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An Algebraic Version of the Central Limit Theorem

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Summary. A non-commutative analogue of the central limit theorem and the weak law of large numbers has been derived, the analogues of integrable functions being non-commutative polynomials. Without the assumption of positivity higher central limit theorems hold which have no analogy in the classical probabilistic case. The treatment includes this classical case and the convergence to so-called "quasi-free states" in the quantum mechanics of bosons [3, 4].

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This paper took its origin from the work of Hepp and Lieb [1] on the laser. There a central limit theorem for stochastic quantities in quantum physics has been derived. The subject of this paper is to generalize Hepp's and Lieb's approach and to bring it into a more abstract frame. The paper covers the algebraic but not the by far more difficult analytical content of Cushen and Hudson's work [2] on the quantum mechanical central limit theorem.

In order to perform the transition to non-commutativity one considers measures not as functionals on continuous functions or as set functions but as functionals on polynomials. Assume a probability measure μ on $\mathbb{R}^d$ such that all moments exist. Then μ defines linear a functional $\hat{\mu}$ on the algebra $\mathfrak{P}$ of all polynomials in d indeterminates by

$$\hat{\mu}(P) = \int \mu(dx)\, P(x).$$

It is well known that in general μ is not completely determined by $\hat{\mu}$. But this does not matter to us here. We want now to formulate the central limit theorem in an algebraic way for functionals on polynomials.

In the central limit theorem one considers usually a product space, e.g. $(\mathbb{R}^d)^N$ for large N, the product measure $\mu^{\otimes N}(dx^1, \ldots, dx^N) = \mu(dx^1) \ldots \mu(dx^N)$ on $(\mathbb{R}^d)^N$. One assumes that μ is a probability measure on $\mathbb{R}^d$ such that the first moments

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$\int x_i \mu(dx) = 0$ for $i = 1, \ldots, d$, $x = (x_1, \ldots, x_d)$. One is interested in the behaviour of functions of

$$x^{(N)} = x^1 + \cdots + x^N$$

where $x^j = (x_1^j, \ldots, x_d^j)$ is the coordinate in the j-th factor in $(\mathbb{R}^d)^N$. The central limit theorem asserts that for any suitable $f: \mathbb{R}^d \to \mathbb{R}$,

$$\int d\mu^{\otimes N} f(x^{(N)}/\sqrt{N}) \to \int g_Q(dx) f(x) \qquad (N \to \infty)$$

where $g_Q(dx)$ is the Gaussian measure on $\mathbb{R}^d$ with the covariance matrix Q,

$$Q_{ik} = \int \mu(dx)\, x_i x_k.$$

Let us formulate this theorem in an algebraic way. We consider the tensor product $\mathfrak{P}^{\otimes N}$ and the functional $\hat{\mu}^{\otimes N}$ on it

$$\hat{\mu}^{\otimes N}(P_1 \otimes \cdots \otimes P_N) = \hat{\mu}(P_1) \ldots \hat{\mu}(P_N).$$

Set

$$x_i^{(N)} = x_i \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes x_i.$$

Then the central limit theorem induces

$$\hat{\mu}^{\otimes N}(P(x_1^{(N)}/\sqrt{N}, \ldots, x_d^{(N)}/\sqrt{N})) \to \hat{g}_Q(P)$$

for any polynomial $P \in \mathfrak{P}$. There $P(x_1^{(N)} \cdot N^{-\frac{1}{2}}, \ldots, x_d^{(N)} \cdot N^{-\frac{1}{2}})$ denotes the polynomial in $\mathfrak{P}^{\otimes N}$ which arises by replacing x_i by $x_i^{(N)} \cdot N^{-\frac{1}{2}}$, $i = 1, \ldots, d$. From this formulation one gets by easy transition to more general cases.

We consider an associative algebra $\mathfrak{A}$ with unity over a field K and a family $(a_i)_{i \in I}$ of elements of $\mathfrak{A}$ and a K-linear functional $\omega: \mathfrak{A} \to K$ with $\omega(1) = 1$. We consider $\mathfrak{A}^{\otimes N}$ with the usual multiplication and the functional $\omega^{\otimes N}: \mathfrak{A}^{\otimes N} \to K$ defined by $\omega^{\otimes N}(f_1 \otimes \cdots \otimes f_N) = \omega(f_1) \ldots \omega(f_N)$ and define

$$a_i^{(N)} = a_i \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes a_i \in \mathfrak{A}^{\otimes N}.$$

The easiest way to formulate non-commutative polynomials is to introduce free algebras. Let $\mathfrak{F}$ be the free algebra generated by x_i, $i \in I$ and let $P \in \mathfrak{F}$ and b_i, $i \in I$ be a family of some elements of an algebra. Then by $P(b_i)$ we understand the polynomial in the b_i which arises by replacing the x_i in P by the b_i.

Let Q be a function $I^s \to K$. Then a Gaussian functional γ_Q on $\mathfrak{F}$ order s with covariance matrix Q is defined by $\gamma_Q(1) = 1$,

$$\gamma_Q(x_{i(1)} \ldots x_{i(k)}) = 0, \qquad \text{if } k \text{ cannot be divided by } s,$$

$$\gamma_Q(x_{i(1)} \ldots x_{i(s)}) = Q_{i(1), \ldots, i(s)},$$

$$\gamma_Q(x_{i(1)} \ldots x_{i(ps)}) = \sum_{\{S_1, \ldots, S_p\} \in \mathscr{P}_s(1, \ldots, ps)} Q_{i(S_1)} \ldots Q_{i(S_p)}$$

where $\mathcal{P}_s$ is the set of all partitions $\{S_1, \ldots, S_p\}$ of $\{1, \ldots, ps\}$ with $\# S_k = s$ for $k = 1, \ldots, p$. If $S_k = \{j_1, \ldots, j_s\}$ and $j_1 < j_2 < \cdots < j_s$, then $Q_{i(S_k)} = Q_{i(j_1), \ldots, i(j_k)}$. From the monomials γ_Q is defined on $\mathfrak{F}$ by linear extension.

If $P, Q \in \mathfrak{F}$, we denote the commutator by $[P, Q] = PQ - QP$. The free Lie algebra $\mathfrak{F}\mathfrak{L}$ generated by x_i can be considered as a K-linear subspace of $\mathfrak{F}$. It is spanned by x_i, $[x_i, x_k]$, $[[x_i, x_k], x_j]$, $[x_i, [x_k, x_j]], \ldots$. We are now able to formulate our results.

Theorem 1. *Let* $\omega(a_{i(1)})$, $\omega(a_{i(1)} a_{i(2)})$, $\omega(a_{i(1)} \ldots a_{i(s-1)}) = 0$ *for all* $i(1), \ldots, i(s-1) \in I$ *and* $1 \leqq s < \infty$ *fixed. Then for* $N \to \infty$

$$\omega^{\otimes N}(P(a_i^{(N)} \cdot N^{-1/s})) \to \gamma_Q(P)$$

where γ_Q *is the Gaussian functional on* $\mathfrak{F}$ *of order* s *with the covariance matrix* Q, $Q(i(1), \ldots, i(s)) = \omega(x_{i(1)} \ldots x_{i(s)})$.

Theorem 2. *The functional* γ_Q *vanishes on the two-sided ideal generated by the elements of the form*

$$P - \gamma_Q(P) 1$$

where P *runs through all homogeneous polynomials of degree* s *in* $\mathfrak{F}\mathfrak{L}$.

Before proving the theorems we want to discuss them, especially the character of γ_Q.

1) $s=1$. This is the generalization of the weak law of large numbers. The matrix Q is one-dimensional and defined by $Q_i = \omega(x_i)$ and $\gamma_Q(x_{i(1)} \ldots x_{i(h)}) = Q_{i(1)} \ldots Q_{i(h)} = \omega(x_{i(1)}) \ldots \omega(x_{i(h)})$. So $\gamma_Q(P) = P(\omega(x_i))$.

2) $s=2$, $K = \mathbb{R}$, $\mathfrak{A}$ is commutative and Q is symmetric and positive. This is the classical case. By Theorem 2 one gets that $\mathfrak{F}$ might be divided without harm by the ideal generated by the commutators $x_i x_k - x_k x_i$ and this is the polynomial algebra $\mathfrak{P}$ in the commuting indeterminates, say again $x_1, \ldots, x_d$. One has by the definition above $\gamma_Q = \hat{g}_Q$, where g_Q is the centered Gaussian measure on $\mathbb{R}^d$ with covariance matrix Q.

3) $s=2$, $K = \mathbb{C}$, $I = \{1, \ldots, 2d\}$, Q is hermitian, $Q_{jk} = G_{jk} + i H_{jk}$. There G is a real symmetric and H a real skew-symmetric matrix. If H is non-degenerate then by a linear transformation of generators H gets the form

$$\frac{1}{2} \begin{pmatrix} 0 & 1 & & & 0 \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ 0 & & & -1 & 0 \end{pmatrix}.$$

By Theorem 2 one obtains that γ_Q vanishes on the ideal $\mathfrak{I}$ generated by

$$x_j x_k - x_k x_j - Q_{jk} + Q_{kj} = [x_j, x_k] - 2i H_{jk}.$$

Therefore $\mathfrak{F}/\mathfrak{I}$ may be interpreted as the algebra $\mathfrak{L}$ generated by $p_1, q_1, \ldots, p_d, q_d$ where all generators commute except that $[p_k, q_k] = i$ for $k = 1, \ldots, d$. These are the well-known canonical commutation relations of quantum mechanics. If the matrix Q is positive definite, then γ_Q can be identified with a so-called "quasi-free state" in quantum mechanics [2, 3].

4) $s \geqq 3$, $K=\mathbb{C}$ and $\mathfrak{A}$ is a *-algebra, $a_i^*=a_i$ and ω is $\geqq 0$, i.e. $\omega(f^*f)\geqq 0$ for $f\in\mathfrak{A}$. Then $\omega(a_i^2)=0$ and Schwarz's inequality implies that $\omega(fa_i)=0$ for all $f\in\mathfrak{A}$. Hence $Q=0$ and $\gamma_Q(1)=0$ and $\gamma_Q(M)=0$ for any non-constant monomial. So assuming positivity one does not get non-trivial results unless $s\leqq 2$.

We now want to prove the theorems.

Lemma 1. *Let $f_1,\dots,f_k\in\mathfrak{A}$, denote*

$$f_i^1=f_i\otimes 1\otimes\cdots\otimes 1,\dots,f_i^N=1\otimes\cdots\otimes 1\otimes f_i\in\mathfrak{A}^{\otimes N}$$

and

$$f_i^{(N)}=f_i^1+\cdots+f_i^N.$$

Then

$$\omega^{\otimes N}(f_1^{(N)}\dots f_k^{(N)})=\sum_{p=1}^{k} N_p \sum_{\{S_1,\dots,S_p\}\in\mathscr{P}(1,\dots,k)} \omega(f_{S_1})\dots\omega(f_{S_p}).$$

There

$$N_p=N(N-1)\dots(N-p+1)$$

and $\mathscr{P}(1,\dots,k)$ is the set of all partitions of $\{1,\dots,k\}$. If $\pi=\{S_1,\dots,S_p\}$ is a special partition and $S_j\in\pi$, $S_j=\{i_i,\dots,i_m\}$ with $i_1<\cdots<i_m$, then $f_{S_j}=f_{i_1}\dots f_{i_m}$ (remark the conservation of order).

Proof of Lemma 1. One has

$$\omega^{\otimes N}(f_1^{(N)}\dots f_k^{(N)})=\sum_{j(1),\dots,j(k)=1}^{N}\omega^{\otimes N}(f_1^{j(1)}\dots f_k^{j(k)}).$$

Consider one fixed function $j\colon\{1,\dots,k\}\to\{1,\dots,N\}$, $l\mapsto j(l)$ as occurring in the right sum and denote by $\pi_j=\{S_1,\dots,S_p\}$ the associated partition of $\{1,\dots,k\}$, i.e. the S_l are the sets where j is constant. Then

$$\omega^{\otimes N}(f_1^{j(1)}\dots f_k^{j(k)})=\omega(f_{S_1})\dots\omega(f_{S_p}).$$

One has still to calculate the number of j with the same $\pi_j=\pi=\{S_1,\dots,S_p\}$. Define $j_0\colon\{1,\dots,k\}\to\{1,\dots,p\}$ by $j_0(l)=r$ for $l\in S_r$. Any j with $\pi_j=\pi$ allows a unique decomposition $j=\alpha_j\circ j_0$ with an injective application $\alpha_j\colon\{1,\dots,p\}\to\{1,\dots,N\}$ and inversely to any such $\alpha\colon\{1,\dots,p\}\to\{1,\dots,N\}$ injective there belongs exactly one $j=\alpha\circ j_0$ with $\pi_j=\pi$. So the number of possible j is equal to the number of injections from $\{1,\dots,p\}\to\{1,\dots,N\}$, i.e. is equal to N_p.

Proof of Theorem 1. Let $M=x_{i(1)}\dots x_{i(k)}$ be a monomial. Then by Lemma 1

$$\omega^{\otimes N}(M(a_i^{(N)}\cdot N^{-1/s}))=\sum_p N^{-k/s}N_p\sum_{\{S_1,\dots,S_p\}}\omega(a_{i(S_1)})\dots\omega(a_{i(S_p)})$$

with $a_{i(S)} = a_{i(l_1)} \dots a_{i(l_m)}$ if $S = \{l_1, \dots, l_m\}$, $l_1 < \dots < l_m$. As $\omega(a_{i(S)}) = 0$ for $\# S < s$ only partitions into sets with $\geq s$ elements may be considered in the right sum, then $ps \leqq k$, $p \leqq \frac{k}{s}$ and $N^{-k/s} N_p \to 0$ for $N \to \infty$ unless $k = ps$. So for $N \to \infty$ the expression vanishes unless k is a multiple of s and it reduces to

$$\sum_{\{S_1, \dots, S_p\},\ \# S_l = s} \omega(a_{i(S_1)}) \dots \omega(a_{i(S_p)}).$$

Corollary of Theorem 1. *If $\mathfrak{A} = \mathfrak{F}$ the free algebra and $Q: I^s \to K$ an application then*

$$\gamma_Q^{\otimes N}(P(x_i^{(N)} \cdot N^{-1/s})) \to \gamma_Q(P)$$

for all $P \in \mathfrak{F}$.

Lemma 2. *With the notations of Lemma 1, if $P \in \mathfrak{F}\mathfrak{L}$, then*

$$P(x_i^{(N)}) = P^{(N)}.$$

Proof of Lemma 2. It is sufficient to prove the lemma for homogeneous polynomials. For degree 0 and 1 it is trivial. A homogeneous polynomial in $\mathfrak{F}\mathfrak{L}$ of degree k is a linear combination of polynomials of the form

$$P = [R, x_j]$$

where R is a homogeneous polynomial of $\mathfrak{F}\mathfrak{L}$ of degree $k-1$. One proceeds by induction and assumes the lemma to be proven for degrees $< k$.

Then by induction and with the notations of Lemma 1

$$\begin{aligned} P(x_i^{(N)}) &= [R(x_i^{(N)}), x_j^{(N)}] = [R^{(N)}, x_j^{(N)}] \\ &= \sum_{p,q} [R^p, x_j^q] = \sum_p [R^p, x_j^p] = \sum_p [R, x_j]^p = [R, x_j]^{(N)} \end{aligned}$$

as R^p and x_j^q commute for $p \neq q$.

Proof of Theorem 2. Let P be a homogeneous polynomial of degree s in $\mathfrak{F}\mathfrak{L}$. We have to show that

$$\gamma_Q(x_{i_1} \dots x_{i_j}(P - \gamma_Q(P)\, 1)\, x_{i_{j+1}} \dots x_{i_k})$$

vanishes.

This expression vanishes anyhow unless k is a multiple of s. By the corollary and lemma 2 there is for $k = ps$

$$\begin{aligned} &\gamma_Q(x_{i_1} \dots x_{i_j} P x_{i_{j+1}} \dots x_{i_k}) \\ &\quad = \lim N^{-(k+s)/s} \gamma_Q^{\otimes N}(x_{i_1}^{(N)} \dots x_{i_j}^{(N)} P(x^{(N)}) x_{i_{j+1}}^{(N)} \dots x_{i_k}^{(N)}) \\ &\quad = \lim N^{-(p+1)} \gamma_Q^{\otimes N}(x_{i_1}^{(N)} \dots x_{i_j}^{(N)} P^{(N)} x_{i_{j+1}}^{(N)} \dots x_{i_k}^{(N)}) \\ &\quad = \lim N^{-(p+1)} \sum_{\{S_1, \dots, S_q\}} N_q \gamma_Q(x_{i(S_1)}) \dots \gamma_Q(x_{i(S_q)}). \end{aligned}$$

Here $\{S_1, \ldots, S_q\}$ runs over all partitions of $\{1, \ldots, j, \Delta, j+1, \ldots, k\}$. We set $x_{i_\Delta} = P$. Then $\gamma_Q(x_{i(S)}) = 0$ unless $\Delta \in S$ or $\Delta \notin S$ and $\# S \geqq s$. So $q \leqq k/s + 1 = p + 1$ and only those terms survive with $q = p + 1$. Hence the limit is equal to

$$\sum_{\{S_1, \ldots, S_{p+1}\}} \gamma_Q(x_{i(S_1)}) \ldots \gamma_Q(x_{i(S_{p+1})})$$

and the S_i have to be either $\{\Delta\}$ or an s-subset of $\{1, \ldots, j, j+1, \ldots, k\}$. Assuming $S_1 = \Delta$ one gets

$$\begin{aligned} &\gamma_Q(P) \sum_{\{S_2, \ldots, S_{p+1}\}} \gamma_Q(x_{i(S_2)}) \ldots \gamma_Q(x_{i(S_{p+1})}) \\ &\quad = \gamma_Q(P)\, \gamma_Q(x_{i_1} \ldots x_{i_j} x_{i_{j+1}} \ldots x_{i_k}) \end{aligned}$$

and hence the theorem.

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