Singular Translates of Measures on Linear Spaces*

By

R. M. DUDLEY

1. Introduction

Given a random element N with values in a linear space S , and a fixed element s of S , we shall give a simple sufficient condition in terms of second moments (Theorem 2 below) for mutual singularity of the distributions of N and $N + s$: that is, that for some disjoint sets A and B , N and $N + s$ should lie almost surely in A and B respectively. One consequence is a simpler proof of a recent result of V. N. SUDAKOV [16] for measures on Hilbert space (see § 5).

For random elements which are Gaussian with mean 0 in a natural sense, our condition is necessary and sufficient for singularity, and there is mutual absolute continuity (equivalence) if it fails (Theorem 3 below). This was essentially known; see the end of § 6.

If N is regarded as “noise”, then our condition can be interpreted as a way of choosing a “signal” s so that N and $N + s$ can almost certainly be distinguished: see FELDMAN [6].

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2. Random linear functionals

It is convenient to formulate our results in terms of “weak distributions” or “random linear functionals”.

Let F be the field of real or complex numbers, and let Q be any linear space over F . Let Q^a be the (algebraic) dual space of Q , i.e. the linear space of all F -linear mappings of Q into F . For any n -tuple $(q_1, \dots, q_n)$ of members of Q and Borel set B in n -space F^n , let

$$S(q_1, \dots, q_n, B) = \{q' \in Q^a : \langle q'(q_1), \dots, q'(q_n) \rangle \in B\}.$$

Let $\mathfrak{L}_Q$ be the σ -algebra of subsets of Q^a generated by all such sets.

Definition. A random linear functional (r.l.f.) on Q is an assignment to each n -tuple $(q_1, \dots, q_n)$ of members of Q a Borel probability measure $\mu(q_1, \dots, q_n)$ () on F^n , such that for any $q_i, r_j \in Q$ and Borel sets $B \subset F^n$ and $C \subset F^m$,

$$S(q_1, \dots, q_n, B) = S(r_1, \dots, r_m, C)$$

implies

$$\mu(q_1, \dots, q_n)(B) = \mu(r_1, \dots, r_m)(C).$$

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Note that the condition $S(q_1, \dots, q_n, B) = S(r_1, \dots, r_m, C)$ amounts to a linear relation or set of relations between (some of) the q_i and r_j , with B and C being related in the same way. Thus the involvement of Q^a could be avoided. However, for our purposes the condition as given seems to be the simplest.

The following theorem, also (like Theorem 3) essentially known, justifies the name "random linear functional":

Theorem 1. *There is a 1-1 correspondence between the set of all r.l.f.'s $\mu(\cdot)(\cdot)$ on a linear space Q and all probability measures P on $\mathfrak{L}_Q$ in Q^a , given by*

$$(*) \quad P(S(q_1, \dots, q_n, B)) = \mu(q_1, \dots, q_n)(B).$$

Proof. First, if $\mu(\cdot)(\cdot)$ is an r.l.f., P is uniquely defined by $(*)$ on sets of the form $S(q_1, \dots, q_n, B)$. Let us show that P extends to a countably additive probability measure on $\mathfrak{L}_Q$. Let I be a (Hamel) basis of Q as a vector space over F . Then Q^a is realized as the set of all F -valued functions on I , i.e. the product space

$$\prod_{i \in I} F_i$$

where each F_i is isomorphic to F .

For each finite subset $J = (q_1, \dots, q_n)$ of I there is a natural projection p_J of Q^a onto $G_J = \prod_{i \in J} F_i$. For each such J , $P_J = P \circ p_J^{-1}$ is the countably additive Borel measure $\mu(q_1, \dots, q_n)$ on G_J . If K includes J and f_{KJ} is the natural projection of G_K onto G_J , then clearly $P_K \circ f_{KJ}^{-1} = P_J^*$.

Thus we have a family of probability spaces (G_J, P_J) with connecting mappings f_{KJ} satisfying all the hypotheses of BOCHNER's theorem (BOCHNER [2], Theorem 5.1.1 p. 120) on inverse limits of measure spaces: the mappings are continuous, the measures are regular Borel measures and we have "sequential maximality". Thus P extends to a countably additive measure on the generated σ -algebra in the inverse limit space, which is Q^a . Since any set $S(r_1, \dots, r_m, B)$ is of the form $S(q_1, \dots, q_n, C)$ for some finite subset $(q_1, \dots, q_n)$ of I , P is defined precisely on $\mathfrak{L}_Q$, independently of I , since a measure on a ring has at most one countably additive extension to the generated σ -ring.

Conversely, given a probability measure P on $\mathfrak{L}_Q$ in Q^a , $(*)$ defines a Borel probability measure $\mu(q_1, \dots, q_n)$ on F^n for each $q_1, \dots, q_n \in Q$, and the consistency condition is easy to verify, so that $\mu(\cdot)(\cdot)$ is a r.l.f., q.e.d.

If P and $\mu(\cdot)(\cdot)$ are related by $(*)$, we shall call P the *distribution* of $\mu(\cdot)(\cdot)$.

3. Quadratic r.l.f.'s: the main theorem

We call a r.l.f. $\mu(\cdot)(\cdot)$ "quadratic" if

$$\int |x|^2 \mu(q)(dx) < +\infty$$

for each $q \in Q$. If $\mu(\cdot)(\cdot)$ is quadratic, let

$$m(q) = \int_F x \mu(q)(dx) = \int_{Q^a} q'(q) dP(q').$$

* The proof can be completed at this point by applying KOLMOGOROV's theorem on product space measures (see M. LOÈVE, Probability Theory, second edition (Princeton, Van Nostrand, 1960) 4.3A, p. 93).

Then clearly $m \in Q^a$. We call m the *mean* of the r.l.f. $\mu(\cdot)(\cdot)$. (Of course the mean is everywhere defined even for certain non-quadratic r.l.f.'s.) We also have a function B on $Q \times Q$ defined by

$$B(q, r) = \int_{Q^a} q'(q) \overline{q'(r)} dP(q') = \int_F \int_F x \bar{y} \mu(q, r) (dx, dy).$$

$B(q, r)$ is clearly linear in q , conjugate-linear in r , nonnegative definite and conjugate-symmetric. Thus we have an "inner product"

$$(q, r)_P = B(q, r)$$

where $(q, q)_P$ may be 0 for $q \neq 0$. It defines a topology on Q by the pseudo-norm $\|q\|_P = (q, q)_P^{1/2}$. We call B the *covariance* of the r.l.f.

If $z \in Q^a$ and P is a probability measure on $\mathfrak{L}_Q$, let P_z be the translate of P by z :

$$P_z(A) = P(A - z), \quad A \in \mathfrak{L}_Q.$$

Here is our condition for singularity of P and P_z :

Theorem 2. *If $\mu(\cdot)(\cdot)$ is a quadratic r.l.f. on a linear space Q with distribution P , $z \in Q^a$, and z is not continuous on Q for $\|\cdot\|_P$, then P and P_z are singular.*

Proof. Suppose z is discontinuous. We show first that there are elements f_n , $n = 1, 2, \dots$, of Q , orthonormal for $(\cdot, \cdot)_P$, such that $|z_n| = |z(f_n)| > 1$ for all n . Since z is discontinuous, there is an $f_1 \in Q$ such that

$$\|f_1\|_P = 1 \quad \text{and} \quad |z(f_1)| > 1.$$

Given $f_1, \dots, f_n$, let p be the orthogonal projection of Q onto the subspace G they span (p is a linear map of Q into itself since G is finite-dimensional). Suppose $p(q) = 0$ and $\|q\|_P = 1$ imply $|z(q)| \leq 1$. Then for any $h \in Q$,

$$\begin{aligned} |z(h)| &= |z(p(h)) + z(h - p(h))| \\ &\leq \left| \sum_{j=1}^n (h, f_j)_P z(f_j) \right| + \|h - p(h)\|_P \\ &\leq \|h\|_P \left(1 + \sum_{j=1}^n |z(f_j)| \right), \end{aligned}$$

contradicting the fact z is discontinuous. Thus the induction can continue.

Now for any n we have

$$\int \left| \sum_{j=1}^n \bar{z}_j q'(f_j) \right|^2 dP(q') = \sum_{j=1}^n |z_j|^2 = K_n \geq n,$$

so that

$$P(q' : \left| \sum_{j=1}^n \bar{z}_j q'(f_j) \right| \leq K_n^{3/4}) \geq 1 - K_n^{-1/2}.$$

On the other hand,

$$\begin{aligned} P_z(q' : \left| \sum_{j=1}^n \bar{z}_j q'(f_j) \right| \leq K_n^{3/4}) &= P(q' : \left| \sum_{j=1}^n \bar{z}_j (q'(f_j) + z_j) \right| \leq K_n^{3/4}) \\ &= P(q' : \left| \left(\sum_{j=1}^n \bar{z}_j q'(f_j) \right) + K_n \right| \leq K_n^{3/4}) \leq \\ &\leq P(q' : \left| \sum_{j=1}^n \bar{z}_j q'(f_j) \right| \geq K_n - K_n^{3/4}). \end{aligned}$$

For $n \geq 16$, $K_n - K_n^{3/4} \geq K_n/2$, so that the latter P -measure is at most $4/K_n$ by the same mean-square argument as above. Thus we have defined a sequence of sets A_n such that $P(A_n) \rightarrow 1$ and $P_z(A_n) \rightarrow 0$ as $n \rightarrow \infty$. Hence P and P_z are singular. (If $B_m = A_{2^m}$, and B is the set of points belonging to infinitely many B_m , then $P(B) = 1$ and $P_z(B) = 0$.) This completes the proof.

4. Distributions on topological linear spaces

If Q is a topological linear space, it is natural to consider r.l.f.'s $\mu(\cdot)(\cdot)$ which are *continuous* in the sense that $q \rightarrow \mu(q)$ is continuous with respect to the topology of weak convergence of measures on F . If Q is a countably-normed nuclear space, then the topological dual space Q' of Q has outer measure 1 for the distribution P of $\mu(\cdot)(\cdot)$ (see MINLOS [11] or GELFAND and VILENKIN [7] for details, and KOLMOGOROV [10] for the shortest proof). If Q is a separable infinite-dimensional Hilbert space, then Q' need not have outer measure 1, e.g. for the "normal" Gaussian r.l.f. with mean 0 and covariance equal to the given inner product.

In any case, Theorem 2 implies that if a quadratic r.l.f. with distribution P , on a topological linear space Q , has a continuous covariance, P and P_z are singular for z not in Q' .

On certain spaces where category arguments are available, a continuous quadratic r.l.f. has a continuous covariance: see e.g. DUDLEY [4] § 4 Lemma 2. However, for other spaces, e.g. an inner product space of countably infinite Hamel dimension, this is not true.

Likewise, for suitable topological linear spaces the set of members of Q^a which are Borel measurable as functions on Q is exactly Q' . A proof for Banach spaces was given by BANACH [1], Chap. V, § 1 Thme. 1 p. 78.

Now suppose μ is a Borel probability measure on a topological linear space S , and let Q_μ be the set of elements q of S' for which

$$\int |q(s)|^2 d\mu(s) < +\infty.$$

If S is a normed space, we can let

$$\nu(A) = \int_A e^{-\|x\|} d\mu(x) / \int_S e^{-\|x\|} d\mu(x)$$

to obtain a Borel probability measure ν such that μ_z and ν_z are equivalent for all z in S , and such that Q_ν is all of S' .

In any case, Q_μ is a linear subspace of S' , and μ defines a quadratic r.l.f. on Q_μ by

$$\mu(q_1, \dots, q_n)(B) = \mu(s : (q_1(s), \dots, q_n(s)) \in B).$$

We now have a

Corollary. *If μ is a Borel probability measure on a topological linear space S , $z \in S$, and z is discontinuous on Q_μ for $(\cdot, \cdot)_\mu$, then μ and μ_z are singular.*

Proof. S is naturally a subspace of the dual space $(Q_\mu)^a$, and we have by Theorem 2 that there is a set B in the σ -algebra $\mathfrak{L}_{Q_\mu}$ such that $P(B) = 1$ and $P_z(B) = 0$ where P is the distribution of the r.l.f. on Q_μ defined by μ . But

$B \cap S$ is a Borel set in S , S has P -outer measure 1 and P reduced to S is μ , so

$$\begin{aligned}\mu(B \cap S) &= P(B) = 1, \quad \mu_z(B \cap S) = 1 - \mu_z(S \sim (B \cap S)) \\ &= 1 - P_z((Q_\mu)^a \sim B) = 0,\end{aligned}$$

q.e.d.

If S is a complete separable metric linear space, and P is a Borel probability measure on S , then P is concentrated in a countable union M of compact sets, and obviously P and P_z are singular for z not in $M - M$. If S is infinite dimensional, then $M - M$ is of first category and so has dense complement. For further information of this type see SUDAKOV [14] and GIRSANOV and MITYAGIN [8].

5. Hilbert spaces

Let H be a separable Hilbert space with inner product $(\cdot, \cdot)$ and μ a Borel probability measure on H . As indicated in § 4, we may assume $s \rightarrow (s, s)$ is integrable for μ , so that Q_μ is the whole topological dual space H' of H , replacing μ by an equivalent measure if necessary. As usual we identify H' with H by

$$f \leftrightarrow f' : (x, f) = f'(x), \quad f \in H, \quad f' \in H'.$$

Now, $(\cdot, \cdot)_\mu$ as defined in § 4 is a nonnegative definite, continuous, conjugate-symmetric sesquilinear form on H , so we have

$$(x, y)_\mu = (Ax, Ay)$$

for some nonnegative, bounded, symmetric (self-adjoint) operator A on H . If $\{f_n\}$ is an orthonormal basis of H , then

$$\sum_{n=1}^{\infty} (Af_n, Af_n) = \int_H \sum_n |(f_n, s)|^2 d\mu(s) = \int_H (s, s) d\mu(s) < \infty.$$

Thus A is of Hilbert-Schmidt type, and there exists a complete orthonormal set $\{g_n\}$ such that $A(g_n) = \lambda_n g_n$, $n = 1, 2, \dots$, where

$$\lambda_n \geq 0, \quad \sum_{n=1}^{\infty} \lambda_n^2 < \infty.$$

According to the Corollary in § 4, μ and μ_z are singular if the linear functional $s \rightarrow (s, z)$ is not continuous for $(\cdot, \cdot)_\mu$. But we have

Lemma 1. *If A is a bounded operator on a Hilbert space H then $|(s, z)| \leq K \|As\|$ for some $K > 0$ and all $s \in H$ if and only if z is in the range of the transpose A^* of A (i.e. the operator A^* with $(Ax, y) = (x, A^*y)$ for all $x, y \in H$).*

Proof. If $z = A^*w$ for some $w \in H$, then

$$|(s, z)| = |(s, A^*w)| = |(As, w)| \leq \|w\| \|As\|$$

for all s in H .

Conversely, if $|(s, z)| \leq K \|As\|$ for some $K > 0$ and all s , then if $As = 0$, $(s, z) = 0$. Thus for some linear functional ψ on the range of A , $(s, z) = \psi(As)$, with $|\psi(x)| \leq K \|x\|$ for all x in the range of A . Thus $\psi(x) = (x, w)$ for some $w \in H$, and $(s, z) = (As, w) = (s, A^*w)$ for all s , so $z = A^*w$, q.e.d.

In our case, A is symmetric, i.e. $A = A^*$. Since A is a compact operator, its range is included in a countable union of compact sets, so that we reobtain the result mentioned at the end of § 4.

SUDAKOV [15], [16] has proved for real Hilbert spaces that if C is a closed, convex, symmetric set such that μ and μ_z are equivalent for all z in C , then

$$C = G + W$$

where G is a finite-dimensional subspace and W is a "Schmidt set", i.e. W is the image of a bounded set for some Hilbert-Schmidt operator. We shall reobtain this result with the additional information that A can be taken as the Hilbert-Schmidt operator for W . To include the case of complex Hilbert spaces, we assume C is "circled", i.e. $x \in C$ and $|\lambda| = 1$, $\lambda \in F$, imply $\lambda x \in C$.

Lemma 2. *Let S and T be Banach spaces, A a compact operator from S to T , and C a closed, convex, circled subset of the range of A . Then for some bounded set $M \subset S$, $C = G + A(M)$ where G is a finite-dimensional subspace.*

Proof. We may assume that A is 1-1 and has dense range (passing to a quotient space or closure respectively if necessary) so that S and T are separable. Let G be the set of $x \in T$ such that $ax \in C$ for all $a > 0$. Then since C is convex and circled, G is a vector subspace of T , closed since C is closed. Thus since C is included in a countable union of compact sets, G is finite-dimensional by category.

If $f \in G$ and $x \in C$ then $(n-1)x/n + nf/n$ belongs to C for all n , so $x + f \in C$. Thus C is a union of cosets (translates) of G . Passing to the quotient space T/G , we may assume $G = \{0\}$, and wish to prove C is of the form $A(M)$, M bounded.

For $n = 1, 2, \dots$, let

$$C_n = \{x \in C : \|x\| \leq n\}, \quad \text{and} \quad \|x\|_n = \inf\{a > 0 : x \in aC_n\}.$$

Then $\|\cdot\|_n$ is a norm with unit ball C_n on the linear subspace

$$T_n = \bigcup_{a>0} aC_n.$$

Now suppose $\{x_r\}$ is a Cauchy sequence in T_n for $\|\cdot\|_n$. Then for some $m > 0$, $x_r \in mC_n$ for all r , and x_r converges for the original norm on T to some x , with $x \in mC_n$ since mC_n is closed. For any $\varepsilon > 0$ there is a k such that $x_r - x_s \in \varepsilon C_n$ for $r, s \geq k$, so $x_r - x \in \varepsilon C_n$ for $r \geq k$ since εC_n is closed, and

$$\lim_{r \rightarrow \infty} \|x_r - x\|_n = 0.$$

Thus $(T_n, \|\cdot\|_n)$ is a Banach space.

Now suppose $u_r \rightarrow u$ in S and $A(u_r) \rightarrow x$ in $(T_n, \|\cdot\|_n)$. Then $A(u_r) \rightarrow A(u)$ for the given norm on T , so $A(u) = x$ since $A(u_r) \rightarrow x$. Thus A^{-1} is a closed operator from all of $(T_n, \|\cdot\|_n)$ to S , thus bounded. Hence for each n , $A^{-1}(C_n)$ is bounded in S , and C_n is compact (since it is closed).

Now if C is not bounded, there are $y_m \in C$ with $\|y_m\| = m$, $m = 1, 2, \dots$. Then $y_m/m \in C_1$ for all m , so the sequence y_m/m has a subsequence convergent to some $y \in C_1$ with $\|y\| = 1$. Now for any $n > 0$, $ny = \lim_{m \rightarrow \infty} ny_m/m \in C$, taking

the limit along our subsequence, so $y \in G$, a contradiction since $y \neq 0$. Thus $C = C_n$ for some n and $A^{-1}(C)$ is bounded, q.e.d.

6. Gaussian distributions

The converse of Theorem 2 is false in general. For example, if Q is one-dimensional and P is an atomic measure concentrated in the point $v \neq 0$, then any $w \in Q'$ is continuous for the covariance but P and P_w are singular for $w \neq 0$. However, the converse holds for certain Gaussian r.l.f.'s (an r.l.f. $\mu(\cdot)(\cdot)$ is Gaussian if each $\mu(q_1, \dots, q_n)$ is a (possibly degenerate) Gaussian measure on n -space).

Theorem 3. *If P is the distribution on Q^a of a Gaussian r.l.f. with mean 0 on a linear space Q , then P and P_z are equivalent if z is continuous for the covariance B of the r.l.f.*

Proof. It suffices to show that P_z is absolutely continuous with respect to P . Suppose there is a set A in $\mathfrak{L}_Q$ such that $P(A) = 0$, $P_z(A) > 0$. Then there is a countable set $q_1, q_2, \dots$, of elements of Q such that A is in the σ -algebra generated by the sets of the form $S(q_n, C)$, $n = 1, 2, \dots$. We may assume that the q_n are linearly independent and then that they are orthogonal for B (by the Gram-Schmidt process).

Now if $B(q_j, q_j) = 0$ for some j , then $z(q_j) = 0$ by continuity and sets of the form $S(q_j, C)$ have measure 1 or 0 according as $0 \in C$ or not, both for P and P_z . Thus, replacing those of measure 1 by all of Q^a and those of measure 0 by the null set, we obtain a set A' in the σ -ring generated by the sets $S(q_n, C)$ for $n \neq j$ with $P_z(A') = P_z(A)$, $P(A') = 0$. Thus we may assume $B(q_j, q_j) > 0$ for all j and normalize to $B(q_j, q_j) = 1$.

Now $\{q_j\}$ is an orthonormal set for B and we have

$$\sum_{j=1}^{\infty} |z_j|^2 < \infty$$

where $z_j = z(q_j)$. From here on we are, in a sense, only proving the known fact that the normal distribution is quasi-invariant: see SEGAL [13], Theorem 3 p. 22.

Let f be the map of Q^a onto a countable product S of copies of F defined by $f(x) = \{x_j\}_{j=1}^{\infty}$ where $x_j = x(q_j)$. Then we obtain Borel measures $P' = P \circ f^{-1}$ and $P'_z = P_z \circ f^{-1}$ on S such that $P'(f(A)) = 0$, $P'_z(f(A)) > 0$. Let us show that actually P'_z is absolutely continuous with respect to P' .

Since our r.l.f. has mean 0 and the q_j are B -orthonormal, P' is a product of Gaussian measures on F with mean 0 and variance 1, and P'_z a product of Gaussian measures with mean z_j and variance 1. Thus a Radon-Nikodym derivative of P'_z with respect to P' is given by the infinite product

$$\prod_{j=1}^{\infty} \exp(-z_j x_j + z_j^2/2)$$

if it converges, i.e. if the sum

$$\sum_{j=1}^{\infty} z_j x_j - z_j^2/2$$

converges. But $z_j^2/2$ is a convergent series of constants and $\sum_{j=1}^{\infty} z_j x_j$ converges

almost everywhere with respect to P' since $z_j x_j$ are independent Gaussian random variables with mean 0 and variance z_j^2 . Thus P'_z is absolutely continuous with respect to P' , and we have reached a contradiction, completing the proof.

The assumption that the r.l.f. has mean 0 in Theorem 3 can be weakened to the assumption that the mean m is itself continuous for the covariance of the r.l.f. defined by P_{-m} (it is automatically continuous for that of P). If not, however, then P and P_{-m} are singular by Theorem 2.

Other forms of Theorem 3 have appeared in CHOVER and FELDMAN [3], FELDMAN [6], and GELFAND and VILENKIN [7], Chap. IV, § 5 Teorema 3.

7. Random processes

It is convenient for many purposes to consider not only ordinary stochastic processes $x(t, \omega)$ but r.l.f.'s on the space $\mathfrak{D}$ of complex-valued C^∞ functions with compact support on the real line. A process $x(t, \omega)$ whose sample functions (ω fixed) are almost all locally integrable defines such a r.l.f. with $\mu(f_1, \dots, f_n)$ being the joint distribution of

$$\int x(t, \omega) f_j(t) dt, \quad j = 1, \dots, n.$$

If an r.l.f. on $\mathfrak{D}$ is continuous for the usual topology of $\mathfrak{D}$ (SCHWARTZ [12]), we call it a *random distribution* (r.d.). It is known that the distribution P on $\mathfrak{D}^a$ of an r.d. gives outer measure 1 to the topological dual space $\mathfrak{D}'$ of $\mathfrak{D}$ (see GELFAND and VILENKIN [7], Glava IV, § 2, No. 4, Teorema 6). Thus P defines a Borel probability measure on $\mathfrak{D}'$ which we shall also call P .

For any quadratic r.d. the covariance $B(\cdot, \cdot)$ is a hypo-continuous bilinear function on $\mathfrak{D}$ (see DUDLEY [4], Lemma 2). Thus according to the "Theorem of kernels" (EHRENPREIS [5], GELFAND and VILENKIN [7], Chap. I, § 1, No. 3, Teorema 5, p. 32) there is a distribution B_2 on the plane such that

$$B(f, g) = B_2(f \otimes \bar{g})$$

where $(f \otimes \bar{g})(s, t) = f(s) \bar{g}(t)$. Here B_2 is conjugate-symmetric under interchange of s and t . We call B_2 the *covariance distribution* of the r.d.

Thus we can say that P and P_z are singular if z is discontinuous for the inner product on $\mathfrak{D}$ defined by B_2 . For a stochastic process $x(t, \omega)$ with locally integrable sample functions, B_2 is the usual covariance function

$$K(s, t) = E(x(s) \overline{x(t)}).$$

Given a non-quadratic r.d. with distribution P , we would like to replace P by an equivalent measure on $\mathfrak{D}'$ defining a quadratic r.d. However, this is not always possible: for example, let P give positive mass to the δ -distribution (unit mass at 0 in the real line) and all its derivatives in the sense of distributions. Then since there is no limit to the order of growth of the derivatives of a C^∞ function at 0, no measure equivalent to P yields a quadratic r.d.

On the other hand, if P is concentrated in a subspace of $\mathfrak{D}'$ whose relative topology is normizable, we can define an equivalent quadratic r.d. as in § 4.

8. Stationary processes

An r.d. is called *stationary* if $\mu(f_1, \dots, f_n) = \mu(g_1, \dots, g_n)$ whenever there is a real number b such that $f_i(x+b) = g_i(x)$, $i = 1, \dots, n$, $f_1, \dots, f_n \in \mathfrak{D}$. Such processes have been discussed by Ito [9] and DUDLEY [4].

For a quadratic stationary r.d. the covariance distribution B_2 is unchanged by the transformation $(s, t) \rightarrow (s + h, t + h)$ for any real h . Thus (see SCHWARTZ [12], tome I, Ch. II, § 6, Thme. VI) there is a distribution C on the real line such that for any $\psi \in \mathfrak{D}(R^2)$,

$$B_2(\psi) = C_{(x)} \left[\int_{-\infty}^{\infty} \psi(x + h, h) dh \right].$$

Thus

$$B(f, g) = C_{(x)} \left[\int_{-\infty}^{\infty} f(x + h) \bar{g}(h) dh \right].$$

The integral can be interpreted as a convolution $f * \check{g}$ where $\check{g}(h) = g(-h)$. It is clear that C is a "positive definite" distribution and hence is the (inverse) Fourier transform of a nonnegative Borel measure m with

$$m([-x, x]) \leq K(1 + x^2)^n$$

for some positive constants K and n (SCHWARTZ [12], tome II, Chap. VII, § 9, Théorème XVIII, p. 132). Thus we have

$$B(f, g) = \int_{-\infty}^{\infty} \tilde{f}(k) \overline{\tilde{g}(k)} dm(k)$$

where $\tilde{f}$ and $\tilde{g}$ are the Fourier transforms of f and g respectively. This form of B was also obtained by K. Ito [9] using another method.

Now suppose that P and P_z are not singular for some $z \in \mathfrak{D}'$. Then according to Theorem 2, $z \in \mathfrak{S}'(R)$ (z is a tempered distribution) and we have

$$z(\varphi) = \int \tilde{\varphi} \eta dm$$

for some function η with $\int |\eta|^2 dm < \infty$. (The fact that $z \in \mathfrak{S}'$ also follows from the fact that P is concentrated in $\mathfrak{S}'$ according to part of the main theorem of DUDLEY [4]). Thus $z = (\eta m)^\sim$ where ηm is the complex-valued Borel measure, finite on compact sets, defined by

$$(\eta m)(B) = \int_B \eta dm.$$

This shows that the spectrum of z is included in that of m , and also gives information on the local behavior of z according to the growth of m at ∞ . We will not try to give more precise information at present.

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Department of Mathematics
University of California
Berkeley 4, California (USA)

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