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On the Asymptotic Behaviour of $\sum f(n_k x)$

Applications

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1. § Introduction

Let $f(x)\,(-\infty<x<\infty)$ be a measurable function such that

$$f(x+1)=f(x), \qquad \int_0^1 f(x)\,dx=0. \tag{1.1}$$

In the first part of this paper (see [1]) we proved some almost sure invariance principles for the sequence $f(n_k x)$ provided that $\{n_k\}$ satisfies

$$n_{k+1}/n_k \geqq q>1 \qquad (k=1,2,\ldots). \tag{1.2}$$

These theorems show that the asymptotic behaviour of $\sum_{k=1}^{N} f(n_k x)$, $N\to\infty$ is the same as that of $\zeta(\tau_N)$, $N\to\infty$ where ζ is a Wiener-process and τ_N is a sequence of random variables which is closely related to the quantities

$$v_{M,N}^{i,k}=2^k \int_{i2^{-k}}^{(i+1)2^{-k}} \left(\sum_{j=M+1}^{M+N} f(n_j x)\right)^2 dx. \tag{1.3}$$

By investigating the behaviour of $v_{M,N}^{i,k}$ in typical cases, in this paper we give some applications of the above mentioned theorems. We shall consider three different applications (which we already mentioned in the Introduction of the first part of our paper): 1. We shall show that if $\{n_k\}$ satisfies a condition slightly stronger than the Hadamard gap condition then the quantities $v_{M,N}^{i,k}$ become asymptotically independent of i, k and thus τ_N become asymptotically constant. Hence in this case the behaviour of $\sum_{k=1}^{N} f(n_k x)$ reduces to that of sums of independent random variables with finite variances. The results obtained in this way simplify, unify and extend several results obtained previously in the literature. 2. We shall show that if $\{n_k\}$ satisfies (1.2) with a large q then $\{f(n_k x)\}$ "almost" satisfies the central limit theorem, the law of the iterated logarithm and the functional versions of these theorems. 3. We shall derive an a.s. invariance principle for the sequence $f(n_k x)$ in the classical case $f(x)=\cos 2\pi x$, assuming only (1.2).

As in [1], we shall assume for f the standard condition

$$|f(x)|\leqq M, \qquad \|f-s_n\|\leqq An^{-\alpha}(\alpha>0,\ n=1,2,\ldots) \tag{1.4}$$

where s_n denotes the n-th partial sum of the Fourier-series of f and $\| \ \|$ is the $L_2(0,1)$ norm. The second relation of (1.4) is equivalent to

$$\frac{1}{2}\sum_{k=n+1}^{\infty}(a_k^2+b_k^2)\leqq A^2 n^{-2\alpha} \tag{1.5}$$

where

$$f\sim a_0+\sum_{k=1}^{\infty}(a_k\cos 2\pi kx+b_k\sin 2\pi kx)$$

is the Fourier-expansion of f. Condition (1.4) is satisfied, e.g., if f is a Lip α function (see [19] p. 241, relation (3.3)) or it is of bounded variation. (In the latter case we have $a_k=O(1/k)$, $b_k=O(1/k)$ and thus (1.5) holds with $\alpha=1/2$.)

As in the first part of our paper, we shall not assume that n_k are integers. From paper [1] we shall use only the theorems and remarks of § 2 and Lemmas (3.3) and (3.6) of § 3. We shall use, without explanation, the notion of quasi-equivalence of sequences of r.v.-s (introduced in [1] § 2) and the symbols $\sim$ and $\asymp$ (see footnote 2 of [1]).

All the results of our paper are probabilistic statements concerning the sequence $f(n_k x)$. The underlying probability space for this sequence is $(\Omega_0, \mathscr{F}_0, P_0)$ where $\Omega_0=[0,1)$, $\mathscr{F}_0$ is the class of Lebesgue-measurable subsets of $[0,1)$ and P_0 is the Lebesgue measure.

2. § Two Preparatory Lemmas

Preparing the applications in §§ 3–5, in this section we prove two simple lemmas of standard character. Let $(\Omega, \mathscr{F}, P)$ be a probability space and let $\{X_n\}$ and $\{\tau_n\}$ be sequences of random variables on $(\Omega, \mathscr{F}, P)$ satisfying the following two conditions:

1. $\{X_n\}$ remains bounded with probability one (its bound may depend on ω).
2. The sequence $\{\tau_n\}$ is positive, strictly increasing and $\tau_n-\tau_{n-1}=O(1)$ a.s. as $n\to\infty$.

We inquire what conclusions can be drawn from the relation

$$X_1+\cdots+X_n=\zeta(\tau_n)+o(n^{1/2}) \quad \text{a.s. as} \quad n\to\infty \tag{2.1}$$

where ζ is a Wiener-process on $(\Omega, \mathscr{F}, P)$ and τ_n satisfies

$$(1-\varepsilon)\,b_n<\tau_n<(1+\varepsilon)\,b_n \quad \text{a.s. for} \quad n\geqq n_0(\omega) \tag{2.2}$$

with $0<\varepsilon<1/4$ and a positive, strictly increasing numerical sequence $b_n\asymp n$. It is fairly obvious that (2.1) and (2.2) imply certain forms of Donsker's invariance principle and Strassen's law of the iterated logarithm for the sequence $\{X_n\}$. Our purpose is to clarify what these "certain forms" are. The answer is given by Lemmas (2.1) and (2.2) below. Before, however, formulating these lemmas, we write (2.1) in a more convenient form. We state

Proposition. *Assume that* $\{X_n\}$ *and* $\{\tau_n\}$ *satisfy* 1., 2. *and* (2.2). *Then relation* (2.1) *is equivalent to*

$$S(t)=\zeta(t)+o(t^{1/2}) \quad \text{a.s. as } t\to\infty \tag{2.3}$$

where $S(t)$ *denotes the (random) function in* $[0,+\infty)$ *which takes the value* $S_n=\sum_{i=1}^{n} X_i$ *at the point* $t=\tau_n$ $(n=0,1,\ldots)$ *and linear in each interval* $[\tau_n,\tau_{n+1}]$ $(n=0,1,\ldots)$. (*We put* $\tau_0=0$.)

Proof. Evidently (2.3) implies (2.1) (take $t=\tau_n$ in (2.3) and use $\tau_n\asymp n$). Conversely, (2.1) implies (2.3) for the values $t=\tau_n$ $(n=0,1,\ldots)$ and it remains to show

$$\sup_{\tau_n\leq t\leq\tau_{n+1}} |S(t)-S(\tau_n)|=o(n^{1/2}) \quad \text{a.s. as } n\to\infty, \tag{2.4}$$

$$\sup_{\tau_n\leq t\leq\tau_{n+1}} |\zeta(t)-\zeta(\tau_n)|=o(n^{1/2}) \quad \text{a.s. as } n\to\infty. \tag{2.5}$$

The first of these relations is trivial from 1. (actually, the left hand side of (2.4) is equal to $|S_{n+1}-S_n|=|X_{n+1}|=O(1)$ a.s.). Relation (2.5) follows from Lemma (3.6) of [1] since $\tau_{n+1}-\tau_n=O(1)$ and $\tau_n=O(n)$.

We can now formulate our basic lemmas.

Lemma (2.1). *Let us assume that* $\{X_n\}$ *and* $\{\tau_n\}$ *satisfy* 1., 2. *and also* (2.1), (2.2). *Put* $S_n=\sum_{i=1}^{n} X_i$ $(S_0=0)$ *and define the random function* $\varphi_n(t)$ $(0\leq t\leq 1)$ *as follows:*

$$\varphi_n(t)=\begin{cases} S_k/\sqrt{b_n} & \text{for } t=b_k/b_n \quad (k=0,1,\ldots,n)^1 \\ \text{linear} & \text{for } t\in[b_k/b_n,\, b_{k+1}/b_n] \quad (k=0,\ldots,n-1). \end{cases} \tag{2.6}$$

Then we have

$$\overline{\lim_{n\to\infty}}\, \rho(\varphi_n,\zeta)\leq C_1\,\varepsilon^{1/4}$$

where $\zeta(t)$ $(0\leq t\leq 1)$ *is Wiener-process and* $\rho(\varphi_n,\zeta)$ *is the Prohorov distance of the processes* $\varphi_n(t)$ *and* $\zeta(t)$[2]. *The constant* C_1 *is absolute.*

To formulate Lemma (2.2) let K denote Strassen's set of functions:

$$K=\left\{x(t)\colon x(t) \text{ is absolutely continuous in } [0,1],\; x(0)=0 \text{ and } \int_0^1 \dot{x}(t)^2\,dt\leq 1\right\}. \tag{2.7}$$

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1 We put $b_0=0$.

2 For two processes $\eta_1(t)$ and $\eta_2(t)$ $(0\leq t\leq 1)$ with continuous paths the Prohorov distance $\rho(\eta_1,\eta_2)$ is defined as the infimum of those positive ε for which $P(\eta_1\in A)\leq P(\eta_2\in A_\varepsilon)+\varepsilon$ and $P(\eta_2\in A)\leq P(\eta_1\in A_\varepsilon)+\varepsilon$ for any Borel-set $A\subset C[0,1]$. A_ε denotes the neighbourhood of A of radius ε (in $C[0,1]$ metric).

Lemma (2.2). *Let us suppose that* $\{X_n\}$ *and* $\{\tau_n\}$ *satisfy* 1., 2. *and also* (2.1), (2.2). *Put* $S_n=\sum_{i=1}^{n} X_i$ $(S_0=0)$ *and define the random function* $\psi_n(t)$ $(0\leqq t\leqq 1)$ *as follows:*

$$\psi_n(t)=\begin{cases} S_k/(2b_n\log\log b_n)^{1/2} & \text{for } t=b_k/b_n \quad (k=0,1,\dots,n) \\ \text{linear} & \text{for } t\in[b_k/b_n,\, b_{k+1}/b_n] \quad (k=0,\dots,n-1). \end{cases} \tag{2.8}$$

Then we have

a) $\overline{\lim_{n\to\infty}}\, d(\psi_n, K)\leqq 28\sqrt{\varepsilon}$ a.s.[3]

b) *For every* $x(t)\in K$ *we have*

$\underline{\lim}_{n\to\infty}\, d(\psi_n, x)\leqq 28\sqrt{\varepsilon}$ a.s.

The proofs of these lemmas are simple routine. To obtain Lemma (2.1) we first establish the following statement:

Lemma (2.3). *Under the conditions of Lemma* (2.1) *we can find Wiener-processes* $v_n(t)$ $(n=1,2,\dots)$ *such that we have*

$$P\Big(\sup_{0\leqq t\leqq 1}|\varphi_n(t)-v_n(t)|\geqq\varepsilon^{1/4}\Big)\leqq C_2\varepsilon \quad (n\geqq n_1). \tag{2.9}$$

C_2 *is an absolute constant.*

Proof. By the Proposition we have

$$S(t)=\zeta(t)+o(t^{1/2}) \quad \text{a.s. as } t\to\infty \tag{2.10}$$

where $S(t)$ is the random function in $[0,+\infty)$ which takes the value S_n at the point $t=\tau_n$ $(n=0,1,\dots)$ and linear in the intervals $[\tau_n,\tau_{n+1}]$ $(n=0,1,\dots)$. Let us introduce the random functions

$$\begin{aligned} \xi_n(t)&=\begin{cases} S_k/\sqrt{b_n} & \text{for } t=\tau_k/\tau_n \quad (k=0,1,\dots,n) \\ \text{linear} & \text{for } t\in[\tau_k/\tau_n,\, \tau_{k+1}/\tau_n] \quad (k=0,\dots,n-1) \end{cases} \\ \theta_n(t)&=\zeta(t\tau_n)/\sqrt{b_n} \\ v_n(t)&=\zeta(tb_n)/\sqrt{b_n} \end{aligned} \tag{2.11}$$

(defined in $0\leqq t\leqq 1$, $0\leqq t<\infty$, $0\leqq t<\infty$, respectively). Since we have $\tau_n\uparrow\infty$ a.s., relations (2.10) and (2.2) imply

$$\sup_{0\leqq t\leqq\tau_n}|S(t)-\zeta(t)|=o(\tau_n^{1/2})=o(b_n^{1/2}) \quad \text{a.s.}$$

or, equivalently,

$$\sup_{0\leqq t\leqq 1}|\xi_n(t)-\theta_n(t)|=o(1) \quad \text{a.s. as } n\to\infty.$$

The latter relation also implies

$$\sup_{0\leqq t\leqq 1}|\xi_n(\lambda(t))-\theta_n(\lambda(t))|=o(1) \quad \text{a.s. as } n\to\infty \tag{2.12}$$

[3] d denotes the $C[0,1]$ metric.

for any function $\lambda(t)$ $(0 \leqq t \leqq 1)$ such that $0 \leqq \lambda(t) \leqq 1$. Let us choose $\lambda(t)=\lambda_n(t)$ where $\lambda_n(t)$ is the function which carries b_k/b_n into τ_k/τ_n for every $0 \leqq k \leqq n$ and linear in the subintervals $[b_k/b_n, b_{k+1}/b_n]$ $(0 \leqq k \leqq n-1)$. Then (2.12) becomes

$$\sup_{0 \leqq t \leqq 1} |\varphi_n(t) - \theta_n(\lambda_n(t))| = o(1) \quad \text{a.s. as } n \to \infty \tag{2.13}$$

and since $v_n(t)$ (defined in (2.11)) is a Wiener-process for any n, (2.6) will follow if we show that

$$P\Big(\sup_{0 \leqq t \leqq 1} |\theta_n(\lambda_n(t)) - v_n(t)| \geqq \tfrac{1}{2}\varepsilon^{1/4}\Big) \leqq C_3 \varepsilon \tag{2.14}$$

for sufficiently large n with an absolute constant C_3. Now, (2.2) easily implies $|\tau_k/\tau_n - b_k/b_n| \leqq 4\varepsilon$ for $n \geqq n_2(\omega)$ and $0 \leqq k \leqq n$ which shows that $|\lambda_n(t) - t| \leqq 4\varepsilon$ for $0 \leqq t \leqq 1$ and $n \geqq n_2(\omega)$. Hence we have by (2.2) and $\lambda_n(t) \leqq t + 4\varepsilon \leqq 2$

$$|\lambda_n(t)\tau_n/b_n - t| \leqq \lambda_n(t)|\tau_n/b_n - 1| + |\lambda_n(t) - t| \leqq 6\varepsilon \quad (n \geqq n_3(\omega)) \tag{2.15}$$

for $0 \leqq t \leqq 1$. Here $n_3(\omega)$ depends on ω but we can find a set B with $P(B) > 1 - \varepsilon$ such that for $\omega \in B$ relation (2.15) is satisfied for any $n \geqq n_4$ where n_4 does not depend on ω. Hence, using Lemma 1 of [9] with $T = 6\varepsilon$, $L = 3$, $\delta = \varepsilon$, $c = \frac{1}{2}\varepsilon^{1/4}$ we get (note that $\lambda_n(t)\tau_n/b_n \leqq t + 6\varepsilon \leqq 1 + 6/4 < 3$ for $n \geqq n_4$ if $\omega \in B$)

$$\begin{aligned} &P\Big(\sup_{0 \leqq t \leqq 1} |\theta_n(\lambda_n(t)) - v_n(t)| \geqq \tfrac{1}{2}\varepsilon^{1/4}\Big) \\ &\quad = P\Big(\sup_{0 \leqq t \leqq 1} |v_n(\lambda_n(t)\tau_n/b_n) - v_n(t)| \geqq \tfrac{1}{2}\varepsilon^{1/4}\Big) \\ &\quad \leqq \varepsilon + P\Big(\sup_{\substack{0 \leqq t_1 < t_2 \leqq 3 \\ |t_2 - t_1| \leqq 6\varepsilon}} |v_n(t_2) - v_n(t_1)| \geqq \tfrac{1}{2}\varepsilon^{1/4}\Big) \\ &\quad \leqq \varepsilon + C_4 \varepsilon^{-3/4} \exp(-1/64\varepsilon^{1/2}) \leqq C_5 \varepsilon \end{aligned}$$

for $n \geqq n_4$ with absolute constants C_4, C_5. (The last inequality follows from the fact that $\varepsilon^{-7/4} \exp(-1/64\varepsilon^{1/2})$ tends to 0 if $\varepsilon \to 0$.) Hence (2.14) is valid and the proof of Lemma (2.3) is completed.

Using Lemma (2.3) we get $P(\varphi_n \in A) \leqq P(v_n \in A_{\varepsilon^{1/4}}) + C_2\varepsilon$ and $P(v_n \in A) \leqq P(\varphi_n \in A_{\varepsilon^{1/4}}) + C_2\varepsilon$ for $n \geqq n_1$ and any Borel-set $A \subset C[0,1]$. Hence, for $n \geqq n_1$, the Prohorov distance of φ_n and v_n is at most $\max(\varepsilon^{1/4}, C_2\varepsilon) \leqq C_6 \varepsilon^{1/4}$. Since v_n is Wiener process for every n, Lemma (2.1) is proved.

To prove Lemma (2.2) we need the following

Lemma (2.4). *Let $\zeta(t)$ be a (separable) Wiener-process and let b_n be a positive numerical sequence such that $b_n \asymp n$. Put*

$$\zeta_n(t) = (2b_n \log\log b_n)^{-1/2} \zeta(b_n t) \quad (0 \leqq t \leqq 1). \tag{2.16}$$

Then, almost surely, the sequence $\zeta_n(t)$ is relatively compact in $C[0,1]$ and its derived set coincides with K. (Actually, $d(\zeta_n, K) \to 0$ a.s. as $n \to \infty$.)

For $b_n = n$, this is Strassen's celebrated theorem for the Wiener-process (see [13]). The proof in the general case is essentially the same.

Fix a $\delta > 0$. From Lemma (2.4) it follows that the relation

$$\sup_{\substack{0 \leqq t_1 < t_2 \leqq 3b_n \\ |t_2 - t_1| \leqq \delta b_n}} |\zeta(t_2) - \zeta(t_1)| \leqq (\sqrt{\delta/3} + 2\delta)\sqrt{6b_n \log\log 3b_n} \tag{2.17}$$

holds almost surely for sufficiently large n. Indeed, let us consider the function $\zeta_n(t)$ in (2.16) but replace b_n by $3b_n$. By Lemma (2.4) we have $d(\zeta_n, K)\leqq\delta$ almost surely for sufficiently large n. Since for any $x(t)\in K$ we have $|x(s_2)-x(s_1)|\leqq\sqrt{s_2-s_1}$ $(0\leqq s_1<s_2\leqq 1)$ we see that the relation

$$\sup_{\substack{0\leqq s_1<s_2\leqq 1\\ |s_2-s_1|\leqq\delta/3}} |\zeta_n(s_2)-\zeta_n(s_1)|\leqq\sqrt{\delta/3}+2\delta$$

holds almost surely for sufficiently large n and this is identical with (2.17).

Turning to the proof of Lemma (2.2), let us put

$$\theta_n^*(t)=(2\log\log b_n)^{-1/2}\,\theta_n(t), \qquad v_n^*(t)=(2\log\log b_n)^{-1/2}\,v_n(t)$$

(v_n^* is only a different notation for the function $\zeta_n(t)$ in (2.16).) Relation (2.13) evidently implies (using $\psi_n(t)=(2\log\log b_n)^{-1/2}\,\varphi_n(t)$)

$$\sup_{0\leqq t\leqq 1} |\psi_n(t)-\theta_n^*(\lambda_n(t))|=o(1) \qquad \text{a.s. as } n\to\infty. \tag{2.18}$$

As we remarked above, $|\lambda_n(t)\tau_n-tb_n|\leqq 6\varepsilon b_n$ and $\lambda_n(t)\tau_n\leqq 3b_n$ are valid for $0\leqq t\leqq 1$ and $n\geqq n_3(\omega)$ (see (2.15)), hence we have, using (2.17),

$$\begin{aligned}
&\sup_{0\leqq t\leqq 1} |\theta_n^*(\lambda_n(t))-v_n^*(t)|\\
&\quad=(2b_n\log\log b_n)^{-1/2}\sup_{0\leqq t\leqq 1}|\zeta(\lambda_n(t)\tau_n)-\zeta(tb_n)|\\
&\quad\leqq(2b_n\log\log b_n)^{-1/2}\sup_{\substack{0\leqq s_1<s_2\leqq 3b_n\\ |s_2-s_1|\leqq 6\varepsilon b_n}}|\zeta(s_2)-\zeta(s_1)|\leqq 2(\sqrt{2\varepsilon}+12\varepsilon)\leqq 28\sqrt{\varepsilon}
\end{aligned} \tag{2.19}$$

for $n\geqq n_5(\omega)$. By (2.18) and (2.19) we have

$$\overline{\lim_{n\to\infty}}\sup_{0\leqq t\leqq 1}|\psi_n(t)-v_n^*(t)|\leqq 28\sqrt{\varepsilon} \qquad \text{a.s.}$$

which evidently implies Lemma (2.2) since, by Lemma (2.4), $v_n^*(t)$ has K as its derived set and $d(v_n^*, K)\to 0$ a.s.

3. § The Asymptotically Independent Case

Let us say that a sequence $n_1<n_2<\cdots$ of positive numbers belongs to class Λ^* if $n_k\to\infty$ and, for every $m\geqq 1$, the set-theoretic union of the sequences $\{n_k\}$, $\{2n_k\},\ldots,\{mn_k\}$ (considered as a new sequence) satisfies the Hadamard gap condition. The purpose of the present section is to investigate the properties of the sequence $f(n_kx)$ provided that $\{n_k\}\in\Lambda^*$. It will turn out that in this case the random variables τ_N occuring in Theorem 2 of [1] become asymptotically constant and thus the asymptotic behaviour of $\sum_{k=1}^{N} f(n_kx)$ as $N\to\infty$ is the same as that of $\zeta(b_N)$ with a certain numerical sequence $b_N\asymp N$. Some typical corollaries of this fact are Donsker's invariance principle, Strassen's law of the iterated logarithm, Kolmogorov-Erdös-Petrovski type upper and lower class tests etc. for the sequence $f(n_kx)$. These results unify and extend several limit theorems obtained earlier in the literature by different methods. For instance, Corollary 2 to Theorem (3.1), if specialized to the ordinary law of the iterated logarithm and integral n_k, implies the theorem of [17], it yields some laws of the iterated logarithm similar to those stated in [6], Chapter 2, § 4 without proof etc. For non-integral n_k even the central limit theorems implied by our results seem to be new. We get, e.g., the interesting result that the sequence $f(q^k x)$ obeys the central limit theorem (with respect to the probability space $(\Omega_0, \mathscr{F}_0, P_0)$) for any real $q>1$. (For a related result see [15].)

The class Λ^* was introduced (for integral n_k) by Gapoškin. The following lemma is due also to him:

Lemma (3.1). *The sequence $\{n_k\}$ of positive numbers belongs to class Λ^* if and only if there exist no subsequences n_{k_i}, n_{s_i} and rational number $r \neq 0$ such that*

$$\lim_{i\to\infty} \frac{n_{k_i}}{n_{s_i}} = r, \qquad \frac{n_{k_i}}{n_{s_i}} \neq r \qquad (i=1,2,\ldots). \tag{3.1}$$

Proof. Let us suppose that $\{n_k\} \notin \Lambda^*$. Then there exists an integer $m \geqq 1$, subsequences n_{j_i}, n_{l_i} and integers $1 \leqq p_i \leqq m$, $1 \leqq q_i \leqq m$ $(i=1,2,\ldots)$ such that

$$\lim_{i\to\infty} \frac{p_i n_{j_i}}{q_i n_{l_i}} = 1, \qquad \frac{p_i n_{j_i}}{q_i n_{l_i}} > 1 \qquad (i=1,2,\ldots). \tag{3.2}$$

Let us choose a sequence $i_1 < i_2 < \cdots < i_k < \cdots$ such that $p_{i_1} = p_{i_2} = \cdots = p$, $q_{i_1} = q_{i_2} = \cdots = q$. Then by (3.2) we have

$$\lim_{k\to\infty} \frac{n_{j_{i_k}}}{n_{l_{i_k}}} = \frac{q}{p}, \qquad \frac{n_{j_{i_k}}}{n_{l_{i_k}}} > \frac{q}{p} \qquad (k=1,2,\ldots)$$

i.e. (3.1) holds with $r = q/p$. Conversely, if (3.1) holds with $r = q/p$ then we have

$$\lim_{i\to\infty} \frac{p n_{k_i}}{q n_{s_i}} = 1, \qquad \frac{p n_{k_i}}{q n_{s_i}} \neq 1 \qquad (i=1,2,\ldots)$$

which shows that the set-theoretic union of the sequences $\{p n_k\}$ and $\{q n_k\}$ does not satisfy the Hadamard gap condition. Hence we have $\{n_k\} \notin \Lambda^*$.

Lemma (3.1) shows that the sequence $\{n_k\}$ belongs to class Λ^* in each of the following cases:

1. $n_k = q^k$ ($q>1$ is arbitrary real number).
2. n_{k+1}/n_k is integer for any $k \geqq 1$.
3. $\lim_{k\to\infty} n_{k+1}/n_k = \infty$.
4. $\lim_{k\to\infty} n_{k+1}/n_k = \alpha$ where α^r is irrational for $r=1,2,\ldots$

We formulate now the main result of this section:

Theorem (3.1). *Let us assume that*

a) *$f(x)$ satisfies* (1.1) *and* (1.4).

b) *The sequence $\{n_k\}$ of positive numbers belongs to class Λ^*.*

c) *There exists a positive constant C_1 such that*

$$\int_0^1 \left(\sum_{j=M+1}^{M+N} f(n_j x) \right)^2 dx \geqq C_1 N \qquad \text{for } M \geqq 0, \quad N \geqq N_0.$$

Then there exists a probability space $(\Omega, \mathscr{F}, P)$ *and a sequence* $X_1, X_2, \ldots$ *of random variables (defined on* $(\Omega, \mathscr{F}, P)$*) such that the sequences* $\{f(n_k x)\}$ *and* $\{X_k\}$ *are quasi-equivalent and*

$$X_1 + \cdots + X_n = \zeta(\tau_n) + o(n^{1/2-\eta}) \quad \text{a.s. as } n \to \infty \tag{3.3}$$

where $\eta > 0$ *is an absolute constant,* ζ *is a Wiener-process on* $(\Omega, \mathscr{F}, P)$ *and* τ_n *is a strictly increasing sequence of random variables (also on* $(\Omega, \mathscr{F}, P)$*) such that* $\tau_n - \tau_{n-1} = O(1)$ a.s. *as* $n \to \infty$ *and*

$$\lim_{n\to\infty} \tau_n / b_n = 1 \quad \text{a.s.}$$

with a strictly increasing positive numerical sequence $b_n \asymp n$.

Corollary 1. *Let us assume that the conditions of Theorem* (3.1) *are satisfied and put* $S_N = \sum_{k=1}^{N} f(n_k x)$. *Then there exists a strictly increasing positive numerical sequence* $b_n \asymp n$ *such that*

$$\varphi_n \Rightarrow \zeta \tag{3.4}$$

where ζ *is the Wiener-process and* φ_n *is the random function defined by* (2.6).[4]

Corollary 2. *Let us assume that the conditions of Theorem* (3.1) *are satisfied and put* $S_N = \sum_{k=1}^{N} f(n_k x)$. *Then there exists a strictly increasing positive numerical sequence* $b_n \asymp n$ *such that if* $\psi_n(t)$ *denotes the random function defined by* (2.8) *then* $\psi_n(t)$ *is relatively compact in* $C[0,1]$ *and its derived set coincides with* K *(defined by* (2.7)*) with probability one.*

In particular,

$$\overline{\lim_{N\to\infty}} (2b_N \log\log b_N)^{-1/2} S_N = 1 \quad \text{a.e.}$$

Corollaries 1 and 2 establish Donsker's invariance principle and Strassen's functional version of the law of the iterated logarithm for the sequence $f(n_k x)$.

Remark 1. Condition b) of Theorem (3.1) requires that, for any $m \geqq 1$, the set-theoretic union of the sequences $\{n_k\}, \{2n_k\}, \ldots, \{mn_k\}$ satisfies the Hadamard gap condition. If $f(x)$ happens to be a trigonometric polynomial of order L:

$$f(x) = \sum_{k=1}^{L} (a_k \cos 2\pi k x + b_k \sin 2\pi k x)$$

then it suffices to require this condition only for $m = L$.

Remark 2. For the sequences τ_n and b_n in Theorem (3.1) we stated $\lim_{n\to\infty} \tau_n/b_n = 1$ a.s. and $b_n \asymp n$. For certain special sequences $\{n_k\} \in \Lambda^*$ we can say more about τ_n and b_n. We mention two such cases.

[4] The symbol $\Rightarrow$ means weak convergence of continuous processes, for definition and properties see [4].

a) If $n_{k+1}/n_k \to \infty$ then we have $b_n \sim \| f \|^2 n$.

b) If $n_k = a^k$ ($a \geqq 2$ is integer) and $\sigma^2 \neq 0$ where

$$\sigma^2 = \| f \|^2 + 2 \sum_{k=1}^{\infty} \int_0^1 f(x) f(a^k x)\, dx\,[5] \tag{3.5}$$

then we have $b_n \sim \sigma^2 n$ and moreover, we can choose $\tau_n = b_n$. From this fact it follows (see [14], pp. 337–338) that in this case the sequence $f(n_k x)$ obeys the Kolmogorov-Erdös-Petrovski integral test in the following form: Let $\varphi(t) > 0$ be an increasing function. Then the set of those $x \in [0, 1)$ for which the relation

$$\sum_{k=1}^{N} f(n_k x) > b_N^{1/2} \varphi(b_N) \qquad \text{for infinitely many } N$$

holds, has Lebesgue-measure 0 or 1 according as the integral

$$\int_1^{\infty} t^{-1} \varphi(t) \exp\{-\tfrac{1}{2} \varphi^2(t)\}\, dt$$

converges or diverges.

To deduce Theorem (3.1) from Theorem 2 of [1] we prove the following

Lemma (3.2). *Let us suppose that the conditions of Theorem* (3.1) *are satisfied and put*

$$a_{M,N} = \int_0^1 \left(\sum_{j=M+1}^{M+N} f(n_j x) \right)^2 dx \qquad (M \geqq 0,\ N \geqq 1). \tag{3.6}$$

Then for any $0 < \varepsilon < 1$ *there exists an* $\omega_0 = \omega_0(\varepsilon)$ *such that for any* $M \geqq 0$, $k \geqq 1$, $0 \leqq i \leqq 2^k - 1$ *we have*

$$(1-\varepsilon)\, a_{M,N} < 2^k \int_{i 2^{-k}}^{(i+1) 2^{-k}} \left(\sum_{j=M+1}^{M+N} f(n_j x) \right)^2 dx < (1+\varepsilon)\, a_{M,N} \tag{3.7}$$

provided that $N \geqq N_0$, $n_M/N \cdot 2^k \geqq \omega_0$.

Remark. If f is a trigonometric polynomial of order L then the assumption $\{n_k\} \in \Lambda^*$ can be weakened, namely, in this case for the validity of Lemma (3.2) it is sufficient to assume that the set-theoretic union of the sequences $\{n_k\}$, $\{2n_k\}, \ldots,$ $\{Ln_k\}$ satisfies the Hadamard gap condition.

Proof of Lemma (3.2). By assumption c) of Theorem (3.1) we have $a_{M,N} \geqq C_1 N$ for $N \geqq N_0$, hence it suffices to show that for any $0 < \varepsilon < 1$ there exists an $\omega_1 = \omega_1(\varepsilon)$ such that the difference

$$\left| 2^k \int_{i 2^{-k}}^{(i+1) 2^{-k}} \left(\sum_{j=M+1}^{M+N} f(n_j x) \right)^2 dx - \int_0^1 \left(\sum_{j=M+1}^{M+N} f(n_j x) \right)^2 dx \right| \tag{3.8}$$

is at most εN provided that $N \geqq N_0$, $n_M/N \cdot 2^k \geqq \omega_1$. Let

$$f \sim \sum_{k=1}^{\infty} (a_k \cos 2\pi k x + b_k \sin 2\pi k x)$$

[5] The series is absolutely convergent by Lemma (5.1) of [1].

be the Fourier expansion of f and write $f=f_1+f_2$ where

$$f_1=\sum_{k=1}^{m}(a_k\cos 2\pi kx+b_k\sin 2\pi kx) \tag{3.9}$$

$$f_2=f-f_1$$

$m\geqq 1$ is an integer to be specified later. Let us also assume that $\|f_2\|\leqq 1$. We shall prove the following two statements:

1. Assume $n_M/2^k\geqq 1$. Then, replacing f by f_1, the expression (3.8) changes at most $C_2\|f_2\|^{1/2}N$ where C_2 is a positive constant depending on $f(x)$ and $\{n_k\}$.
2. We have

$$\left|2^k\int_{i2^{-k}}^{(i+1)2^{-k}}\left(\sum_{j=M+1}^{M+N}f_1(n_jx)\right)^2dx-\int_0^1\left(\sum_{j=M+1}^{M+N}f_1(n_jx)\right)^2dx\right|\leqq C_3\frac{2^k}{n_M}N^2 \tag{3.10}$$

where C_3 is a positive constant depending on $f(x)$, $\{n_k\}$ and m.

Choosing m in such a way that $\|f_2\|\leqq 1$ and $C_2\|f_2\|^{1/2}\leqq\varepsilon/2$ hold, the above two statements imply that the expression (3.8) is at most $\varepsilon N/2+C_4 2^k N^2/n_M$ (where C_4 depends on $f(x)$, $\{n_k\}$ and ε) whence the statement of the lemma follows.

To prove statement 1, let Q and Q' denote, respectively, the first term of the difference (3.8) and the expression obtained from Q by replacing f by f_1. The substitution $t=2^kx$ shows that

$$Q=\int_i^{i+1}\left(\sum_{j=M+1}^{M+N}f(m_jt)\right)^2dt,\qquad Q'=\int_i^{i+1}\left(\sum_{j=M+1}^{M+N}f_1(m_jt)\right)^2dt$$

where $m_j=n_j/2^k$ $(M+1\leqq j\leqq M+N)$. By assumption b) of Theorem (3.1) we have $n_{k+1}/n_k\geqq q$ $(k=1,2,\ldots)$ with a certain $q>1$ and thus, using Minkowski's inequality and Corollary 1 after Lemma (3.3) of [1] we get for $n_M/2^k\geqq 1$

$$|Q^{1/2}-Q'^{1/2}|\leqq\left(\int_i^{i+1}\left(\sum_{j=M+1}^{M+N}f_2(m_jt)\right)^2dt\right)^{1/2}\leqq C_5\|f_2\|^{1/2}N^{1/2} \tag{3.11}$$

where C_5 is a positive constant depending on $f(x)$ and $\{n_k\}$. (Observe that the Fourier-series of f_2 is

$$f_2\sim\sum_{k=1}^{\infty}(\tilde a_k\cos 2\pi kx+\tilde b_k\sin 2\pi kx)$$

where $\tilde a_k=\tilde b_k=0$ for $1\leqq k\leqq m$ and $\tilde a_k=a_k$, $\tilde b_k=b_k$ for $k>m$. This shows that f_2 also satisfies the second relation of (1.4) with the same A, α.) A further application of Corollary 1 after Lemma (3.3) of [1] yields for $n_M/2^k\geqq 1$ (note that $\|f_1\|\leqq\|f\|$ and f_1 also satisfies the second relation of (1.4) with the same A, α)

$$Q^{1/2}+Q'^{1/2}\leqq C_6N^{1/2} \tag{3.12}$$

where C_6 depends on $f(x)$ and $\{n_k\}$. Multiplying relations (3.11) and (3.12) together, we get

$$|Q-Q'|\leqq C_5C_6\|f_2\|^{1/2}N.$$

A similar argument shows that the second term of the difference (3.8) changes also at most $C_7 \|f_2\|^{1/2} N$ when we replace f by f_1. Hence statement 1. is proved.

To prove statement 2. let $n_1^{(m)} < n_2^{(m)} < \cdots < n_k^{(m)} < \cdots$ be the set-theoretic union of the sequences $\{n_k\}$, $\{2n_k\}$, ..., $\{mn_k\}$. By assumption b) of Theorem (3.1) the sequence $\{n_k^{(m)}\}$ satisfies the Hadamard gap condition i.e. there exists a $1 < q^* < 2$ (depending on m) such that

$$n_{k+1}^{(m)} / n_k^{(m)} \geqq q^* \qquad (k=1, 2, \ldots). \tag{3.13}$$

By (3.9) we have

$$f_1(n_j x) = \sum_{k=1}^{m} (a_k \cos 2\pi k n_j x + b_k \sin 2\pi k n_j x)$$

and thus

$$\sum_{j=M+1}^{M+N} f_1(n_j x) = \sum_{r=1}^{H} (c_r \cos 2\pi \lambda_r x + d_r \sin 2\pi \lambda_r x) \tag{3.14}$$

where $H \leqq mN$, every λ_r belongs to the sequence $\{n_k^{(m)}\}$ and, choosing the indices in such a way that $\lambda_1 < \lambda_2 < \cdots < \lambda_H$ holds, we have $\lambda_1 = n_{M+1}$. By (3.13) we have

$$\lambda_s - \lambda_r \geqq (q^*-1)\lambda_r \geqq (q^*-1)\lambda_1 > (q^*-1) n_M \qquad (1 \leqq r < s \leqq H). \tag{3.15}$$

It is also easy to see that

$$\bar{M} = \max\{|c_1|, |d_1|, \ldots, |c_H|, |d_H|\} \leqq C_8 \tag{3.16}$$

where C_8 is a constant depending on $f(x)$, $\{n_k\}$ and m. Indeed, the trigonometric sums $f_1(n_\nu x)$ and $f_1(n_\mu x)$ $(\nu < \mu)$ in the left hand side of (3.14) can overlap[6] only if $n_\mu \leqq m n_\nu$, i.e. overlapping is impossible if $\mu - \nu \geqq l$ where l is the smallest integer such that $(q^*)^l > m$. (Note that (3.13) implies $n_{k+1}/n_k \geqq q^*$ for $k=1, 2, \ldots$.) This remark shows that $\bar{M} \leqq l M^*$ where $M^* = \max\{|a_1|, |b_1|, \ldots, |a_m|, |b_m|\}$ and thus (3.16) is valid.

Squaring (3.14) and using well-known trigonometric identities we get

$$\left(\sum_{j=M+1}^{M+N} f_1(n_j x)\right)^2 = \tfrac{1}{2} \sum_{r=1}^{H} (c_r^2 + d_r^2) + I \tag{3.17}$$

where

$$I = \sum_i (e_i \cos 2\pi \rho_i x + f_i \sin 2\pi \rho_i x) \tag{3.18}$$

is a trigonometric polynomial such that $|e_i| \leqq \bar{M}^2$, $|f_i| \leqq \bar{M}^2$ and the ρ_i-s are of the form $\rho_i = \lambda_r + \lambda_s$ $(1 \leqq r \leqq s \leqq H)$ and $\rho_i = \lambda_s - \lambda_r$ $(1 \leqq r < s \leqq H)$. By (3.15) we have

$$\rho_i \geqq (q^*-1) n_M \qquad (i=1, 2, \ldots). \tag{3.19}$$

6 We say that two trigonometric sums overlap if they contain a sine or cosine term with the same frequency.

It is also evident that the sum in (3.18) contains at most $8H^2 \leqq 8m^2N^2$ terms. Let (α, β) be any interval. Using (3.16), (3.19), the facts

$$\left|\int_\alpha^\beta \cos \gamma x\, dx\right| \leqq 2/|\gamma|, \qquad \left|\int_\alpha^\beta \sin \gamma x\, dx\right| \leqq 2/|\gamma|$$

and the remarks above, we get

$$\left|\int_\alpha^\beta I\, dx\right| \leqq \sum_i \left(\frac{2\bar{M}^2}{2\pi\rho_i} + \frac{2\bar{M}^2}{2\pi\rho_i}\right) \leqq \frac{\bar{M}^2}{(q^*-1)\, n_M} \sum_i 1 \leqq \frac{\bar{M}^2}{(q^*-1)\, n_M} 8m^2N^2 \leqq C_9 \frac{N^2}{n_M} \tag{3.20}$$

where C_9 depends on $f(x)$, $\{n_k\}$ and m. Integrating (3.17) on $(i2^{-k}, (i+1)2^{-k})$ and $(0, 1)$ and using (3.20) we obtain

$$2^k \int_{i2^{-k}}^{(i+1)2^{-k}} \left(\sum_{j=M+1}^{M+N} f_1(n_j x)\right)^2 dx = \tfrac{1}{2} \sum_{r=1}^{H} (c_r^2 + d_r^2) + J,$$

$$\int_0^1 \left(\sum_{j=M+1}^{M+N} f_1(n_j x)\right)^2 dx = \tfrac{1}{2} \sum_{r=1}^{H} (c_r^2 + d_r^2) + J',$$

$$|J| \leqq C_9 \frac{N^2}{n_M} 2^k, \qquad |J'| \leqq C_9 \frac{N^2}{n_M}.$$

The latter relations evidently imply (3.10).

If $f(x)$ is a trigonometric polynomial of order L then choosing $m = L$ in the proof above we shall have $f_1 = f$, $f_2 = 0$. Hence the Remark after Lemma (3.2) follows immediately from statement 2. above. (Observe that in the proof of statement 2. we did not make use of the full strength of $\{n_k\} \in \Lambda^*$ but only the fact that the set-theoretic union of $\{n_k\}, \{2n_k\}, \ldots, \{mn_k\}$ (m is the number for which statement 2. is formulated) satisfies the Hadamard gap condition.)

For the numbers $a_{M,N}$ in (3.6) we have

$$C_1 N \leqq a_{M,N} \leqq C_2 N \qquad (M \geqq 0,\ N \geqq N_0) \tag{3.21}$$

by condition c) of Theorem (3.1) and Lemma (3.3) of [1] (C_1, C_2 are positive constants independent of M, N). Hence Theorem (3.1) follows immediately from Theorem 2 of [1] via Lemma (3.2) above (see Remark 2 after Theorem 3 in [1]). Observe also that the sequences $\{X_n\}$, $\{\tau_n\}$ occuring in Theorem (3.1) satisfy conditions 1. and 2. in § 2 (Condition 1. follows from the quasi-equivalence of $\{X_k\}$ and $\{f(n_k x)\}$ and the first relation of (1.4)). Thus Corollaries 1 and 2 follow from Theorem (3.1) by means of Lemmas (2.1) and (2.2). (Note that the weak convergence of continuous processes is equivalent to convergence in the Prohorov metric.)

It remains to prove Remarks 1 and 2 after Theorem (3.1). Remark 1 is evident by the Remark after Lemma (3.2). To get part a) of Remark 2 let us note that in the case $n_{k+1}/n_k \to \infty$ the numbers $a_{M,N}$ in (3.6) satisfy not only (3.21) but also the stronger relation $a_{M,N} \sim \|f\|^2 N$ as $N \to \infty$, uniformly in M (this follows from Corollary 2 to Lemma (3.3) in [1]). Hence relation $b_n \sim \|f\|^2 n$ follows from Remark 4 after Theorem 3 in [1]. To get part b) let us observe that for any $k \geqq 1$,

$0 \leq i \leq 2^k - 1, M \geq k, N \geq 1$ we have

$$2^k \int_{i2^{-k}}^{(i+1)2^{-k}} \left(\sum_{j=M+1}^{M+N} f(2^j x) \right)^2 dx = \int_i^{i+1} \left(\sum_{j=M+1}^{M+N} f(2^{j-k} t) \right)^2 dt$$

$$= \int_0^1 \left(\sum_{j=M+1}^{M+N} f(2^{j-k} t) \right)^2 dt = \int_0^1 \left(\sum_{r=1}^{N} f(2^r t) \right)^2 dt$$

by the periodicity of f and the stationarity of the sequence $f(2^n x)$. Hence, putting

$$a_{M,N} = a_N = \int_0^1 \left(\sum_{r=1}^{N} f(2^r t) \right)^2 dt$$

and noting that $a_N \sim \sigma^2 N$ where σ^2 is the number defined by (3.5) with $a=2$ (see e.g. [7], Lemma (4.1)), part b) of Remark 2 (in the case $a=2$) follows from Remark 5 after Theorem 3 in [1]. For $a>2$ the proof is similar (see Remark 6 after Theorem 3 in [1]).

4. § Some ε-Limit Theorems

It is well known that the lacunarity condition

$$n_{k+1}/n_k \geq q > 1 \qquad (k=1, 2, \ldots)$$

does not imply the central limit theorem and the law of the iterated logarithm for the sequence $f(n_k x)$ even if q is large and f satisfies strong smoothness conditions. This is shown, e.g., by the example of Erdös and Fortet (see [8]):

$$f(x) = \cos 2\pi x + \cos 2\pi m x, \qquad n_k = m^k - 1 \tag{4.1}$$

in which case we have

$$\lim_{N\to\infty} P\left(N^{-1/2} \sum_{k=1}^{N} f(n_k x) < t \right) = (2\pi)^{-1/2} \int_0^1 ds \int_{-\infty}^{t/\sqrt{2}\,|\cos(m-1)\pi s|} e^{-u^2/2} du^{7}$$

and

$$\limsup_{N\to\infty} (2N \log\log N)^{-1/2} \sum_{k=1}^{N} f(n_k x) = \sqrt{2} \cos(m-1)\pi x \qquad \text{a.e.}$$

Let us note, however, that for the function f in (4.1) the sequence $f(n_k x)$ satisfies both the central limit theorem and the law of the iterated logarithm provided that $\{n_k\}$ satisfies

$$n_{k+1}/n_k > 2m \qquad (k=1, 2, \ldots).$$

(For integral n_k this follows, e.g., from the results of [2, 10], the extension for non-integral n_k is also easy.) This gives us some hope that even if $f(n_k x)$ does not imitate the behaviour of independent random variables for a given f and q, the situation becomes better if we increase q (by keeping f fixed). In this section we shall see that this is really valid. As a matter of fact, it is not true that for any f (satisfying

7 $t/0$ denotes $+\infty$, 0 and $-\infty$ according as $t>0$, $t=0$ and $t<0$.

certain smoothness conditions) there exists a q_0 such that the sequence $f(n_k x)$ obeys the central limit theorem and the law of the iterated logarithm if $q \geqq q_0$. We shall show, however, that the "deviation" from central limit and iterated logarithm behaviour of the sequence $f(n_k x)$ tends to 0 if $q \to \infty$ (and f is being kept fixed). The same holds for Donsker's invariance principle and Strassen's version of the law of the iterated logarithm.

Theorem (4.1). *Let $f(x)$ satisfy* (1.1) *and* (1.4). *Then for any given $\varepsilon > 0$ there exists a $q_0 = q_0(\varepsilon, f)$ such that if $\{n_k\}$ satisfies* (1.2) *with $q \geqq q_0$ then the following two statements hold* $\left(\textit{we put } S_N = \sum_{k=1}^{N} f(n_k x)\right)$:

a) $\overline{\lim_{N \to \infty}} \sup_t |P(S_N / \sigma \sqrt{N} < t) - \Phi(t)| \leqq \varepsilon,$

b) $1 - \varepsilon \leqq \overline{\lim_{N \to \infty}} (2\sigma^2 N \log \log N)^{-1/2} S_N \leqq 1 + \varepsilon \quad$ a.e.

where $\sigma = \| f \| \neq 0$ ($\Phi(t)$ denotes the distribution function of the standard normal distribution).

The functional version of Theorem (4.1) can also be formulated. It is not evident, however, what to call the "functional version" of Theorem (4.1) i.e. how to define the notions "the sequence $f(n_k x)$ satisfies Donsker's invariance principle with accuracy ε" and "the sequence $f(n_k x)$ satisfies Strassen's law of the iterated logarithm with accuracy ε". The following definitions are quite natural but not the only possible ones.

Definition 1. Let $Y_1, Y_2, \ldots$ be a sequence of random variables, $S_n = \sum_{i=1}^{n} Y_i$ $(S_0 = 0)$ and define the random function $\varphi_n(t)$ $(0 \leqq t \leqq 1)$ as follows:

$$\varphi_n(t) = \begin{cases} S_k / \sqrt{n} & \text{for } t = k/n \quad (k = 0, 1, \ldots, n) \\ \text{linear} & \text{for } t \in [k/n, (k+1)/n] \quad (k = 0, 1, \ldots, n-1). \end{cases}$$

Let $\varepsilon > 0$ be fixed. We say that the sequence Y_k obeys Donsker's invariance principle with accuracy ε if

$$\overline{\lim_{n \to \infty}} \rho(\varphi_n, \zeta) \leqq \varepsilon$$

where $\zeta(t)$ $(0 \leqq t \leqq 1)$ is the Wiener-process and ρ is the Prohorov-distance[8].

Definition 2. Let $Y_1, Y_2, \ldots$ be a sequence of random variables, $S_n = \sum_{i=1}^{n} Y_i$ $(S_0 = 0)$ and define the random function $\psi_n(t)$ $(0 \leqq t \leqq 1)$ as follows:

$$\psi_n(t) = \begin{cases} S_k / (2n \log \log n)^{1/2} & \text{for } t = k/n \quad (k = 0, 1, \ldots, n) \\ \text{linear} & \text{for } t \in [k/n, (k+1)/n] \quad (k = 0, 1, \ldots, n-1). \end{cases}$$

[8] See footnote 2.

Let $\varepsilon > 0$ be fixed and let K denote the set of functions defined by (2.7). We say that the sequence Y_k obeys Strassen's law of the iterated logarithm with accuracy ε if

a) $\overline{\lim}_{n\to\infty} d(\psi_n, K) \leqq \varepsilon$ a.s.[9]

b) For any $x(t) \in K$ we have

$$\underline{\lim}_{n\to\infty} d(\psi_n, x) \leqq \varepsilon \quad \text{a.s.}$$

Theorem (4.2). *Let $f(x)$ satisfy* (1.1) *and* (1.4) *and assume, for simplicity, that $\|f\|=1$. Then for any given $\varepsilon>0$ there exists a $q_0 = q_0(\varepsilon, f)$ such that if $\{n_k\}$ satisfies* (1.2) *with $q \geqq q_0$ then the sequence $f(n_k x)$ obeys both Donsker's invariance principle and Strassen's law of the iterated logarithm with accuracy ε.*

Remark. The example of Erdös and Fortet mentioned above shows that q_0 in Theorems (4.1) and (4.2) depends on f strongly.

It is easy to see that Theorem (4.2) implies Theorem (4.1). On the other hand, Theorem (4.2) follows immediately (via Lemmas (2.1) and (2.2)) from the following general theorem which is the main result of this section:

Theorem (4.3). *Let $f(x)$ satisfy* (1.1) *and* (1.4) *and assume, for simplicity, that $\|f\|=1$. Then for any given $\varepsilon>0$ there exists a $q_0 = q_0(\varepsilon, f)$ such that if $\{n_k\}$ satisfies* (1.2) *with $q \geqq q_0$ then for the sequence $f(n_k x)$ we have the following result:*

There exists a new probability space $(\Omega, \mathscr{F}, P)$ and a sequence $X_1, X_2, \ldots$ of random variables (defined on $(\Omega, \mathscr{F}, P)$) such that the sequences $\{f(n_k x)\}$ and $\{X_k\}$ are quasi-equivalent and

$$X_1 + \cdots + X_n = \zeta(\tau_n) + o(n^{1/2-\eta}) \quad \text{a.s. } as\ n\to\infty$$

where $\eta>0$ is an absolute constant, ζ is a Wiener-process on $(\Omega, \mathscr{F}, P)$ and τ_n is a positive, strictly increasing sequence of random variables (also on $(\Omega, \mathscr{F}, P)$) such that $\tau_n - \tau_{n-1} = O(1)$ a.s. *as $n\to\infty$ and*

$$1-\varepsilon \leqq \liminf_{n\to\infty} \tau_n/n \leqq \limsup_{n\to\infty} \tau_n/n \leqq 1+\varepsilon \quad \text{a.s.}$$

Theorems (4.1) and (4.2) show that if $f(x)$ and $\{n_k\}$ satisfy (1.1), (1.4) and $n_{k+1}/n_k \to \infty$ then the sequence $f(n_k x)$ satisfies the central limit theorem, the law of the iterated logarithm and the functional versions of these theorems exactly (i.e. with $\varepsilon=0$). Similarly, in this case the conclusion of Theorem (4.3) holds with $\varepsilon=0$. These remarks can be obtained also from the results of the preceding section (see Remark 2 after Theorem (3.1)).

Proof of Theorem (4.3). The substitution $t=2^k x$ shows that

$$2^k \int_{i2^{-k}}^{(i+1)2^{-k}} \left(\sum_{j=M+1}^{M+N} f(n_j x) \right)^2 dx = \int_i^{i+1} \left(\sum_{j=M+1}^{M+N} f(m_j t) \right)^2 dt \tag{4.2}$$

where $m_j = 2^{-k} n_j$. By Lemma (3.3) of [1] there exists a $q_0 = q_0(\varepsilon, f)$ such that if $m_{M+1} \geqq 1$ and $m_{j+1}/m_j \geqq q_0$ for $M+1 \leqq j \leqq M+N-1$ (i.e., equivalently, $n_{M+1} \geqq 2^k$ and $n_{j+1}/n_j \geqq q_0$ for $M+1 \leqq j \leqq M+N-1$) then for the right hand side of (4.2)

9 See footnote 3.

we have (using $\|f\|=1$)

$$(1-\varepsilon)N<\int_{i}^{i+1}\left(\sum_{j=M+1}^{M+N} f(m_j t)\right)^2 dt<(1+\varepsilon)N.$$

Hence Theorem (4.3) follows from Theorem 1 of [1].

We mention one more consequence of Theorem (4.1) which extends some results of [11, 16]:

Corollary. *Let $f(x)$ satisfy (1.1) and (1.4) and let $\{n_k\}$ be a sequence of positive numbers satisfying (1.2). Then we have*

$$\overline{\lim_{N\to\infty}}\,(2N\log\log N)^{-1/2}\sum_{k=1}^{N} f(n_k x)\leqq C \qquad \text{a.e.}$$

where C is a constant depending on $f(x)$ and q.

Indeed, any sequence $\{n_k\}$ satisfying (1.2) can be decomposed into finitely many sequences each of which satisfies (1.2) with an (arbitrarily prescribed) large $q=q_1$. Applying statement b) of Theorem (4.1) (e.g. with $\varepsilon=1/2$ which is possible if q_1 in the decomposition is chosen large enough) and using the fact that $\overline{\lim_{n\to\infty}}(a_n+b_n)\leqq\overline{\lim_{n\to\infty}}\,a_n+\overline{\lim_{n\to\infty}}\,b_n$ we immediately get the Corollary.

Remark. There is an other way to get Theorems (4.1) and (4.2) of this section. Let us write $f=f_1+f_2$ where f_1 is a trigonometric polynomial and $\|f_2\|$ is small. It is easy to see that if $n_{k+1}/n_k\geqq q$ where q is large enough then $f_1(n_k x)$ is a multiplicative system. (For a definition see [2].) Since for multiplicative systems the law of the iterated logarithm is valid (see e.g. [2]), this holds in particular for $f_1(n_k x)$. On the other hand, analyzing the proof of the theorem in [16] we see that if $\|f_2\|$ is small enough then

$$\overline{\lim_{N\to\infty}}\,(2N\log\log N)^{-1/2}\left|\sum_{k=1}^{N} f_2(n_k x)\right|$$

is also small. From these facts and $f=f_1+f_2$ it follows that $f(n_k x)$ obeys the law of the iterated logarithm with accuracy ε (i.e. statement b) of Theorem (4.1) holds) if q is large enough. Since Donsker's invariance principle and Strassen's law of the iterated logarithm are also valid for multiplicative systems (see [2, 10]), the remaining parts of Theorems (4.1) and (4.2) can also be obtained in this way. This derivation, however, presupposes the validity of some functional limit theorems for multiplicative systems the proofs of which are rather involved. Our method used in this section is simpler, unified and assumes Donsker's and Strassen's theorems only for the Wiener-process.

5. § The Trigonometric Case

In this section we shall investigate what form of the a.s. invariance principle can be stated in the classical case $f(x)=\cos 2\pi x$. We know that under quite general conditions the asymptotic behaviour of $\sum_{k=1}^{N} f(n_k x)$, $N\to\infty$ is the same as that of $\zeta(\tau_N)$, $N\to\infty$ where ζ is a Wiener process and τ_N is a certain sequence of random variables. In § 3 we showed that if $\{n_k\}\in\Lambda^*$ (which is a stronger assumption than the Hadamard gap condition) then τ_N are asymptotically constant: $\tau_N\sim b_N$ with a certain numerical sequence $b_N\asymp N$. The example of Erdös and Fortet (see § 4) shows that this is not necessarily valid if we assume only the Hadamard gap condition for $\{n_k\}$. (In fact, in this case even the central limit theorem can fail to hold.) In the sequel we shall show that the case $f(x)=\cos 2\pi x$ is exceptional, namely, in this case the Hadamard gap condition is sufficient to imply the a.s. invariance principle for the sequence $f(n_k x)$ with asymptotically constant τ_N (even with constant τ_N). A result of this type follows at once from Remark 1 after Theorem (3.1). Indeed, $f(x)=\cos 2\pi x$ is a trigonometric polynomial of order 1 and thus for this function f the statement of Theorem (3.1) holds assuming only the Hadamard gap condition for $\{n_k\}$ (instead of $\{n_k\}\in\Lambda^*$). The following theorem shows a little more, namely that in the case $f(x)=\cos 2\pi x$ the random variables τ_N can actually be chosen constant:

Theorem (5.1). *Let $\{n_k\}$ be a sequence of positive numbers satisfying* (1.2). *Then there exists a probability space $(\Omega,\mathscr{F},P)$ and a sequence $X_1, X_2, \ldots$ of random variables (defined on $(\Omega,\mathscr{F},P)$) such that the sequences $\{\cos 2\pi n_k x\}$ and $\{X_k\}$ are quasi-equivalent and*

$$X_1+\cdots+X_n=\zeta(n/2)+o(n^{1/2-\eta}) \qquad \text{a.s. } as\ n\to\infty$$

where ζ is a Wiener-process on $(\Omega,\mathscr{F},P)$ and $\eta>0$ is an absolute constant.

Proof. By the identity $2\cos\alpha\cos\beta=\cos(\alpha+\beta)+\cos(\alpha-\beta)$ we have

$$2^k\int_{i2^{-k}}^{(i+1)2^{-k}}\left(\sum_{j=M+1}^{M+N}\cos 2\pi n_j x\right)^2 dx=I_1+I_2 \tag{5.1}$$

where

$$I_1=2^k\sum_{j=M+1}^{M+N}\int_{i2^{-k}}^{(i+1)2^{-k}}\cos^2 2\pi n_j x\,dx=\frac{N}{2}+2^k\sum_{j=M+1}^{M+N}\int_{i2^{-k}}^{(i+1)2^{-k}}\tfrac{1}{2}\cos 4\pi n_j x\,dx,$$

$$I_2=2^k\sum_{M+1\leqq\mu<\nu\leqq M+N}\int_{i2^{-k}}^{(i+1)2^{-k}}(\cos 2\pi(n_\nu+n_\mu)x+\cos 2\pi(n_\nu-n_\mu)x)\,dx.$$

Using (1.2) and the fact that

$$\left|\int_\alpha^\beta\cos\gamma x\,dx\right|\leqq 2/|\gamma|$$

we get

$$\left|I_1-\frac{N}{2}\right|\leqq 2^k\sum_{j=M+1}^{M+N}\frac{2}{8\pi n_j}\leqq 2^k\frac{2}{8\pi n_{M+1}}(1+q^{-1}+q^{-2}+\cdots)\leqq C_1\frac{2^k}{n_M},$$

$$|I_2|\leqq 2^k\sum_{M+1\leqq\mu<\nu\leqq M+N}\left(\frac{2}{2\pi(n_\nu+n_\mu)}+\frac{2}{2\pi(n_\nu-n_\mu)}\right)$$

$$\leqq 2^k\sum_{M+1\leqq\mu<\nu\leqq M+N}\left(\frac{1}{n_\nu}+\frac{1}{n_\nu(1-1/q)}\right)\leqq 2^k N\sum_{\nu=M+1}^{M+N}\frac{C_2}{n_\nu}\leqq C_3\frac{N\cdot 2^k}{n_M}$$

where C_1, C_2, C_3 depend only on q. Hence the left hand side of (5.1) is

$$\frac{N}{2}+O\left(\frac{N\cdot 2^k}{n_M}\right)$$

and thus Theorem (5.1) follows from Theorem 3 of [1].

Theorem (5.1) is not the best result in the field. We can state a better result under the same conditions and, on the other hand, we can weaken the assumption that $\{n_k\}$ satisfies the Hadamard gap condition. As to the first line of generalization, we mention the following result of Philipp and Stout giving an a.s. invariance principle for lacunary trigonometric sums with weights:

Theorem (*see* [12]). *Let* $\{n_k\}$ *be a sequence of positive numbers satisfying* (1.2). *Let further* $\{a_k\}$ *be a sequence of real numbers such that, putting* $A_N^2=\frac{1}{2}\sum_{k=1}^{N} a_k^2$, *we have* $A_N\to\infty$ *and* $a_N=O(A_N^{1-\delta})$ *with a constant* $0<\delta\leqq 1$. *Then there exists a probability space* $(\Omega,\mathscr{F},P)$ *and a sequence* $X_1,X_2,\ldots$ *of random variables on* $(\Omega,\mathscr{F},P)$ *such that the sequences* $\{X_k\}$ *and* $\{a_k\cos 2\pi n_k x\}$ *are equivalent and*

$$X_1+\cdots+X_n=\zeta(A_n^2)+o(A_n^{1-c\delta}) \qquad \text{a.s. } as\ n\to\infty$$

where ζ *is a Wiener-process on* $(\Omega,\mathscr{F},P)$ *and* $c>0$ *is an absolute constant.*

The other line of generalization is motivated by a remarkable theorem of Erdös (see [5]). Erdös' theorem states that the sequence $\cos 2\pi n_k x$ satisfies the central limit theorem provided that $\{n_k\}$ is a sequence of integers satisfying

$$n_{k+1}/n_k\geqq 1+c_k/\sqrt{k}, \qquad c_k\to\infty. \tag{5.2}$$

Erdös also remarks that this theorem is best possible i.e. for any fixed $c>0$ there is a sequence $\{n_k\}$ of integers which satisfies $n_{k+1}/n_k\geqq 1+c/\sqrt{k}$ and the sequence $\cos 2\pi n_k x$ does not obey the central limit theorem. (For some other results related to Erdös' theorem see [18] and the bibliography given there.) In view of Erdös' theorem one can expect that if $\{n_k\}$ satisfies (5.2) then the sequence $\cos 2\pi n_k x$ obeys an almost sure invariance principle with constant τ_N. In this direction we have proved the following theorem:

Theorem (*see* [3]). *Let* $\{n_k\}$ *be a sequence of integers satisfying*

$$n_{k+1}/n_k\geqq 1+1/k^\alpha, \qquad \alpha<1/2. \tag{5.3}$$

Then we have the conclusion of Theorem (5.1) (*with the minor modification that now the constant* $\eta>0$ *can depend on* α).

By the remarks above the last theorem is not valid for $\alpha=1/2$. It is very likely that the theorem remains valid if (5.3) is replaced by (5.2).

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