}_N^* - \mathbf{u}_N^*)$ *has asymptotically a multinormal distribution with a null mean vector and dispersion matrix* $\boldsymbol{\Gamma}_N = ((\gamma_{N,jk}))_{j,k=1,\ldots,p}$, *defined by* (5.18), *provided* $\boldsymbol{\Gamma}_N$ *is positive definite for all N sufficiently large.*

*Proof.* We write $T_{N,j}^*$ equivalently as

$$T_{N,j}^* = \int_0^\infty J_{N,j}\left(\frac{N}{N+1} H_{N,j}(x)\right) dF_{N,j}(x), \quad j=1,\ldots,p. \tag{5.5}$$

Also, we write

$$\begin{aligned} J_{N,j}\left(\frac{N}{N+1} H_{N,j}\right) &= J_j\left(\frac{N}{N+1} H_{N,j}\right) \\ &\quad + \left[J_{N,j}\left(\frac{N}{N+1} H_{N,j}\right) - J_j\left(\frac{N}{N+1} H_{N,j}\right)\right], \end{aligned} \tag{5.6}$$

$$\begin{aligned} J_j\left(\frac{N}{N+1} H_{N,j}\right) &= J_j(H_{j(N)}^*) + [H_{N,j} - H_{j(N)}^*]\, J_j'[H_{j(N)}^*] \\ &\quad - \frac{1}{N+1} H_{N,j} J_j'(H_{j(N)}^*) + \left\{J_j\left(\frac{N}{N+1} H_{N,j}\right) - J_j(H_{j(N)}^*)\right. \\ &\quad \left. - \left[\frac{N}{N+1} H_{N,j} - H_{j(N)}^*\right] J_j'(H_{j(N)}^*)\right\}, \end{aligned} \tag{5.7}$$

and finally, $dF_{N,j} = dF_{j(N)}^* + d[F_{N,j} - F_{j(N)}^*]$, $j=1,\ldots,p$. Then, from the preceding three equations we have

$$T_{N,j}^* = \left(\frac{1}{N}\right) \sum_{\alpha=1}^N B_{Nj}(X_{j\alpha}) + \sum_{r=1}^4 C_{r,Nj}, \tag{5.8}$$

where $B_{Nj}(X_{j\alpha}) = B_{Nj}^{(1)}(X_{j\alpha}) + B_{Nj}^{(2)}(X_{j\alpha})$, and

$$B_{Nj}^{(1)}(X_{j\alpha}) = J_j[H_{j(N)}^*(X_{j\alpha})]\, c(X_{j\alpha}); \tag{5.9}$$

$$B_{Nj}^{(2)}(X_{j\alpha}) = \int_0^\infty [c(x - |X_{j\alpha}|) - H_{j,\alpha}(x)]\, J_j'[H_{j(N)}^*(x)]\, dF_{j(N)}^*(x); \tag{5.10}$$

$$C_{1,Nj} = \int_0^1 \left[J_{N,j}\left(\frac{N}{N+1} H_{N,j}\right) - J_j\left(\frac{N}{N+1} H_{N,j}\right)\right] dF_{Nj} = o_p(N^{-\frac{1}{2}}), \quad \text{by (2.6)}; \tag{5.11}$$

$$C_{2,Nj} = \left[\frac{1}{(N+1)}\right] \int_0^\infty H_{N,j} J_j'(H_{j(N)}^*)\, dF_{Nj}, \tag{5.12}$$

$$C_{3,Nj} = \int_0^\infty [H_{N,j} - H_{j(N)}^*]\, J_j'(H_{j(N)}^*)\, d[F_{Nj} - F_{j(N)}^*], \tag{5.13}$$

and

$$C_{4,Nj}=\int_0^\infty \left\{J_j\left(\frac{N}{N+1}H_{N,j}\right)-J_j(H^*_{j(N)}) -\left[\frac{N}{N+1}H_{N,j}-H^*_{j(N)}\right]J'_j(H^*_{j(N)})\right\}dF_{Nj}. \tag{5.14}$$

As in Sen [8], it follows that $C_{r,Nj}$, $r=1,2,3,4$, $j=1,\dots,p$ are all $o_p(N^{-\frac{1}{2}})$. Hence, $N^{\frac{1}{2}}(\mathbf{T}^*_N-\boldsymbol{\mu}^*_N)'$ and $\left[N^{-\frac{1}{2}}\sum_{\alpha=1}^{N}\{B_{Nj}(X_{j\alpha})-\mu_{Nj,\alpha}\},\ j=1,\dots,p\right]$ have the same limiting distribution, if they have any at all. Thus, it suffices to show that $\mathbf{B}_N(\mathbf{X}_\alpha)=[B_{N1}(\mathbf{X}_\alpha),\dots,B_{Np}(\mathbf{X}_\alpha)]'$, $\alpha=1,\dots,N$ satisfy the conditions of the (multivariate) central limit theorem. With this end in view, we consider first the moments of these variables. By definition in (5.9) and (5.10), we have

$$E\{B^{(1)}_{Nj}(X_{j\alpha})\}=\int_0^\infty J_j[H^*_{N,j}(x)]\,dF_{j,\alpha}(x)=\mu_{Nj,\alpha}, \tag{5.15}$$

$$E\{B^{(2)}_{Nj}(X_{j\alpha})\}=0, \qquad \text{for all } j=1,\dots,p;\ \alpha=1,\dots,n. \tag{5.16}$$

Also, upon writing $H^o_{jk,\alpha}(x,y)=P\{|X_{j\alpha}|\leqq x, |X_{k\alpha}|\leqq y\}=F_{jk,\alpha}(x,y)-F_{jk,\alpha}(-x,y)-F_{jk,\alpha}(x,-y)+F_{jk,\alpha}(-x,-y)$, $x\geqq 0$, $y\geqq 0$, it follows that

$$\begin{aligned}
\gamma_{Njk,\alpha}&=\operatorname{cov}[B_{Nj}(X_{j\alpha}),B_{Nk}(X_{k\alpha})]\\
&=\sum_{r=1}^{2}\sum_{s=1}^{2}\operatorname{cov}[B^{(r)}_{Nj}(X_{j\alpha}),B^{(s)}_{Nk}(X_{k\alpha})]\\
&=\int_0^\infty\int_0^\infty J_j[H^*_{j(N)}(x)]\,J_k[H^*_{k(N)}(y)]\,dF_{jk,\alpha}(x,y)-\mu_{Nj,\alpha}\mu_{Nk,\alpha}\\
&\quad+\int_{x=0}^\infty\int_{y=0}^\infty\int_{z=-\infty}^\infty J_j[H^*_{j(N)}(y)]\,[c(x-|z|)-H_{k,\alpha}(x)]\,J'_k[H^*_{k(N)}(x)]\\
&\quad\cdot dF^*_{k(N)}(x)\,dF_{jk,\alpha}(y,z)\\
&\quad+\int_{x=0}^\infty\int_{y=0}^\infty\int_{z=-\infty}^\infty J_k[H^*_{k(N)}(y)]\,[c(x-|z|)-H_{j,\alpha}(x)]\,J'_j[H^*_{j(N)}(x)]\\
&\quad\cdot dF^*_{j(N)}(x)\,dF_{kj,\alpha}(y,z)\\
&\quad+\int_0^\infty\int_0^\infty[H^o_{jk,\alpha}(x,y)-H_{j,\alpha}(x)\,H_{k,\alpha}(y)]\,J'_j[H^*_{j(N)}(x)]\,J'_k[H^*_{k(N)}(y)]\\
&\quad\cdot dF^*_{j(N)}(x)\,dF^*_{k(N)}(y)
\end{aligned} \tag{5.17}$$

for $j,k=1,\dots,p$, $\alpha=1,\dots,N$. Let then

$$\boldsymbol{\Gamma}_N=((\gamma_{Njk}));\qquad \gamma_{Njk}=N^{-1}\sum_{\alpha=1}^{N}\gamma_{Njk,\alpha},\qquad j,k=1,\dots,p. \tag{5.18}$$

Thus, from the preceding four equations, we obtain that

$$E\left\{N^{-1}\sum_{\alpha=1}^{N} B_{Nj}(X_{j\alpha})\right\}=\mu_{Nj}^{*}, \quad j=1,\ldots,p; \tag{5.19}$$

$$N\operatorname{cov}\left\{N^{-1}\sum_{\alpha=1}^{N} B_{Nj}(X_{j\alpha}),\ N^{-1}\sum_{\alpha=1}^{N} B_{Nk}(X_{k\alpha})\right\}=\gamma_{Njk}, \tag{5.20}$$

for $j,k=1,\ldots,p$. Now, by (5.4), $\mu_{Nj}^{*}$ depends on $(F_1,\ldots,F_N)$ only through the cdf $F_{j(N)}^{*}$, $j=1,\ldots,p$. But, $\boldsymbol{\Gamma}_N$ depends on $F_1,\ldots,F_N$ in a rather involved way. We shall show next that $\boldsymbol{\Gamma}_N$ satisfies certain interesting matrix inequality. For this, let

$$\begin{aligned}
\gamma_{jk}(F_{(N)}^{*})=&\int_0^\infty\int_0^\infty J_j(H_{j(N)}^{*})\,J_k(H_{k(N)}^{*})\,dF_{jk(N)}^{*}(x,y)-\mu_{Nj}^{*}\,\mu_{Nk}^{*}\\
&+\int_{x=0}^{\infty}\int_{y=0}^{\infty}\int_{z=-\infty}^{\infty} J_j[H_{j(N)}^{*}(y)]\,[c(x-|z|)-H_{k(N)}^{*}(x)]\,J_k'[H_{k(N)}^{*}(x)]\\
&\cdot dF_{k(N)}^{*}(x)\,dF_{jk(N)}^{*}(y,z)\\
&+\int_{x=0}^{\infty}\int_{y=0}^{\infty}\int_{z=-\infty}^{\infty} J_k[H_{k(N)}^{*}(y)]\,[c(x-|z|)-H_{j(N)}^{*}(x)]\,J_j'[H_{j(N)}^{*}(x)]\\
&\cdot dF_{((N)}^{*}(x)\,dF_{kj(N)}^{*}(y,z)\\
&+\int_0^\infty\int_0^\infty [H_{jk(N)}^{o*}(x,y)-H_{j(N)}^{*}(x)\,H_{k(N)}^{*}(y)]\,J_j'[H_{j(N)}^{*}(x)]\,J_k[H_{k(N)}^{*}(y)]\\
&\cdot dF_{j(N)}^{*}(x)\,dF_{k(N)}^{*}(y),
\end{aligned} \tag{5.21}$$

where $H_{jk(N)}^{o*}=N^{-1}\sum_{\alpha=1}^{N} H_{jk,\alpha}^{o}$; $j,k=1,\ldots,p$. Let then

$$\boldsymbol{\Gamma}(F_{(N)}^{*})=((\gamma_{jk}(F_{(N)}^{*}))), \tag{5.22}$$

so that $\boldsymbol{\Gamma}(F_{(N)}^{*})$ is the value of $\boldsymbol{\Gamma}_N$ when $F_1\equiv\cdots\equiv F_N\equiv F_{(N)}^{*}$. Also, let

$$\begin{aligned}
\beta_{Nj,\alpha}^{(1)}&=\mu_{Nj,\alpha}-\mu_{Nj}^{*},\\
\beta_{Nj,\alpha}^{(2)}&=\int_0^\infty [H_{j\alpha}(x)-H_{j(N)}^{*}(x)]\,J_j'[H_{j(N)}^{*}(x)]\,dF_{j(N)}^{*}(x),
\end{aligned} \tag{5.23}$$

for $j=1,\ldots,p$, $\alpha=1,\ldots,N$. Thus, $N^{-1}\sum_{\alpha=1}^{N}\beta_{Nj,\alpha}^{(i)}=0$ for $i=1,2$; $j=1,\ldots,p$. Finally, let

$$\eta_{jk,N}=N^{-1}\sum_{\alpha=1}^{N}[\beta_{Nj,\alpha}^{(1)}+\beta_{Nj,\alpha}^{(2)}]\,[\beta_{Nk,\alpha}^{(1)}+\beta_{Nk,\alpha}^{(2)}], \quad j,k=1,\ldots,p. \tag{5.24}$$

$$\mathbf{H}_N=((\eta_{jk,N})). \tag{5.25}$$

Note that by definition, $\mathbf{H}_N$ is a positive semi-definite matrix; it ceases to be positive definite only when $\beta_{Nj,\alpha}^{(1)}+\beta_{Nj,\alpha}^{(2)}=0$ for all $\alpha=1,\ldots,N$ and for at least one $j(=1,\ldots,p)$.

**Lemma 5.2.** $\boldsymbol{\Gamma}_N=\boldsymbol{\Gamma}(F_{(N)}^{*})-\mathbf{H}_N$; $0\leqq|\boldsymbol{\Gamma}(F_{(N)}^{*})|<\infty$, *for all* $F_1,\ldots,F_N$.

To prove the Lemma, we consider first the following identities whose proofs are evident and therefore omitted.

$$\begin{aligned}\left(\frac{1}{N}\right)\sum_{\alpha=1}^{N}[c(x-|z|)-H_{k,\alpha}(x)]\,dF_{jk,\alpha}(y,z)&=[c(x-|z|)-H^*_{k(N)}(x)]\,dF^*_{jk(N)}(y,z)\\&\quad-\left(\frac{1}{N}\right)\sum_{\alpha=1}^{N}[H_{k,\alpha}(x)-H^*_{k(N)}(x)]\,d[F_{jk,\alpha}(x)-F^*_{jk(N)}(x)];\end{aligned}\tag{5.26}$$

$$\begin{aligned}\left(\frac{1}{N}\right)\sum_{\alpha=1}^{N}[H^o_{jk,\alpha}(x,y)-H_{j,\alpha}(x)\,H_{k,\alpha}(y)]&=[H^{o*}_{jk(N)}(x,y)-H^*_{j(N)}(x)\,H^*_{k(N)}(y)]\\&\quad-\left(\frac{1}{N}\right)\sum_{\alpha=1}^{N}[H_{j,\alpha}(x)-H^*_{j(N)}(x)]\,[H_{k,\alpha}(y)-H^*_{k(N)}(y)];\end{aligned}\tag{5.27}$$

$$\left(\frac{1}{N}\right)\sum_{\alpha=1}^{N}\mu_{Nj,\alpha}\mu_{Nk,\alpha}-\mu^*_{Nj}\mu^*_{Nk}=\left(\frac{1}{N}\right)\sum_{\alpha=1}^{N}(\mu_{Nj,\alpha}-\mu^*_{Nj})(\mu_{Nk,\alpha}-\mu^*_{Nk});\tag{5.28}$$

$$\int_{y=0}^{\infty}\int_{z=-\infty}^{\infty}J_j[H^*_{N(j)}(y)]\,d[F_{jk,\alpha}(y,z)-F^*_{jk(N)}(y,z)]=\beta^{(1)}_{Nj,\alpha},\tag{5.29}$$

for all $j,k=1,\ldots,p$. The first part of the lemma then follows from (5.17) and (5.18) after using (4.1) and (5.22) through (5.29). Since $\boldsymbol{\Gamma}(F^*_{(N)})$ is a covariance matrix, the proof of the second part follows directly by showing that $\gamma_{jj}(F^*_{(N)})<\infty$ for all $j=1,\ldots,p$. This directly follows from (5.21) by noting that for $j=k$, $y=z$ and

$$dF^*_{j(N)}(x)\leqq dH^*_{j(N)}(x)\qquad\text{for all }\ x\geqq 0;\tag{5.30}$$

$$\iint_{0<u<v<1}u^r(1-v)^s\,|J_j^{(r)}(u)|\,|J_j^{(s)}(v)|\,du\,dv<\infty,\tag{5.31}$$

for all $r,s=0,1$, where (5.31) follows from assumption (c) of Section 2. Hence the lemma.

Now, it follows from Lemma 3.2 of Sen [8] that for $\delta(>0)$, defined by (2.7),

$$N^{-1}\sum_{i=1}^{N}E\{|B_{Nj}(X_{j\alpha})|^{2+\delta}\}<\infty,\qquad\text{for all }\ j=1,\ldots,p,\tag{5.32}$$

uniformly in $\mathbf{F}_N\in\mathscr{F}_{pN}$. Hence, it remains only to show that for any non-null $\mathbf{l}$, $\mathbf{l}'\mathbf{B}_N(\mathbf{X}_\alpha)$, $\alpha=1,\ldots,N$ satisfy the Lindeberg condition of the classical central limit theorem. Now, for any non-null $\mathbf{l}$,

$$N^{-1}\sum_{\alpha=1}^{N}\operatorname{Var}(\mathbf{l}'\mathbf{B}_N(X_\alpha))=\mathbf{l}'\boldsymbol{\Gamma}_N\mathbf{l}>0,\tag{5.33}$$

as $\boldsymbol{\Gamma}_N$ is assumed to be positive definite (in the sense that the characteristic roots of $\boldsymbol{\Gamma}_N$ are all bounded away from zero). Also, by (5.32), we have for $\delta$, defined by (2.7),

$$\begin{aligned}N^{-1}\sum_{\alpha=1}^{N}E\{|\mathbf{l}'\{\mathbf{B}_N(\mathbf{X}_\alpha)-E\,\mathbf{B}_N(\mathbf{X}_\alpha)\}|^{2+\delta}\}&\leqq 2^{1+\delta}N^{-1}\sum_{\alpha=1}^{N}E\{|\mathbf{l}'\mathbf{B}_N(\mathbf{X}_\alpha)|^{2+\delta}\}\\&\leqq(2p)^{1+\delta}\sum_{j=1}^{p}|l_j|^{2+\delta}\left\{N^{-1}\sum_{\alpha=1}^{N}E\{|B_{Nj}(X_{j\alpha})|\}^{2+\delta}\right\}<\infty.\end{aligned}\tag{5.34}$$

Now (5.33) and (5.34) imply the Lindeberg condition. Hence Theorem 5.1 generalizes the results of Sen [8] to the multivariate case and of Sen and Puri [9] to the case when the cdf's are not all identical. When $F_1(\mathbf{x}) \equiv \cdots \equiv F_N(\mathbf{x}) \equiv F(\mathbf{x})$, it follows from (5.23) that $\beta_{Nj,\alpha}^{(i)} = 0$ for $i = 1, 2$ and $j = 1, \ldots, p$, and hence, from Lemma 5.2, we obtain the following:

**Lemma 5.3.** *If $F_1 \equiv \cdots \equiv F_N \equiv F$, then $\boldsymbol{\Gamma}_N = \boldsymbol{\Gamma}(F)$, where $\boldsymbol{\Gamma}(F)$ is defined by* (5.22) *with $F_{(N)}^* \equiv F$.*

In fact, we have a stronger result, as follows:

**Lemma 5.4.** *If $F_\alpha$ have univariate marginals all symmetric about* 0, *for $\alpha = 1, \ldots, N$, then $\mathbf{H}_N = \mathbf{0}^{p \times p}$ for all $F_1, \ldots, F_N$.*

*Proof.* If $F_{j\alpha}$ is symmetric about 0, then $H_{j\alpha}(x) = 2F_{j\alpha}(x) - 1$, $x \geqq 0$, for all $\alpha = 1, \ldots, N$. Hence $H_{j(N)}^*(x) = 2F_{j(N)}^*(x) - 1$ for all $x \geqq 0$, $j = 1, \ldots, p$. Consequently, writing $\beta_{Nj,\alpha}^{(1)}$ as

$$\int_0^\infty J_j[H_{j(N)}^*(x)]\, d[F_{j,\alpha}(x) - F_{j(N)}^*(x)] = \tfrac{1}{2} \int_0^1 J_j[H_{j(N)}^*(x)] \cdot d[H_{j,\alpha}(x) - H_{j(N)}^*(x)],$$

and integrating it by parts and finally adding to $\beta_{Nj,\alpha}^{(2)}$, we obtain $\beta_{Nj,\alpha}^{(1)} + \beta_{Nj,\alpha}^{(2)} = 0$ for all $j = 1, \ldots, p$, $\alpha = 1, \ldots, N$. Hence, by (5.24), $\mathbf{H}_N = \mathbf{0}$. Q.E.D.

It also follows that when $F_{j\alpha}$'s are all symmetric about 0,

$$\mu_{Nj}^* = \int_0^\infty J_j[H_{j(N)}^*(x)]\, dF_{j(N)}^*(x) = \tfrac{1}{2} \int_0^1 J_j(u)\, du, \quad j = 1, \ldots, p. \tag{5.35}$$

$$\gamma_{jj}(F_{(N)}^*) = \tfrac{1}{4} \int_0^1 J_j^2(u)\, du, \quad j = 1, \ldots, p \tag{5.36}$$

are all independent of $F_{(N)}^*$. However, $\gamma_{jk}(F_{(N)}^*)$ depends on the unknown cdf $F_{(N)}^*$ for all $j \neq k = 1, \ldots, p$. We consider next the following lemma, whose proof is omitted.

**Lemma 5.5.** *If $F_\alpha$, $\alpha = 1, \ldots, N$ are all diagonally symmetric about* $\mathbf{0}$, *then $\boldsymbol{\Gamma}(F_{(N)}^*) = \frac{1}{4}\mathbf{v}_N$, where $\mathbf{v}_N$ is defined by* (4.10) *and* (4.11).

We are now in a position to consider also a class of asymptotically distribution-free tests for $H_o$ in (2.2) based on $\mathbf{T}_N^*$ in (5.1). Let $\hat{\boldsymbol{\Gamma}}(F_{(N)}^*)$ be any consistent estimate of $\boldsymbol{\Gamma}(F_{(N)}^*)$ [in the sense that $\hat{\boldsymbol{\Gamma}}(F_{(N)}^*) \xrightarrow{P} \boldsymbol{\Gamma}(F_{(N)}^*)$ as $N \to \infty$]. Then, using Theorem 5.1, Lemma 5.3, and (5.35), we propose the test statistic

$$\mathscr{L}_N^* = N(\mathbf{T}_N^* - \boldsymbol{\mu}_o^*)' [\hat{\boldsymbol{\Gamma}}(F_{(N)}^*)]^{-1} (\mathbf{T}_N^* - \boldsymbol{\mu}_o^*), \tag{5.37}$$

where

$$\boldsymbol{\mu}_o^* = (\mu_{1o}^*, \ldots, \mu_{po}^*); \quad \mu_{jo}^* = \tfrac{1}{2} \int_0^1 J_j(u)\, du, \quad j = 1, \ldots, p. \tag{5.38}$$

It follows from Theorem 5.1 that under $H_o$ in (2.2), $\mathscr{L}_N^*$ has asymptotically a $\chi^2$ distribution with $p$ d.f., where of course, by virtue of Lemma 5.5 and Theorem 4.1, $\boldsymbol{\Gamma}(F_{(N)}^*)$ is assumed to be positive definite. We shall now study the large sample properties of the tests based on $\mathscr{L}_N$ in (2.10) and $\mathscr{L}_N^*$ in (5.37), and show that they are asymptotically power-equivalent.

For this purpose, we consider the following sequence of admissible alternative hypotheses $\{H_N\}$ which specifies that

$$H_N:\ F_\alpha(\mathbf{x})=F_{\alpha,N}(x)=F_{\alpha,o}(\mathbf{x}-N^{-\frac{1}{2}}\,\mathbf{d}_\alpha),\qquad d_\alpha=(d_{1\alpha},\dots,d_{p\alpha})',\qquad \alpha=1,\dots,N,\quad (5.39)$$

where $d_{j\alpha}$'s are all real and finite and $F_{\alpha o}$, $\alpha=1,\dots,N$ are all diagonally symmetric about $\mathbf{0}$. Furthermore, it is assumed that the score $E_{N,\alpha}^{(j)}$ is the expected value of the $\alpha$-th smallest observation of a sample size $N$ drawn from a distribution $\Psi_j(x)$, where

$$\Psi_j(x)=\begin{cases}2\Psi_j^*(x)-1, & x\geqq 0,\\ 0, & x<0.\end{cases}\quad \text{and}\quad \Psi_j^*(x)+\Psi_j^*(-x)=1\qquad \text{for all } x.\quad (5.40)$$

(5.40) implies that $J_j(0)=0$ and

$$J_j(u)=\Psi_j^{-1}(u)=\Psi_j^{*\,-1}\left(\frac{1+u}{2}\right)=J_j^*\left(\frac{1+u}{2}\right),\qquad 0<u<1,\ j=1,\dots,p.\quad (5.41)$$

Thus, in this case (2.7) holds. We also denote $F_{j(N)}^*$ as in (4.1) and let

$$f_{j(N)}^*=\left(\frac{d}{dx}\right)F_{j(N)}^*=\left(\frac{1}{N}\right)\sum_{\alpha=1}^{N}\left(\frac{d}{dx}\right)F_{j,\alpha}(x),\qquad j=1,\dots,p.\qquad (5.42)$$

Furthermore, we assume that

$$f_{j(N)}^*(x)\,J^{*\prime}[F_{j(N)}^*(x)]\qquad \text{is bounded as } x\to\pm\infty,\ j=1,\dots,p.\qquad (5.43)$$

Finally, we let

$$\mathbf{c}_{(N)}^*=(c_{1(N)}^*,\dots,c_{p(N)}^*)';$$
$$c_{j(N)}^*=\left(\frac{1}{2N}\right)\sum_{\alpha=1}^{N}d_{j\alpha}\int_{-\infty}^{\infty}J_j^{*\prime}[F_{j(N)}^*(x)]\,[f_{j(N)}^*(x)]^2\,dx,\qquad (5.44)$$

$j=1,\dots,p$; it is of course assumed that $F_\alpha$'s are all absolutely continuous density functions. Let us define

$$v_{N,jk}^*=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}J_j^*[F_{j(N)}^*(x)]\,J_k^*[F_{k(N)}^*(y)]\,dF_{jk(N)}^*(x,y),\qquad (5.45)$$

for $j,k=1,\dots,p$, and let $\mathbf{v}_N^*=((v_{N,jk}^*))$. Then the following Corollary is an immediate consequence of Theorem 5.1, Lemma 5.5, (5.41), and the fact that $\mathbf{v}_N^*=\mathbf{v}_N$ for diagonally symmetric $F_{(N)}^*$.

**Corollary 5.1.** *Under* $\{H_N\}$ *in* (5.39), *the conditions of Theorem* 5.1, (5.41), *and* (5.43), $N^{\frac{1}{2}}(\mathbf{T}_N-\mu_o^*)$ (*where* $\boldsymbol{\mu}_o^*$ *is defined by* (5.38)) *has asymptotically a p-variate normal distribution with mean vector* $\frac{1}{2}\,\mathbf{c}_{(N)}^*$ *and dispersion matrix* $\frac{1}{4}\,\mathbf{v}_N^*$ *where* $\mathbf{v}_N^*$ *has the elements* $v_{Njk}^*$, *defined by* (5.45). *In particular, when* $F_1\equiv\cdots\equiv F_N\equiv F$, $\mathbf{c}_{(N)}^*=(\bar{d}_{1(N)}B_1,\dots,\bar{d}_{p(N)}B_p)'$ *and* $v_{Njk}^*$ *is given by* (5.45), *where*

$$\bar{d}_{j(N)}=\frac{1}{N}\sum_{\alpha=1}^{N}d_{j\alpha}\quad\text{and}\quad B_j=\int_{-\infty}^{\infty}\left(\frac{d}{dx}\right)J_j^*[F_j(x)]\,dF_j(x),\qquad j=1,\dots,p.\quad (5.46)$$

**Corollary 5.2.** *The limiting distribution of* $N^{\frac{1}{2}}(\mathbf{T}_N-\boldsymbol{\mu}_N^*)$ *is singular if one or more* $J_j^*[F_{j(N)}^*(X_{j\alpha})]$ *can be expressed as a linear function of* $J_k^*[F_{k(N)}^*(X_{k\alpha})]$, $k=j=1,\ldots,p$.

The proof is simple and is omitted.

We shall now consider the large sample distribution of $\mathcal{L}_N$ in (2.10), when the null hypothesis (2.2) may not hold. To justify the limiting distribution theory, it will be assumed in the sequel that as $N\to\infty$, $F_{(N)}^*(\mathbf{x})=\left(\frac{1}{N}\right)\sum_{\alpha=1}^{N}F_\alpha(\mathbf{x})$ converges to a limiting cdf $F^*(\mathbf{x})$. Under fairly general conditions, such an assumption may be made, and we refer to Sen (1968) for various common models where this assumption is justified. Also, in (5.39) we make the following assumption for simplification of the results:

$$\text{Either } \mathbf{d}_\alpha=\mathbf{d},\ \forall\alpha \text{ and the } F_\alpha \text{ possibly differ, or } F_\alpha=F,\ \forall\alpha, \text{ and } \bar{d}_{j(N)}, \text{ defined by (5.46), tends to } \bar{d}_j, \text{ as } N\to\infty;\ j=1,\ldots,p. \tag{5.47}$$

We also define $B_j$ as in (5.46), and let

$$\mathbf{c}^*=(c_1^*,\ldots,c_p^*)';\quad c_j^*=d_jB(F_j^*)\quad\text{or}\quad \bar{d}_jB_j,\quad j=1,\ldots,p, \tag{5.48}$$

where $B(F_j^*)$ is defined as in (5.46) with $F$ replaced by $F^*$. Finally, let $\mathbf{v}^*=((v_{jk}^*))$, where $v_{jk}^*$ is defined by (5.45) with $F_{(N)}^*$ replaced by $F^*$. Then, we have the following

**Theorem 5.6.** *If* (i) $\lim_{N\to\infty}F_{(N)}^*\equiv F^*$ *exists and* (ii) *the conditions of Corollary* 5.5.1 *are satisfied, then the limiting distribution of* $\mathcal{L}_N$ *in* (2.10) *is noncentral chi-square with p d.f. and the non-centrality parameter*

$$\Delta_S=\mathbf{c}^{*\prime}\,\mathbf{v}^{*-1}\,\mathbf{c}^*. \tag{5.49}$$

The proof of the theorem follows from (5.45), Corollary 5.1, and some straightforward computations. The details are omitted.

Now, in (5.37), we have considered $\hat{\boldsymbol{\Gamma}}$ to be any consistent estimate of $\boldsymbol{\Gamma}(F_{(N)}^*)$. Since, under (5.38), $\boldsymbol{\Gamma}_N$ in (5.17) is asymptotically equivalent to $\boldsymbol{\Gamma}(F_{(N)}^*)$, of which $\hat{\boldsymbol{\Gamma}}$ is a consistent estimator, it follows from Theorem 5.6 that $\mathcal{L}_N^*$ has asymptotically the same distribution as of $\mathcal{L}_N$, and $\mathcal{L}_N^*\overset{P}{\sim}\mathcal{L}_N$. Hence the permutation test based on $\mathcal{L}_N$ and the asymptotically distribution-free test based on $\mathcal{L}_N^*$ are asymptotically power equivalent. Moreover, the choice of $\hat{\boldsymbol{\Gamma}}$ is of no importance in the limit.

## 6. Asymptotic Relative Efficiency and Robust Efficiency of $\mathcal{L}_N$

If $F_1=\cdots=F_N=F$ is a multi-normal cdf, the optimum invariant test for $H_0$ in (2.2) is based on the Hotelling $T_N^2$-statistic

$$\begin{aligned}T_N^2&=N\bar{\mathbf{X}}_N'S_N^{-1}\bar{\mathbf{X}}_N;\qquad \bar{\mathbf{X}}_N=N^{-1}\sum_{\alpha=1}^{N}\mathbf{X}_\alpha,\\ \mathbf{S}_N&=(N-1)^{-1}\sum_{\alpha=1}^{N}(\mathbf{X}_\alpha-\bar{\mathbf{X}}_N)(\mathbf{X}_\alpha-\bar{\mathbf{X}}_N)'.\end{aligned} \tag{6.1}$$

Suppose now, the null hypothesis in (2.2) holds but the $F_\alpha$ are neither all identical nor normal, but have finite moments up to the order $2+\delta$, $\delta>0$. Define by $\boldsymbol{\Sigma}_\alpha$

the covariance matrix of $\mathbf{X}_\alpha$, $\alpha=1,\dots,N$, and let $\Sigma^*_{(n)}=\left(\frac{1}{N}\right)\sum_{\alpha=1}^{N}\Sigma_\alpha$. Using then the multivariate central limit theorem, it follows that under $H_o$ in (2.2)

$$\mathscr{L}(N^{\frac{1}{2}}\bar{\mathbf{X}}_N)\to N(\mathbf{0},\Sigma^*_{(N)}). \tag{6.2}$$

Also writing

$$\mathbf{S}_N=\frac{N}{N-1}\left\{\frac{1}{N}\sum_{\alpha=1}^{N}\mathbf{X}_\alpha\mathbf{X}'_\alpha-\bar{\mathbf{X}}_N\bar{\mathbf{X}}'_N\right\}, \tag{6.3}$$

using (6.2) and Markov's law of large numbers, it follows that $S_N\xrightarrow{P}\Sigma^*_{(N)}$. Consequently, omitting some routine computations we find that when $F_1,\dots,F_N$ have all finite moments up to the order $2+\delta$, $\delta>0$, then

$$\mathscr{L}(T^2_N|H_o)\to\chi^2_p. \tag{6.4}$$

In this sense, Hotelling's $T^2_N$-test has also the same rejection rule as the test based on $\mathscr{L}_N$ or $\mathscr{L}^*_N$. We shall now consider the sequence of alternative hypotheses in (5.39) [under the further simplifications in (5.47)] and study the asymptotic distribution of $T^2_N$. Since, for such a sequence of alternative hypotheses,

$$\mathscr{L}(N^{\frac{1}{2}}\bar{\mathbf{X}}_N|H_N)\to N(\bar{\mathbf{d}},\Sigma^*), \tag{6.5}$$

[where $\bar{\mathbf{d}}=(\bar{d}_1,\dots,\bar{d}_p)'$ has the elements defined by (5.47), and $\Sigma^*$ is the covariance matrix of $F^*=\lim_{N\to\infty}F^*_{(N)}$], and even for such alternatives $\mathbf{S}_N\xrightarrow{P}\Sigma^*_{(N)}\to\Sigma^*$, it follows that $T^2_N$ has asymptotically a non-central chisquare distribution with non-centrality parameter

$$\Delta_{T^2}=\bar{\mathbf{d}}'(\Sigma^*)^{-1}\bar{\mathbf{d}}. \tag{6.6}$$

Thus, $T^2_N$ and the two other test statistics $\mathscr{L}_N$ and $\mathscr{L}^*_N$ have asymptotically the same distribution (namely, non-central $\chi^2$ with $p$ d.f.) differing only in their non-centrality parameters, a comparison of the non-centrality parameters reveals their asymptotic efficiencies. Hence, denoting by $e_{T_N,T^*_N}$ the asymptotic relative efficiency (A.R.E.) of a test $T_N$ with respect to a second test $T^*_N$, we have from Theorem 5.6 and (6.6) that

$$e_{\mathscr{L}_N,T^2_N}=[(\mathbf{c}^{*\prime}\boldsymbol{\nu}^{*-1}\mathbf{c}^*)/(\bar{\mathbf{d}}'(\Sigma^*)^{-1}\bar{\mathbf{d}})]. \tag{6.7}$$

It is quite clear that the A.R.E. in (6.7), in general, depends not only on $\boldsymbol{\nu}^*$ and $\Sigma$ but also on $\mathbf{c}^*$ and $\bar{\mathbf{d}}$. Consider the two situations in (5.47). In the first situation, where $\mathbf{d}_\alpha=\mathbf{d}$ for all $\alpha$ but $F_\alpha$'s possibly differ, (6.7) reduces to

$$e^{(1)}_{\mathscr{L}_N,T^2_N}=[(\mathbf{d}'(\mathbf{T}^*)^{-1}\mathbf{d})/(\mathbf{d}'\Sigma^{*-1}\mathbf{d})], \tag{6.8}$$

where

$$\mathbf{T}^*=((\tau^*_{jk}));\quad \tau^*_{jk}=\nu^*_{jk}/[B(F^*_j)\,B(F^*_k)],\quad j,k=1,\dots,p, \tag{6.9}$$

and $B(F^*_j)$'s are defined by (5.46). Thus, in this case, the A.R.E depends on $\mathbf{d}$, $\mathbf{T}^*$, and $\Sigma^*$. In the second case, when $F_\alpha\equiv F$ but $\mathbf{d}_\alpha$'s are not all equal, (6.7) reduces to

$$e^{(2)}_{\mathscr{L}_N,T^2_N}=[(\bar{\mathbf{d}}'\mathbf{T}^{-1}\bar{\mathbf{d}})/(\bar{\mathbf{d}}'\Sigma^{-1}\bar{d})], \tag{6.10}$$

where $\boldsymbol{\Sigma}$ is the common covariance matrix and $\mathbf{T}=\mathbf{T}^*|_{F^*\equiv F}$. Thus, in this case, the A.R.E. depends on $\bar{\mathbf{d}}$, $\mathbf{T}$, and $\boldsymbol{\Sigma}$. In any case, unlike the univariate situation, the A.R.E. is not independent of $\mathbf{d}$ or $\bar{\mathbf{d}}$, in general. However, a classical theorem due to Courant provides bounds for the variation of the values of (6.8) and (6.10) over $\mathbf{d}$ or $\mathbf{d}^*$.

**Theorem 6.1.** *The maximum and minimum values of* $e^{(1)}_{\mathscr{L}_N, T_N^2}$ *are given by the maximum and minimum eigen values of* $\boldsymbol{\Sigma}^*(\mathbf{T}^*)^{-1}$. *Similarly, the maximum and minimum values of* $e^{(2)}_{\mathscr{L}_N, T_N^2}$ *are given by the maximum and minimum eigen values of* $\boldsymbol{\Sigma}\mathbf{T}^{-1}$.

Now the bounds for $\boldsymbol{\Sigma}^*(\mathbf{T}^*)^{-1}$ have been studied in detail by Sen and Puri [9] for the special case $F_1=\cdots=F_N=F$. It follows that the same results hold in this general case provided we simply replace $F$ by $F^*$. For brevity, the details are therefore omitted. Further, generalizing the results of Sen [8] it can be shown that in many cases (such as coordinate wise independent or equally correlated variates), these A.R.E. values are higher in the heterogeneous case than in the homogeneous case, and this accounts for robust efficiency of $T_N^2$.

## 7. Robust Estimation of the Common Median (Vector)

Here we assume that $F_1, \ldots, F_N$ all have the common median $\boldsymbol{\theta}$ but these may be otherwise quite arbitrary. Thus, each $F_\alpha$ is assumed to be diagonally symmetric about $\boldsymbol{\theta}$. Define $T_{N,j}, j=1, \ldots, p$ as in (2.5), and let

$$\theta^*_{j,N}=\sup\{\theta\colon T_{N,j}(X_{j1}-\theta, \ldots, X_{jN}-\theta)>0\}, \tag{7.1}$$

$$\theta^{**}_{j,N}=\inf\{\theta\colon T_{N,j}(X_{j1}-\theta, \ldots, X_{jN}-\theta)<0\}, \quad j=1, \ldots, p, \tag{7.2}$$

where $T_{N,j}(X_{j1}-\theta, \ldots, X_{jN}-\theta)$ stands for the statistic $T_{N,j}$ computed for the values expressed in the arguments. As in [4], we consider

$$\hat{\boldsymbol{\theta}}_N=(\hat{\theta}_{1,N}, \ldots, \hat{\theta}_{p,N})'; \quad \hat{\theta}_{j,N}=\tfrac{1}{2}(\theta^*_{j,N}+\theta^{**}_{j,N}), \quad j=1, \ldots, p, \tag{7.3}$$

as a suitable estimate of $\boldsymbol{\theta}$. The estimate in (7.3) forms a general class of estimates, two important members of which are the Wilcoxon scores estimator and the normal scores estimator obtained respectively by taking $E^{(j)}_{N,\alpha}=\dfrac{\alpha}{N+1}$ and the expected value of the $\alpha$-th smallest observation of a sample of size $N$ from a chi-distribution with 1 degree of freedom.

The properties of the continuity (absolute continuity) of the distribution of $\hat{\boldsymbol{\theta}}_N$, its invariance under translation, and the diagonal symmetry of the distribution of $\hat{\boldsymbol{\theta}}_N$ around $\boldsymbol{\theta}$ have all been studied in detail in [4] for the particular case $F_1=\cdots=F_N=F$. Since the same proof goes through readily in the heterogeneous case, we refer to Theorems 3.1, 3.2, and 3.4 of [4] for details of these properties. Further, if we proceed exactly on the same line as in Theorems 4.3 and 4.4 of [4] and use our Theorem 5.1 with (5.41) (where $d_\alpha=1$, $\forall\,\alpha$), we readily arrive at the following.

**Theorem 7.1.** *Under the assumptions of Theorem* 5.1, $N^{\frac{1}{2}}(\hat{\boldsymbol{\theta}}_N-\boldsymbol{\theta})$ *has asymptotically a p-variate normal distribution with null mean vector and dispersion matrix* $\mathbf{T}^*$, *defined by* (6.7).

Now to compare $\hat{\boldsymbol{\theta}}_N$ and $\bar{\mathbf{X}}_N$, we employ the measure of "generalized variance" due to Wilks [10] and explained in detail in Section 5 of [4]. Thus, if we assume that $\lim_{N\to\infty} F^*_{(N)} = F^*$ exists, so that $\Sigma^*_{(N)} \to \Sigma^*$ as $N\to\infty$, we obtain from (6.2) and Theorem 7.1, that the A.R.E. of $\hat{\boldsymbol{\theta}}_N$ with respect to $\bar{\mathbf{X}}_N$ is equal to

$$e_{\boldsymbol{\theta},\bar{\mathbf{X}}} = \{\|\Sigma^*\|/\|T^*\|\}^{1/p}, \tag{7.4}$$

where $\|A\|$ refers to the determinant of a square matrix $A$. This again agrees with the expression for the A.R.E. in the particular case $F_1 = \cdots = F_N = F$, studied in detail in [4], with the only change that $F$ has to be replaced by $F^*$. As such, the various bounds for (7.4) studied in [4] remain valid even in the general case of heterogeneous distributions.

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