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The Converse of the Hartman-Wintner Theorem

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Let $\langle X_t \rangle$ be a sequence of independent and identically distributed random variables. Hartman and Wintner [1] proved that if $E(X_t)=0$ and $E(X_t^2)=\sigma^2<\infty$, then

$$\overline{\lim_{n\to\infty}} \frac{X_1+\cdots+X_n}{(2n\sigma^2 \log\log n)^{\frac{1}{2}}} = -\underline{\lim_{n\to\infty}} \frac{X_1+\cdots+X_n}{(2n\sigma^2 \log\log n)^{\frac{1}{2}}} = 1 \quad \text{a.s.} \tag{1}$$

Strassen [2] proved the converse, using quite sophisticated methods. In view of the importance of the law of the iterated logarithm, it may be worth giving a simple and elementary proof of Strassen's result. We show that if there exists a number K such that

$$\overline{\lim_{n\to\infty}} \frac{|X_1+\cdots+X_n|}{(2nK^2 \log\log n)^{\frac{1}{2}}} = 1 \quad \text{a.s.}, \tag{2}$$

then $E(X_t^2)<\infty$. It follows immediately that $E(X_t)=0$, and the value of $E(X_t^2)$ is determined by (1). As noticed by Strassen, the reduction of the general case to that of symmetrically distributed variables presents no difficulty.

We assume then that the distribution of X_t is symmetric and show that (2) and $E(X_i^2)=\infty$ are inconsistent. Given K, and assuming $E(X_t^2)=\infty$, choose $T>0$ so that

$$P[|X_t|\leqq T]>\tfrac{1}{2}, \tag{3}$$

and so that the conditional expectation of X_t^2 given $|X_t|\leqq T$ is bigger than, or equal to, $16K^2$. We define the sequences $\langle\kappa(i)\rangle$ and $\langle\lambda(j)\rangle$ as follows:

$$\kappa(1)=\min(t\colon |X_t|\leqq T); \qquad \kappa(i+1)=\min\big(t>\kappa(i)\colon |X_t|\leqq T\big);$$
$$\lambda(1)=\min(t\colon |X_t|>T); \qquad \lambda(j+1)=\min\big(t>\lambda(j)\colon |X_t|>T\big).$$

Put $\xi_i=X_{\kappa(i)}$, $\eta_j=X_{\lambda(j)}$. Since, by the Borel-Cantelli lemma, there are a.s. infinitely many values of t for which $|X_t|\leqq T$ and infinitely many values for which $|X_t|>T$, the random sequences $\langle\kappa(i)\rangle$ and $\langle\lambda(j)\rangle$ and, therefore, also $\langle\xi_i\rangle$ and $\langle\eta_j\rangle$ are a.s. well defined.

Let p and q be arbitrarily fixed positive integers. Given the increasing sequences $\langle k(1),\ldots,k(r)\rangle$ and $\langle l(1),\ldots,l(s)\rangle$ such that $r\geqq p$, $s\geqq q$, $r=p$ or $s=q$, and that $\langle k(1),\ldots,k(r), l(1),\ldots,l(s)\rangle$ is a permutation of $\langle 1,\ldots,r+s\rangle$, let $A\big(k(1),\ldots,k(r); l(1),\ldots,l(s)\big)$ denote the event "$|X_{k(1)}|\leqq T,\ldots,|X_{k(r)}|\leqq T, |X_{l(1)}|>T,\ldots,|X_{l(s)}|>T$". These events are pairwise disjoint, and their union has probability 1. Given $A\big(k(1),\ldots,k(r); l(1),\ldots,l(s)\big)$, the conditional joint probability distribution of $\xi_1,\ldots,\xi_r,\eta_1,\ldots,\eta_s$, that is of $X_{k(1)},\ldots,X_{k(r)},X_{l(1)},\ldots,X_{l(s)}$, is easily written, and factorizes into univariate probability distributions. One deduces from it, as a marginal distribution, that of $\xi_1, \ldots, \xi_p, \eta_1, \ldots, \eta_q$. Since it does not depend on $A(k(1), \ldots, k(r); l(1), \ldots, l(s))$, it is simply the joint probability distribution of these variables, which are thus found to be independent.

In view of $16K^2 \leqq E(\xi_i^2) \leqq T^2$, by the Hartman-Wintner theorem, the set S of indices m for which

$$|\xi_1 + \cdots + \xi_m| > (16K^2 m \log\log m)^{\frac{1}{2}} \tag{4}$$

is a.s. infinite. Let B denote the event "$(\xi_1 + \cdots + \xi_m)(\eta_1 + \cdots + \eta_{\kappa(m)-m}) \geqq 0$ for infinitely many m in S". In view of the symmetric distribution and of the independence of $\xi_1 + \cdots + \xi_m$ and $\eta_1 + \cdots + \eta_{\kappa(m)-m}$, $P(B) \geqq \frac{1}{2}$. But, by (3) and the strong law of large numbers, the set of values of m for which $\kappa(m) \geqq 2m$ is a.s. finite. Thus, with probability $\geqq \frac{1}{2}$, there remain infinitely many m in S for which both $\kappa(m) < 2m$ and

$$(\xi_1 + \cdots + \xi_m)(\eta_1 + \cdots + \eta_{\kappa(m)-m}) > 0,$$

and for which, by (4),

$$\begin{aligned}|X_1 + \cdots + X_{\kappa(m)}| &= |\xi_1 + \cdots + \xi_m + \eta_1 + \cdots + \eta_{\kappa(m)-m}| > (16K^2 m \log\log m)^{\frac{1}{2}} \\ &> \left(4K^2 \kappa(m) \log\log \kappa(m)\right)^{\frac{1}{2}},\end{aligned}$$

which contradicts (2).

References

1. Hartman, P. and Wintner, A.: On the law of the iterated logarith. Amer. J. math. **63**, 169–176 (1941).
2. Strassen, V.: A converse to the law of the iterated logarithm. Z. Wahrscheinlichkeitstheorie verw. Geb. **4**, 265–268 (1966).

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