Z. Wahrscheinlichkeitstheorie verw. Gebiete
42, 135–140 (1978)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

An Algebraic Central Limit Theorem in the Anti-Commuting Case

W. von Waldenfels

Institut für Angewandte Mathematik der Universität,
Im Neuenheimer Feld 294, 6900 Heidelberg 1, Federal Republic of Germany

In a previous paper [2] an algebraic version of the central limit theorem was derived which contains the classical central limit theorem and the convergence to "quasi-free states" in the quantum mechanics of bosons. In the following paper these results shall be generalized in order to cover the convergence to "fermion quasi-free states". In the same way higher central limit theorems can be derived which become trivial in the case of positivity. The method of proof is somewhat different from [3]. The results are more general, as one does not presuppose a Clifford algebra, but the structure of Clifford algebra comes out in the limit theorem.

We consider an associative algebra $\mathfrak{A}$ with 1 over a field K and denote by $\bigotimes^N \mathfrak{A}$ the tensor product of N copies of $\mathfrak{A}$ as a vector space without any multiplicative structure. If in $\bigotimes^N \mathfrak{A}$ the product

$$(f_1 \otimes \cdots \otimes f_n), (g_1 \otimes \cdots \otimes g_N) \mapsto f_1 g_1 \otimes \cdots \otimes f_N g_N$$

is introduced, $\bigotimes^N \mathfrak{A}$ becomes an associative K-algebra and is then denoted by $\bigotimes^N_\sigma \mathfrak{A}$, called $\bigotimes^N \mathfrak{A}$ with the symmetric multiplication, emphasizing the fact that e.g. $f \otimes 1 \otimes \cdots \otimes 1$ and $1 \otimes g \otimes 1 \otimes \cdots \otimes 1$ commute.

Let $\mathfrak{A}$ be a semigraded algebra, i.e. $\mathfrak{A} = \mathfrak{A}_0 \oplus \mathfrak{A}_1$, $\mathfrak{A}_i \mathfrak{A}_j \subset \mathfrak{A}_k$, with $k \equiv i + j \bmod 2$, then in $\bigotimes^N \mathfrak{A}$ an antisymmetric multiplication [1] can be introduced. Define $\varepsilon: \mathfrak{A}_0 \cup \mathfrak{A}_1 \to \{0, 1\}$, $\varepsilon(x) = 0$, if $x \in \mathfrak{A}_0$, $\varepsilon(x) = 1$, if $x \in \mathfrak{A}_1$. Let $f_1, \ldots, f_N$, $g_1, \ldots, g_N$ belong to $\mathfrak{A}_0$ or $\mathfrak{A}_1$, then put

$$(f_1 \otimes \cdots \otimes f_N)(g_1 \otimes \cdots \otimes g_N) = \pm f_1 g_1 \otimes \cdots \otimes f_N g_N$$

where the $+$sign stands, if $\sum_{1 \leq i < j \leq N} \varepsilon(f_j)\, \varepsilon(g_i)$ is even and the $-$sign stands if this sum is odd. With this multiplication $\bigotimes^N \mathfrak{A}$ forms an associative semigraded algebra [1] which is denoted by $\bigotimes^N_\alpha \mathfrak{A}$, where α stands for antisymmetric, for e.g. $f \otimes 1 \otimes \cdots \otimes 1$ and $1 \otimes g \otimes \cdots \otimes 1$ anticommute if f and g belong to $\mathfrak{A}_1$. An important example is the free algebra $\mathfrak{F}$ generated by x_i, $i \in I$. This algebra is semigraded, with $\mathfrak{F}_0$ spanned by the monomials of even degree, and $\mathfrak{F}_1$ spanned by the monomials of odd degree.

0044-3719/78/0042/0135/$01.20

We prove first a combinatorial lemma which generalizes that of [2]. If $f \in \mathfrak{A}$, denote

$$f_1^{(N)} = f \otimes 1 \otimes \cdots \otimes 1, \ldots, f_N^{(N)} = 1 \otimes \cdots \otimes 1 \otimes f \in \bigotimes^N \mathfrak{A}$$

and

$$f^{(N)} = f_1^{(N)} + f_2^{(N)} + \cdots + f_N^{(N)}.$$

Lemma 1. *Let $\omega: \mathfrak{A} \to K$ be linear and $\omega(1) = 1$. Assume that $\mathfrak{A}$ is semigraded and that in $\bigotimes^n \mathfrak{A}$, $n \in N$, either the symmetric or the antisymmetric multiplication has been introduced. Let $f_1, \ldots, f_k \in \mathfrak{A}$. Then*

$$\omega^{\otimes N}(f_1^{(N)} \ldots f_k^{(N)}) = \sum_{p=1}^{k} \binom{N}{p} \sum_{\varphi \in F(k,p)} \omega^{\otimes p}(f_{1,\varphi(1)}^{(p)} \cdots f_{k,\varphi(k)}^{(p)})$$

where $\omega^{\otimes N}: \mathfrak{A}^{\otimes N} \to K$ is defined by linear extension from $\omega^{\otimes N}(f_1 \otimes \cdots \otimes f_N) = \omega(f_1) \ldots \omega(f_N)$ and $F(n, p)$ denotes the set of all functions from $\{1, \ldots, n\}$ onto $\{1, \ldots, p\}$.

Proof. The lemma uses only the following coherence property of the multiplication laws of $\bigotimes_\sigma^N \mathfrak{A}$ resp. $\bigotimes_\alpha^N \mathfrak{A}$. Let $\mathfrak{S}$ be the class of all finite ordered sets. Let $S, S' \in \mathfrak{S}$ and $\varphi: S \to S'$ be order-preserving and injective. Define the mapping $\eta(\varphi): \bigotimes^S \mathfrak{A} \to \bigotimes^{S'} \mathfrak{A}$ by

$$\eta(\varphi)(\bigotimes_{\alpha \in S} f_\alpha) = \bigotimes_{\beta \in S'} \tilde{f}_\beta$$

where $\tilde{f}_\beta = f_\alpha$ if $\beta = \varphi(\alpha)$ and $\tilde{f}_\beta = 1$ if $\beta \notin \varphi(S)$.

We say a family of multiplication laws in $\bigotimes^S \mathfrak{A}$, $S \in \mathfrak{S}$, is *coherent* if $\eta(\varphi)$: $\bigotimes^S \mathfrak{A} \to \bigotimes^{S'} \mathfrak{A}$ is a multiplicative homomorphism, for every order-preserving injection φ. It is easy to prove that the families $\bigotimes_\sigma^S \mathfrak{A}$, $S \in \mathfrak{S}$ and $\bigotimes_\alpha^S \mathfrak{A}$, $S \in \mathfrak{S}$ have this coherence property.

Even without any coherence property one has

$$\omega^{\otimes S} \circ \eta(\varphi) = \omega^{\otimes S'}.$$

Multiplying out the left-hand side below one sees

$$f_1^{(N)} \ldots f_k^{(N)} = \sum_{\psi} f_{1,\psi(1)}^{(N)} \cdots f_{k,\psi(k)}^{(N)}$$

where ψ runs over all mappings $\psi: \{1, \ldots, k\} \to \{1, \ldots, N\}$. Fix ψ and set $S = \operatorname{Im} \psi = \{\gamma_1, \ldots, \gamma_p\}$, $\gamma_1 < \cdots < \gamma_p$. Let ϑ be the mapping $\{1, \ldots, p\} \to \{1, \ldots, N\}$, $i \mapsto \gamma_i$. As ϑ is order-preserving and injective, $\eta(\vartheta)$ is a multiplicative homomorphism from $\bigotimes^p \mathfrak{A}$ into $\bigotimes^N \mathfrak{A}$ with either symmetric or antisymmetric multiplication. For $j \in S$

$$f_{i,j}^{(N)} = \eta(\vartheta) f_{i,\vartheta^{-1}(j)}^{(p)}$$

and

$$f_{i,\psi(i)}^{(N)} = \eta(\vartheta) f_{i,\varphi(j)}^{(p)}$$

with $\varphi = \vartheta^{-1} \circ \psi$. Hence

$$f_{1,\psi(1)}^{(N)} \dots f_{k,\psi(k)}^{(N)} = \eta(\vartheta)(f_{1,\varphi(1)}^{(p)} \dots f_{k,\varphi(k)}^{(p)})$$

and

$$\begin{aligned}\omega^{\otimes N}(f_{1,\psi(1)}^{(N)} \dots f_{k,\psi(k)}^{(N)}) &= \omega^{\otimes p}(f_{1,\varphi(1)}^{(p)} \dots f_{k,\varphi(k)}^{(p)}) \\ &= C_\varphi, \text{ say.}\end{aligned}$$

Fix φ. Then for any subset of $\{1, \dots, N\}$ with p elements there can be defined ψ and ϑ such that $\varphi = \vartheta^{-1} \circ \psi$. So to any φ there belong $\binom{N}{p}$ mappings ψ. Hence the result sought

$$\omega^{\otimes N}(f_1^{(N)} \dots f_k^{(N)}) = \sum_{p=1}^{k} \binom{N}{p} \sum_{\varphi} C_\varphi.$$

Lemma 2. *Let $\omega: \mathfrak{A} \to K$ be linear and $\omega(1) = 1$. Assume that in $\bigotimes^N \mathfrak{A}$ either the symmetric or the antisymmetric multiplication has been introduced. Let $f_1, \dots, f_k \in \mathfrak{A}$ and assume that*

$$\omega(f_{i_1}),\ \omega(f_{i_1} f_{i_2}), \dots, \omega(f_{i_1} \dots f_{i_{s-1}}) = 0$$

for $1 \leqq i_1 < i_2 < \dots < i_{s-1} \leqq k$. Then as $N \to \infty$

$$\omega^{\otimes N}((f_1^{(N)} \cdot N^{-1/s}) \dots (f_k^{(N)} \cdot N^{-1/s}))$$

goes to 0 if k is not a multiple of s and to

$$\frac{1}{p!} \sum_{\varphi \in F(k,p,s)} \omega^{\otimes p}(f_{1,\varphi(1)}^{(p)} \dots f_{k,\varphi(k)}^{(p)}) \qquad \text{if } k = ps$$

where $F(k, p, s)$ is the set of all mappings φ from $\{1, \dots, k\}$ onto $\{1, \dots, p\}$ such that $\# \varphi^{-1}(i) = s$ for $i = 1, \dots, p$.

Proof. We use the following fact which holds for symmetric and antisymmetric multiplications in $\bigotimes^N \mathfrak{A}$:

$$\omega^{\otimes p}(f_{1,\varphi(1)}^{(p)} \dots f_{k,\varphi(k)}^{(p)})$$

vanishes if there exists an i, $1 \leqq i \leqq p$, such that $\# \varphi^{-1}(i) < s$. By Lemma 1

$$\begin{aligned}&\omega^{\otimes N}((f_1^{(N)} \cdot N^{-1/s}) \dots (f_k^{(N)} \cdot N^{-1/s})) \\ &\quad = N^{-k/s} \sum_{p=1}^{k} \binom{N}{p} \sum_{\varphi \in F(k,p)} \omega^{\otimes p}(f_{1,\varphi(1)}^{(p)} \dots f_{k,\varphi(k)}^{(p)})\end{aligned}$$

and in the right-hand sum only those φ appear for which $\# \varphi^{-1}(i) \geqq s$ for $i = 1, \dots, p$. This implies that $sp \leqq k$, $p \leqq k/s$. For $N \to \infty$ and k not a multiple of s, the highest term in N vanishes at least like $N^{-1/s}$, as $\binom{N}{p}$ behaves like $N^p/p!$. If k is a multiple of s, $k = p_0 s$, only the term with $p = p_0$ survives and gives the stated result.

From the Lemmata 1 and 2 immediately follow Lemma 1 and Theorem 1 of [2] and hence the rest of the results for symmetric multiplication. In the following therefore we treat only $\bigotimes_\alpha^N \mathfrak{A}$. We introduce some notation. Let a_i, $i \in I$, be a family of elements of an algebra A, assume I to be ordered and $S \subset I$, $S = \{i_1, \ldots, i_k\}$ with $i_1 < i_2 < \cdots < i_k$ a finite subset. Then we denote by a_S the product $a_S = a_{i_1} \ldots a_{i_k}$. If S is a finite ordered set and $(S_1, \ldots, S_p)$ a sequence of subsets of S with $S_i \cap S_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^{p} S_i = S$, then $\begin{pmatrix} S \\ S_1 \ldots S_p \end{pmatrix}$ denotes the permutation of S, which is defined in the following way: map the first element of S onto the first element of S_1, the second element of S onto the second element of S_1 and so on until S_1 is exhausted. Then map the following element of S onto the first element of S_2 and so on.

If φ is a mapping from $\{1, \ldots, k\}$ onto $\{1, \ldots, p\}$ and $f_1, \ldots, f_k$ belong to $\mathfrak{A}_1$, then

$$\omega^{\otimes p}(f_{1,\varphi(1)}^{(p)} \ldots f_{k,\varphi(k)}^{(p)}) = \operatorname{sg} \begin{pmatrix} 1 & \ldots k \\ S_1 & \ldots S_p \end{pmatrix} \omega(f_{S_1}) \ldots \omega(f_{S_p})$$

with $S_i = \varphi^{-1}(i)$, $i = 1, \ldots, p$.

Lemma 3. *Assume that in $\bigotimes^N \mathfrak{A}$ the antisymmetric multiplication has been introduced, that $f_1, \ldots, f_k \in \mathfrak{A}_1$, and that $\omega(f_S) = 0$ for $S \subset \{1, \ldots, k\}$, $\# S < s$. Then*

$$\omega^{\otimes N}((f_1^{(N)} \cdot N^{-1/s}) \ldots (f_k^{(N)} \cdot N^{-1/s}))$$

converges as $N \to \infty$ to 0 if k is not a multiple of s and to

$$\omega(f_1 \ldots f_p) \quad \text{if } k = p$$

and to 0 if $k = sp$, s odd and greater than 1, $p \geqq 2$, and to

$$\sum_{\{S_1, \ldots, S_p\} \in \mathscr{P}_s(1, \ldots, ps)} \operatorname{sg} \begin{pmatrix} 1 & \ldots ps \\ S_1 & \ldots S_p \end{pmatrix} \omega(f_{S_1}) \ldots \omega(f_{S_p})$$

if s even, $p \geqq 1$. Here $\mathscr{P}_s(1, \ldots, ps)$ is the set of all partitions of $\{1, \ldots, ps\}$ into subsets of s elements. As $\operatorname{sg} \begin{pmatrix} 1 & \ldots ps \\ S_1 & \ldots S_p \end{pmatrix}$ does not depend upon the order of $S_1, \ldots, S_p$, this number depends only on the partition $\{S_1, \ldots, S_p\}$.

Proof. This lemma follows from the fact that

$$\operatorname{sg} \begin{pmatrix} S \\ S_1 \ldots S_p \end{pmatrix}, \qquad \# S_i = s, \qquad S = \{1, \ldots, k\}$$

changes sign upon interchanging S_i, S_j, $i \neq j$, if s is odd and does not change sign if s is even. This yields immediately the case of even s. That the expression of Lemma 2 vanishes for $p \geqq 2$ follows from the equation $\sum_{\pi \in \mathfrak{S}_p} \operatorname{sg} \pi = 0$ for $p \geqq 2$.

Let a_i, $i \in I$, be a family of elements of $\mathfrak{A}$, $a_i \in \mathfrak{A}_1$. Denote by $\mathfrak{F}$ the free algebra generated by x_i, $i \in I$. Let $Q: I^s \to K$ be an s-dimensional matrix, s even. Define the antigaussian functional α_Q of order s on $\mathfrak{F}$ with respect to Q by

$$\alpha_Q(1)=1,$$
$$\alpha_Q(x_{i_1}\dots x_{i_k})=0, \text{ if } k \text{ is not a multiple of } s,$$
$$\alpha_Q(x_{i_1}\dots x_{i_{ps}})=\sum_{\{S_1,\dots,S_p\}\in\mathscr{P}_s(1,\dots,ps)} \operatorname{sg}\begin{pmatrix}1 & \dots ps\\ S_1 & \dots S_p\end{pmatrix} Q_{i(S_1)}\dots Q_{i(S_p)}$$

where $Q_{i(S_1)}=Q_{i(j_1)\dots i(j_s)}$ if $S_1=\{j_1,\dots,j_s\}$, $j_1<\dots<j_s$. Then one has

Theorem 1. *Let $\omega(a_{i_1})=0,\dots,\omega(a_{i_1}\dots a_{i_{s-1}})=0$ for $i_1,\dots,i_{s-1}\in I$ and let s be even. Then*

$$\omega^{\otimes N}(P(a_i^{(N)}\cdot N^{-1/s}))\to\alpha_Q(P)$$

as $N\to\infty$, where P is any polynomial and $P(a_i^{(N)}\cdot N^{-1/s})$ signifies that in P the x_i have been replaced by $a_i^{(N)}\cdot N^{-1/s}$.

We did not formulate the case of odd s in such a solemn way, as this case is trivial by Lemma 3.

Let $\mathfrak{A}$ be a semigraded algebra. Define in $\mathfrak{A}$ a non-associative multiplication $\circ$ by linear extension from the definition

$$f_1\circ f_2=f_1 f_2-(-1)^{\varepsilon(f_1)\varepsilon(f_2)}f_2 f_1$$

if f_i belong to $\mathfrak{A}_0$ or to $\mathfrak{A}_1$. So $f_1\circ f_2$ is the commutator unless both f_1 and f_2 belong to $\mathfrak{A}_1$. In that case it is the anticommutator.

Let $\mathfrak{F}_c\subset\mathfrak{F}$ be the $\circ$-subalgebra generated by x_i, $i\in I$. So $\mathfrak{F}_c$ is the linear span of 1, x_i, $x_i\circ x_j$, $(x_i\circ x_j)\circ x_k$, $x_i\circ(x_j\circ x_k)$, etc. $\mathfrak{F}_c$ is nothing else than the Lie superalgebra generated by x_i [4].

Theorem 2. *The antigaussian functional $\alpha_Q:\mathfrak{F}\to K$ of order s vanishes on the two-sided ideal generated by the elements of the form*

$$P-\alpha_Q(P),$$

where P runs through all homogeneous polynomials of degree s in $\mathfrak{F}_c$.

Proof. One gets immediately for $\mathfrak{A}=\mathfrak{F}$ that

$$\alpha_Q^{\otimes N}(P(x_i^{\otimes N}\cdot N^{-1/s}))\to\alpha_Q(P)$$

for $P\in\mathfrak{F}$. As $\bigotimes_\alpha^n\mathfrak{A}$ is again semigraded, if $\mathfrak{A}$ is semigraded [1], we can define $\circ$ in $\bigotimes_\alpha^n\mathfrak{A}$ as well as in $\mathfrak{A}$. We state then that $(W_1\circ W_2)^{(N)}=W_1^{(N)}\circ W_2^{(N)}$ for monomials $W_1, W_2\in\mathfrak{F}$, for

$$\begin{aligned}W_1^{(N)}\circ W_2^{(N)} &= \sum_{i,j=1}^{N} W_{1,i}^{(N)}\circ W_{2,j}^{(N)}=\sum_{i=1}^{N} W_{1,i}^{(N)}\circ W_{2,i}^{(N)}\\ &=\sum_{i=1}^{N}(W_1\circ W_2)_i^{(N)}=(W_1\circ W_2)^{(N)}\end{aligned}$$

hence for $P\in\mathfrak{F}$ one has

$$P(x_i^{(N)})=P^{(N)}.$$

Then for $k=ps$

$$\begin{aligned}
&\alpha_Q(x_{i(1)} \dots x_{i(j)} P x_{i(j+1)} \dots x_{i(k)}) \\
&\quad = \lim_{N\to\infty} \alpha_Q^{\otimes N}((x_{i(1)}^{(N)} \cdot N^{-1/s}) \dots (x_{i(j)}^{(N)} \cdot N^{-1/s}) \\
&\qquad \cdot P(x^{(N)} \cdot N^{-1/s})(x_{i(j+1)}^{(N)} \cdot N^{-1/s}) \dots (x_{i(k)}^{(N)} \cdot N^{-1/s})) \\
&\quad = \lim N^{-(p+1)} \alpha_Q^{\otimes N}(x_{i(1)}^{(N)} \dots x_{i(j)}^{(N)} P^{(N)} x_{i(j+1)}^{(N)} \dots x_{i(k)}^{(N)}) \\
&\quad = \lim N^{-(p+1)} \sum_{q=1}^{k+1} \binom{N}{q} \\
&\qquad \cdot \sum_{\varphi} \alpha_Q^{\otimes q}(x_{i(1),\varphi(1)}^{(q)} \dots x_{i(j),\varphi(j)}^{(q)} P_{\varphi(\Delta)}^{(q)} x_{i(j+1),\varphi(j+1)}^{(q)} \dots x_{i(k),\varphi(k)}^{(q)})
\end{aligned}$$

where the last sum runs over all φ from $\{1, \dots, j, \Delta, j+1, \dots, k\}$ onto $\{1, \dots, q\}$, and Δ has been inserted to take care of P.

The term corresponding to φ vanishes unless $\varphi^{-1}(i)$ contains $\geqq s$ elements or Δ for $1 \leqq i \leqq p$. Hence as $N \to \infty$ only those φ survive for which $\varphi^{-1}(i) \subset \{1, \dots, j, j+1, \dots, k\}$ and $\# \varphi^{-1}(i) = s$ or $\varphi^{-1}(i) = \{\Delta\}$.

The limit becomes

$$\begin{aligned}
&\frac{1}{(p+1)!} \sum_{\varphi} \alpha_Q^{\otimes (p+1)}(x_{i(1),\varphi(1)}^{(p+1)} \dots x_{i(j),\varphi(j)}^{(p+1)} P_{\varphi(\Delta)}^{(p+1)} x_{i(j+1),\varphi(j+1)}^{(p+1)} \dots x_{i(k),\varphi(k)}^{(k+1)}) \\
&\quad = \alpha_Q(P)\, \alpha_Q(x_{i(1)} \dots x_{i(j)} x_{i(j+1)} \dots x_{i(k)})
\end{aligned}$$

as s is even, P is an even polynomial and $P_{\varphi(\Delta)}$ commutes with $x_{i(l)}$. For $k=ps$ one has therefore

$$\alpha_Q(x_{i(1)} \dots x_{i(j)}(P - \alpha_Q(P))\, x_{i(j+1)} \dots x_{i(k)}) = 0.$$

For k not divisible by p this formula is trivial. Hence the theorem.

Discussion of the Results. The most interesting case is $s=2$. If $s \geqq 3$, $K=\mathbb{C}$ and $\mathfrak{A}$ is a *-algebra, $a_i^* = a_i$ and $\omega \geqq 0$, then $Q=0$ as has been pointed out in [2]. If $s=2$, then theorem 2 reads that α_Q vanishes on the ideal I generated by $x_i x_j + x_j x_i - (Q_{ij} + Q_{ji})$. So $\mathfrak{F}/I$ is the Clifford algebra generated by ξ_i, $i \in I$, and the quadratic form defined by the matrix $(Q_{ij} + Q_{ji})_{i,j}$. If $K=\mathbb{C}$ and $\mathfrak{F}$ is considered as a *-algebra as in [2], then the connection with the quantum mechanics of fermions is well-known. α_Q can then be interpreted as a "quasi-free state" and Theorem 1 and 2 together yield a central limit theorem for non-commuting quantities.

References

1. Chevalley, C.: The Construction and Study of Certain Important Algebras. Publ. Math. Soc. Japan, **1** (1955)
2. Giri, N., von Waldenfels, W.: An Algebraic Version of the Central Limit Theorem. Z. Wahrscheinlichkeitstheorie verw. Gebiete **42**, 129 (1978)
3. Hudson, R.L.: A Quantum Mechanical Central Limit Theorem for Anticommuting Observables. J. Appl. Probability **10**, 502–509 (1973)
4. Kac, V.G.: A sketch of Lie superalgebra theory. Commun. Math. Phys. **53**, 31–64 (1977)

Received April 29, 1977