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R-Theory for Markov Chains on a Topological State Space. II

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§ 1. Introduction

This paper continues the investigation begun in [4]. We consider a discrete time irreducible Markov chain $\{X_n\}$ on a measure space $(\mathscr{X}, \mathscr{F})$, with stationary transition probabilities $P(x, \cdot)$. Writing $G_z(x, A)$ for the generating function $\sum_1^\infty z^n P^n(x, A)$ of the n-step transition probabilities, we follow Tweedie [6] in using the following form of irreducibility: *there is a probability measure M on $\mathscr{F}$ for which $M(A)>0$ implies that $G_{\frac{1}{2}}(x, A)>0$ for all $x \in \mathscr{X}$, and $M(A)=0$ implies $M\{y: G_{\frac{1}{2}}(y, A)>0\}=0$.* We denote $\{A \in \mathscr{F}: M(A)>0\}$ by $\mathscr{F}^+$; "almost all x" shall henceforth be with respect to this M.

Tweedie [6] has proved the following solidarity results for such a chain (cf. Theorems A and C of [4]):

(i) There is an $R \geqq 1$ and a class $\mathscr{C}_R \subseteq \mathscr{F}^+$ such that for each $A \in \mathscr{C}_R$ the radius of convergence of $G_z(x, A)$ is R, for almost all x. Members of $\mathscr{C}_R$ are called *R-sets*; $\mathscr{X}$ can be partitioned into countably many R-sets.

(ii) If $\{X_n\}$ is aperiodic, then there is a class $\mathscr{C}_L \subseteq \mathscr{C}_R$ (the class of *L-sets*) such that for every $A \in \mathscr{C}_L$ and almost all $x \in \mathscr{X}$, $\lim_{n \to \infty} R^n P^n(x, A) = \pi(x, A)$ exists and is finite, and $\mathscr{X}$ can be partitioned into countably many L-sets. Either $\pi(x, A)=0$ a.e. for every $A \in \mathscr{C}_L$ (the *R-null* case) or $\pi(x, A)>0$ a.e. for every $A \in \mathscr{C}_L$ (the *R-positive* case).

It is of interest to be able to identify specific R-sets or L-sets. In [4] we showed that for a strongly continuous chain on a topological space $\mathscr{X}$ (i.e. when $P(x, A)$ is a continuous function of x for every $A \in \mathscr{F}$) every compact member of $\mathscr{F}^+$ is both an R-set and an L-set. We also showed by example that by itself weak continuity of the transition probabilities would not suffice to draw the same conclusion, but we found auxiliary conditions *on the chain* which were sufficient

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when combined with weak continuity. In this paper we improve these weakly continuous results by replacing these restrictions on $\{X_n\}$ with restrictions on the irreducibility measure M. (Our proofs can also be adapted to simplify the arguments given in [4] in the strongly continuous case.) A more general form of weak continuity will be used: *we assume that* $P(x, g)$ $(=\int P(x, dy)\, g(y))$ *is a lower semi-continuous (l.s.c.) function of* x *for every bounded l.s.c. function* g. This is obviously equivalent to $P(x, h)$ being upper semi-continuous (u.s.c.) for every bounded u.s.c. function h. It implies that $P(x, f)$ is a continuous function of x for every bounded continuous f, and is in fact equivalent to this when $\mathscr{X}$ is a completely regular space and each $P(x, \cdot)$ is τ-smooth (see [5]). Also, if $\mathscr{X}$ is locally compact and the $P(x, \cdot)$ are Radon measures, then our weak continuity is equivalent to $P(x, f)$ being l.s.c. for every continuous f of compact support, or vanishing at infinity; so our assumption is weaker than, for example, 1.1 of [9]. Other similar equivalent continuity conditions can be derived from Theorem 8.1 of [5].

We also need the concepts of support, second category and regularity. The *support* of M (written supp M) is defined to be the closed subset of $\mathscr{X}$ consisting of those points for which every open neighbourhood is in $\mathscr{F}^+$; we assume that $M(\operatorname{supp} M)=1$, which is automatically true for example if M is τ-smooth [5], or if $\mathscr{X}$ is a separable metric space. A space is said to be of *second category* if it cannot be expressed as a countable union of sets whose closures have empty interiors; M is *regular* if $M(A)=\sup\{M(F)\colon A\supseteq \text{closed } F\}$ for every $A\in\mathscr{F}$.

It is always assumed that $\mathscr{F}$ is the Borel σ-field of $\mathscr{X}$.

Our two main results are:

Theorem 1. *If the Markov chain is weakly continuous, and if* M *is regular and the support of* M *is of second category (under the relativised topology) then every relatively compact* $A\in\mathscr{F}^+$ *is an* R*-set.*

Theorem 2. *If* $\{X_n\}$ *and* M *satisfy the conditions of Theorem 1 and if* $\{X_n\}$ *is also aperiodic, then every relatively compact* $A\in\mathscr{F}^+$ *is an* L*-set.*

These are clearly improvements over [4] since the conditions do not involve quantities whose existence depends upon the deeper properties of the chain viz. R-subinvariant measures as in Theorem 2 of [4]. Also we immediately deduce that *for weakly continuous chains on locally compact spaces with regular irreducibility measure, or complete metric spaces, every relatively compact* $\mathscr{F}^+$ *set is an* R*-set, and an* L*-set if the chain is aperiodic.* This is so since closed subsets of such spaces are always of second category under their relative topologies. For the metric space, the regularity of M is also automatic.

The proofs of these theorems, and related results, are given in the next two sections. In the final section we show that even the weak continuity assumption on $P(x, \cdot)$ can be relaxed: it suffices to have $P(x, \cdot)$ merely bounding a weakly continuous family of transition measures for our conclusions to hold.

It is interesting to note that Cogburn [1] has also found weak continuity and a second category assumption on supp M to be natural conditions for showing that compact sets have other solidarity properties, rather different from those of being R-sets and L-sets.

§ 2. R-Sets

The radius R in (i) is defined by $R=\sup\{r>0: G_r(x,A)<\infty$ for some $x\in\mathscr{X}$ and $A\in\mathscr{F}^+\}$. If $G_R(x,A)<\infty$ for some $x\in\mathscr{X}$ and $A\in\mathscr{F}^+$ then we say that the chain is *R-transient*, and we take $\mathscr{C}_R$ as the class of those $A\in\mathscr{F}^+$ for which $G_R(x,A)<\infty$ for some $x\in\mathscr{X}$ (and hence almost all x [6]). Otherwise we call $\{X_n\}$ *R-recurrent*, and $\mathscr{C}_R$ is taken to consist of all those $A\in\mathscr{F}^+$ for which $G_z(x,A)$ has radius of convergence R for some $x\in\mathscr{X}$ (and hence again for almost all x). It is non-trivial to show that $\mathscr{C}_R\neq\varnothing$ in the R-recurrent case (cf. Lemma 2 below).

The class $\mathscr{C}_R$ enjoys the following properties, which we shall employ in the sequel:

R1: If $A\in\mathscr{F}^+$ and $A\subseteq B$ for some $B\in\mathscr{C}_R$, then $A\in\mathscr{C}_R$;

R2: If $A, C\in\mathscr{F}^+$ differ only by an M-null set and $C\subseteq B$ for some $B\in\mathscr{C}_R$, then $A\in\mathscr{C}_R$.

The second property holds because $M(A\setminus C)=0$ implies $P^n(x, A\setminus C)=0$ for all n and almost all x.

To prove that a relatively compact $A\in\mathscr{F}^+$ is an R-set, we observe from $R1$ that it suffices to show there is some $B\in\mathscr{C}_R$ which contains A. Since A is contained in a compact set (the closure $\bar{A}$, if $\mathscr{X}$ is Hausdorff) we need only cover the space $\mathscr{X}$ with an increasing sequence of *open* R-sets. The following lemma shows that it is enough to find a single open R-set; we also give the analogous result for closed sets, which will be needed later.

Lemma 1. *Suppose the chain is weakly continuous.*

(i) *If there exists an open R-set U then there is a sequence of open R-sets* $U_n\uparrow\mathscr{X}$

(ii) *If there exists a closed R-set F then there is a sequence of closed R-sets* $F_n\uparrow\mathscr{X}$.

Proof. As in [4] it is easy to prove that $G_\theta(x,g)$ is a l.s.c. function of x for every bounded l.s.c. g and $1>\theta>0$. Also $G_\theta(x,\mathscr{X})$ is continuous (in fact constant), hence $G_\theta(x,h)$ is u.s.c. for bounded u.s.c. h. Thus $U_n=\{y: G_{\frac{1}{2}}(y,U)>n^{-1}\}$ defines a sequence of open sets, and $F_n=\{y: G_{\frac{1}{2}}(y,F)\geqq n^{-1}\}$ a sequence of closed sets. Since $M(U)>0$, $G_{\frac{1}{2}}(y,U)>0$ for every $y\in\mathscr{X}$ and hence $U_n\uparrow\mathscr{X}$. As a consequence $M(U_n)\uparrow M(\mathscr{X})$ so that $U_n\in\mathscr{F}^+$ for large enough values of n. Similar results hold for the F_n.

The Chapman-Kolmogorov identity shows that, for any $m,n\geqq 1$ and $r,\theta>0$,

$$\theta^m r^{n+m} P^{n+m}(x,B)=\int r^n P^n(x,dy)\, r^m\theta^m P^m(y,B).$$

Summing this over n gives

$$\theta^m G_r(x,B)\geqq\theta^m\sum_{n=1}^{\infty} r^{n+m}P^{n+m}(x,B)=\int G_r(x,dy)\,(r\theta)^m\,P^m(y,B)$$

and a second summation over m leads to the following inequality, valid for $0<\theta<1$ and $r>0$:

$$\theta(1-\theta)^{-1}\,G_r(x,B)\geqq\int G_r(x,dy)\,G_{r\theta}(y,B). \tag{2.1}$$

If $1\leqq r<R$ take $\theta=(2r)^{-1}$, $B=U$ and then choose x so that the left hand side is finite (possible since U is an R-set). Then the right hand side is bounded below

by $n^{-1} G_r(x, U_n)$ for every n. It follows that $U_n \in \mathscr{C}_R$ eventually, and similarly for the F_n. $\square$

In general proving the existence of even one open R-set is difficult, being equivalent to seeking an R-set whose interior belongs to $\mathscr{F}^+$. However the next lemma shows that it is relatively easy to find a closed R-set.

Lemma 2. *If M is regular then there exists a closed R-set.*

Proof. Suppose there exists an R-set A. Since M is regular we can choose a closed $F \subseteq A$ with $F \in \mathscr{F}^+$. This is the required closed set, from $R1$.

In the R-transient case the existence of the R-set A is trivial, but not so for the R-recurrent case. We give a constructive proof for this, as an alternative to the non-constructive proof given in [6].

Choose a sequence $r_k \uparrow R$. We construct inductively a decreasing sequence $\{A_k\}$ in $\mathscr{F}^+$ for which $M(A_k \setminus A_{k+1}) \leqq 2^{-(k+1)} M(A_1)$ and $G_{r_k}(x, A_k) < \infty$ for almost all x. Suppose $A_1, \ldots, A_{k-1}$ have been so constructed. By definition of R we can find $B \in \mathscr{F}^+$ for which $G_{r_k}(x, B) < \infty$ for almost all x. Define $B_n = \{y: G_{\frac{1}{2}}(y, B) \geqq n^{-1}\}$. Since $B_n \uparrow \mathscr{X}$ and $G_{r_k}(x, B_n) < \infty$ for almost all x by (2.1), we have only to set $A_k = A_{k-1} \cap B_n$ for a large enough value of n. Let $A = \bigcap_{k=1}^{\infty} A_k$. Then $M(A) = M(A_1) - \sum_{k=1}^{\infty} M(A_k \setminus A_{k+1}) \geqq \frac{1}{2} M(A_1) > 0$. Thus $A \in \mathscr{F}^+$; and since $G_z(x, A)$ has radius of convergence greater than or equal to R for almost all x by construction, it follows that A is an R-set. $\square$

With these two lemmas we can now find conditions for the existence of an open R-set.

Lemma 3. *If M is regular and* supp M *is of second category there exists an open R-set.*

Proof. Let F be the closed R-set given by Lemma 2. Then from Lemma 1 (ii) there is a sequence of closed R-sets $F_n \uparrow \mathscr{X}$.

Now the second category set supp M can be expressed as the countable union of closed sets $\bigcup_{n=1}^{\infty} (\operatorname{supp} M \cap F_n)$; hence we can find an N such that $\operatorname{supp} M \cap F_N$ contains an interior point in the space supp M. Thus there is an open subset U of $\mathscr{X}$ for which $\varnothing \neq U \cap \operatorname{supp} M \subseteq \operatorname{supp} M \cap F_N$. As $M(U \setminus \operatorname{supp} M) = 0$, and U contains a support point of M, U is an open R-set, by $R2$.

We now have enough for

Proof of Theorem 1. Since M is regular and supp M is of second category, Lemma 3 and Lemma 1 (i) imply that $\mathscr{X}$ has a countable cover of increasing open R-sets U_n. If $A \in \mathscr{F}^+$ is contained in some compact set K, then $A \subseteq K \subseteq U_n$ for one of the members of this open cover. Thus A is an R-set, from $R1$. $\square$

§ 3. *L*-Sets

In this section we assume that $\{X_n\}$ is aperiodic and R-recurrent. This is the only case we need consider; for if the chain is R-transient, then we can take $\mathscr{C}_L = \mathscr{C}_R$ and Theorem 2 follows from Theorem 1.

We shall make use of the *R-invariant function* f which satisfies

$$f(x)=R\int_{\mathscr{X}} P(x,dy)f(y) \quad \text{for almost all } x \tag{3.1}$$

and the *R-invariant measure* Q which satisfies

$$Q(A)=R\int_{\mathscr{X}} Q(dy)P(y,A) \quad \text{for all } A\in\mathscr{F}. \tag{3.2}$$

Under the assumption of R-recurrence, Theorems 3 and 4 of [6] prove that (up to constant multiples and definition on a null set) f is the unique non-negative $\mathscr{F}$-measurable function satisfying (3.1) for which $\int f\,dM>0$ (in fact $f(x)>0$ for almost all x), whilst Q is the unique (up to constant multiples) non-trivial σ-finite measure on $\mathscr{F}$ satisfying (3.2); Q is equivalent to M.

From Proposition 2.2 of [4] we have the following sufficient condition involving this f and Q for identifying L-sets: *for an aperiodic R-recurrent chain, $A\in\mathscr{F}$ is an L-set if*

$$\operatorname{ess\,inf}\{f(x): x\in A\}>0 \tag{3.3}$$

and

$$0<Q(A)<\infty. \tag{3.4}$$

The first condition means that there is a $\delta>0$ such that $M\{x\in A: f(x)<\delta\}=0$.

In the R-recurrent case we shall write $\mathscr{C}'_L$ for the class of sets satisfying (3.3) and (3.4). Note that $\mathscr{C}'_L\subseteq\mathscr{C}_L$; in general the inclusion is strict. The class $\mathscr{C}'_L$ has the following closure properties, analogous to $R1$ and $R2$:

$L1$: If $A\in\mathscr{F}^+$ and $A\subseteq B$ for some $B\in\mathscr{C}'_L$, then $A\in\mathscr{C}'_L$;

$L2$: If $A, C\in\mathscr{F}^+$ differ only by an M-null set and $C\subseteq B$ for some $B\in\mathscr{C}'_L$, then $A\in\mathscr{C}'_L$.

Lemma 4. *Suppose $\{X_n\}$ is weakly continuous.*

(i) *If there is an open set U in $\mathscr{C}'_L$ then there is a sequence of open sets $U_n\uparrow\mathscr{X}$, with each $U_n\in\mathscr{C}'_L$.*

(ii) *If there is a closed set F in $\mathscr{C}'_L$ then there is a sequence of closed sets $F_n\uparrow\mathscr{X}$, with each $F_n\in\mathscr{C}'_L$.*

Proof. Define U_n, F_n as in Lemma 1. Upon iterating (3.1) and (3.2) we obtain for any $0<\theta<1$, as in the proof of (2.1),

$$\theta(1-\theta)^{-1}f(x)=\int_{\mathscr{X}} G_{R\theta}(x,dy)f(y) \quad \text{for almost all } x \tag{3.5}$$

and

$$\theta(1-\theta)^{-1}Q(A)=\int_{\mathscr{X}} Q(dy)G_{R\theta}(y,A) \quad \text{for all } A\in\mathscr{F}. \tag{3.6}$$

So taking $\theta=(2R)^{-1}$ in (3.5) we obtain, for almost all $x\in U_n$

$$\begin{aligned} f(x)&\geqq\int_U G_{\frac{1}{2}}(x,dy)f(y)\\ &\geqq n^{-1}\cdot\operatorname{ess\,inf}\{f(y): y\in U\}\end{aligned}$$

so that (3.3) is satisfied for all n. Similarly, from (3.6) we obtain

$$\infty > \int_{U_n} Q(dy)\, G_{\frac{1}{2}}(y, U)$$
$$\geqq n^{-1} \cdot Q(U_n)$$

so that (3.4) is satisfied for large enough n (notice that $M(U_n)>0$ is equivalent to $Q(U_n)>0$).

The same argument holds for the sequence $\{F_n\}$. □

Having proved this lemma, which shows that it suffices merely to find one open set in $\mathscr{C}'_L$ in order to cover $\mathscr{X}$ with an increasing sequence of such open L-sets, we can proceed easily with

Proof of Theorem 2. We first show that the regularity of M implies the existence of a closed set in $\mathscr{C}'_L$.

Since Q is σ-finite there is a sequence of $\mathscr{F}$ sets $B_n \uparrow \mathscr{X}$ with $Q(B_n)<\infty$. Also, since $f>0$ a.e., the sequence of $\mathscr{F}$ sets $A_n=\{x: f(x)\geqq n^{-1}\}\uparrow \mathscr{X}\setminus N$ where N is an M-null set. Thus $M(A_n\cap B_n)\uparrow M(\mathscr{X})$. Hence from regularity we can find an n and a closed set $F\subseteq A_n\cap B_n$ for which $M(F)\geqq \frac{1}{2}M(A_n\cap B_n)>0$. It follows from $L1$ that $F\in\mathscr{C}'_L$.

Now from Lemma 4 (ii), there is a sequence $F_n\uparrow\mathscr{X}$ with each F_n closed and in $\mathscr{C}'_L$. As in the proof of Lemma 3, $\operatorname{supp} M$ being of second category implies that we can find an N and an open set U such that $\varnothing \neq U\cap\operatorname{supp} M\subseteq \operatorname{supp} M\cap F_N$. Since $M(U\setminus\operatorname{supp} M)=0$ and U contains a support point it follows from $L2$ that $U\in\mathscr{C}'_L$. From Lemma 4 (i) there is a sequence U_n of open sets in $\mathscr{C}'_L$ such that $U_n\uparrow\mathscr{X}$.

Finally, since any relatively compact set in $\mathscr{F}^+$ is contained in some compact set, which in turn must be contained in one of these U_n's, another application of $L1$ shows that the theorem holds. □

§ 4. The Second Category Assumption

In proving Theorems 1 and 2 we showed that if a weakly continuous chain with a regular M of second category support is not R-positive then there is a sequence $\{U_k\}$ of open sets covering $\mathscr{X}$ for which $\lim_{n\to\infty} R^n P^n(x, U_k)=0$. In this section we show the importance of the category assumption by exhibiting a weakly continuous chain which is not 1-positive, whose regular M measure *has support which is not of second category,* and for which

$$\liminf_{n\to\infty} P^n(x, U)>0 \tag{4.1}$$

for every non-empty open U and $x\in\mathscr{X}$. Thus no open set is an L-set, and the method of proof for Theorems 1 and 2 breaks down. We have not been able to construct a relatively compact set which is not an L-set though, so the assertion of Theorem 2 may still be true in this case without the category assumption.

We take as the space $\mathscr{X}$ the set $\mathscr{Q}$ of "rational points" in $\mathscr{R}$, the circle of unit circumference, and give $\mathscr{Q}$ the usual relative topology. Let $\{t_j\}$ be an enumeration of $\mathcal{Q}$, and $\{p_j\}$ be any discrete probability distribution for which $p_j > 0$ for all j. Define a random walk on $\mathcal{R}$ by the transition probabilities $P(x, x+t_j) = p_j$.

This random walk on $\mathcal{R}$ does not admit an irreducibility measure. However we note that $P(x, \mathcal{Q}) = 1$ when $x \in \mathcal{Q}$, so that $\mathcal{Q}$ is stochastically closed in $\mathcal{R}$. Also, since $p_j > 0$ for each j, the random walk restricted to $\mathcal{Q}$ *is* irreducible with M as any probability measure equivalent to counting measure on $\mathcal{Q}$. As

$$\operatorname{supp} M = \mathcal{Q} = \bigcup_{j=1}^{\infty} \{t_j\}$$

and each singleton $\{t_j\}$ is closed with empty interior, $\operatorname{supp} M$ is *not* of second category.

The random walk on $\mathcal{Q}$ is weakly continuous; for if g is any bounded continuous function on $\mathcal{Q}$, and if $x_j \to x$ in $\mathcal{Q}$ then

$$\begin{aligned} \int_{\mathcal{Q}} P(x_j, dy)\, g(y) &= \sum_k P(x_j, x_j + t_k)\, g(x_j + t_k) \\ &= \sum_k p_k\, g(x_j + t_k) \\ &\to \sum_k p_k\, g(x + t_k) \\ &= \int_{\mathcal{Q}} P(x, dy)\, g(y) \end{aligned}$$

by dominated convergence. (The random walk on $\mathcal{R}$ is of course similarly weakly continuous, but we do not use this fact.)

To prove that the random walk on $\mathcal{Q}$ is not 1-positive, it suffices to show that

$$\lim_{n \to \infty} P^n(x, s) = 0, \qquad x, s \in \mathcal{Q} \tag{4.2}$$

since $\mathcal{Q}$ is countable and each singleton is in $\mathcal{F}^+$. We prove both (4.1) and (4.2) (and hence that no open set is an L-set) by using known results for the random walk on the whole of $\mathcal{R}$.

Let h be any bounded uniformly continuous real function on $\mathcal{Q}$, and write $\tilde{h}$ for its unique continuous extension to $\mathcal{R}$. For the random walk on $\mathcal{R}$ itself it is known [2, p. 274] that for any $x \in \mathcal{R}$

$$\lim_{n \to \infty} \int_{\mathcal{R}} P^n(x, dy)\, \tilde{h}(y) = \int_{\mathcal{R}} \tilde{h}(y)\, dy. \tag{4.3}$$

Now if $x \in \mathcal{Q}$, the fact that $P^n(x, \cdot)$ is concentrated on $\mathcal{Q}$ means that

$$\int_{\mathcal{R}} P^n(x, dy)\, \tilde{h}(y) = \int_{\mathcal{Q}} P^n(x, dy)\, h(y)$$

and so from (4.3)

$$\lim_{n \to \infty} \int_{\mathcal{Q}} P^n(x, dy)\, h(y) = \int_{\mathcal{R}} \tilde{h}(y)\, dy. \tag{4.4}$$

For any $s \in \mathcal{Q}$ and $\varepsilon > 0$ there is a bounded uniformly continuous non-negative function h_ε with $h_\varepsilon(s) = 1$ such that $\int_{\mathcal{R}} \tilde{h}_\varepsilon(y)\, dy < \varepsilon$. Hence (4.2) follows from (4.4), and the random walk on $\mathcal{Q}$ is indeed not 1-positive.

On the other hand, for any non-empty open subset U of $\mathcal{Q}$ we can find a non-zero uniformly continuous h with $0 \leqq h \leqq 1_U$, and so again from (4.4)

$$\begin{aligned}\liminf_{n\to\infty} P^n(x, U) &\geqq \liminf_{n\to\infty} P^n(x, h)\\ &= \int_{\mathcal{R}} \tilde{h}(y)\,dy\\ &> 0.\end{aligned}$$

§ 5. Weakly Continuous Components

Examining the proofs of Sections 2 and 3 it becomes evident that the only points at which weak continuity of the $P(x, \cdot)$ was applied were in the proofs of Lemmas 1 and 4. More specifically, the weak continuity was only used to prove that $G_{\frac{1}{2}}(x, A)$ is l.s.c. if A is open, and u.s.c. if A is closed. Another point to notice is that the crucial relations (2.1), (3.5) and (3.6) were only needed to put one sided bounds on various quantities. These observations lead us to the following useful concept.

We say that $P(x, \cdot)$ has a *weakly continuous component* if there is a family of measures $T(x, \cdot)$ for which

(i) $P(x, \cdot) \geqq T(x, \cdot) \geqq 0$ for each $x \in \mathcal{X}$.

(ii) $T(x, \cdot)$ is a weakly continuous function of x (i.e. $T(x, g)$ is l.s.c. for each bounded l.s.c. g).

(iii) $T(x, \cdot)$ is irreducible in the sense that $M(A) > 0$ implies $H_{\frac{1}{2}}(x, A) > 0$ for every $x \in \mathcal{X}$, where we write $H_z(x, A)$ for the generating function $\sum_{n=1}^{\infty} z^n T^n(x, A)$.

Notice that $T(x, \mathcal{X})$ need not be constant, although it is a continuous function of x because of (ii).

It is a very simple matter to modify the proofs of Theorems 1 and 2 to obtain the stronger versions

Theorem 1'. *If the chain has a weakly continuous component, and if M is regular and the support of M is of second category, then every relctively compact member of $\mathcal{F}^+$ is an R-set.*

Theorem 2'. *If the chain satisfies the conditions of Theorem 1' and is aperiodic, then every relatively compact member of $\mathcal{F}^+$ is an L-set.*

We shall not prove these theorems but merely note that in Lemmas 1 and 4 we should consider $U_n' = \{y : H_{\frac{1}{2}}(y, U) > n^{-1}\}$ and $F_n' = \{y : H_{\frac{1}{2}}(y, F) \geqq n^{-1}\}$. Combined with the modified form of (2.1)

$$\theta(1-\theta)^{-1} G_r(x, B) \geqq \int_{\mathcal{X}} G_r(x, dy)\, H_{r\theta}(y, B)$$

and similar inequalities to replace (3.5) and (3.6), the proofs then carry through with very little change.

In [7] the idea of continuous components is exploited. It is also clear, although we did not realize it at the time, that Example 3 of [4] works because the chain constructed there has a strongly continuous component. That example illustrates the behaviour we envisage of a chain with a well-behaved component; and in general, it is easy to see that if $\{X_n\}$ is irreducible in the sense of the introduction, and has a transition law that can be written as

$$P(x,\cdot)=\alpha(x)P_1(x,\cdot)+[1-\alpha(x)]P_2(x,\cdot),$$

where $\{P_1(x,\cdot)\}$ is irreducible, weakly continuous and $\alpha(x)$ is a continuous $(0,1]$-valued function of x, then $T(x,\cdot)=\alpha(x)P_1(x,\cdot)$ is irreducible and so is a weakly continuous component of $P(x,\cdot)$.

We conclude with easy examples to show that even when M has all the desired properties one still needs some continuity condition on $\{P(x,\cdot)\}$ to achieve the desired results.

Example 1. Let $\{X_n\}$ be a renewal-type chain on $\mathscr{X}_1=\{0;1,\frac{1}{2},\frac{1}{3},\ldots\}$; that is, $\Pr\{X_{n+1}=(j+1)^{-1}\mid X_n=j^{-1}\}=\alpha_j=1-\Pr\{X_{n+1}=0\mid X_n=j^{-1}\}$ for $j=1,2,\ldots$ and $\Pr\{X_{n+1}=1\mid X_n=0\}=\alpha_0=1-\Pr\{X_{n+1}=0\mid X_n=0\}$, where $1>\alpha_j>0$ for all j. Equipping $\mathscr{X}_1$ with the relativised topology from the real line makes it compact, as also is each of $K_m=\{0;j^{-1}:j\geqq m\}$. Each K_m is also recurrent in the classical sense in that $G_1(x,K_m)=\infty$ for any x. Further, the irreducibility measure M_1 defined by $M_1\{0\}=\frac{1}{2}$ and $M_1\{j^{-1}\}=2^{-(j+1)}$ is regular and has the whole of $\mathscr{X}_1$ as support. Thus $\operatorname{supp} M_1$ is of second category (since $\mathscr{X}_1$ is compact).

By a suitable choice of $\{\alpha_j\}$ though, the chain can be made to have any desired radius of convergence greater than or equal to one (cf. Section 5 of [8]), which means that there will be compact sets which are not R-sets or L-sets. Notice that this means the chain cannot have a weakly continuous component – a result which can also be proved directly.

Example 2. To see that irreducibility of the weakly continuous component is vital in Theorems 1′ and 2′, we construct another chain from the $\{X_n\}$ of Example 1 and a second chain $\{Z_n\}$ which is a random walk on the left half line; that is, the transition law of $\{Z_n\}$ is described by taking a probability measure μ on the real line (which we assume equivalent to Lebesgue measure) and defining for $A\subseteq(-\infty,0)$ and $x\leqq 0$

$$\Pr\{Z_{n+1}\in A\mid Z_n=x\}=\mu(A-x)$$

and

$$\Pr\{Z_{n+1}=0\mid Z_n=x\}=\mu[-x,\infty).$$

The chain $\{Z_n\}$ can easily be shown to be weakly (and in fact, strongly, cf. Example 1 of [4]) continuous.

Now form a chain $\{W_n\}$ on $\mathscr{X}=\mathscr{X}_1\cup\mathscr{X}_2$, where $\mathscr{X}_1$ is as in Example 1 and $\mathscr{X}_2=(-\infty,0]$, by putting for a measurable $B\subseteq\mathscr{X}$

$$\Pr\{W_{n+1}\in B\mid W_n=j^{-1}\}=\Pr\{X_{n+1}\in B\cap\mathscr{X}_1\mid X_n=j^{-1}\}\qquad\text{for } j=1,2,\ldots,$$

$$\Pr\{W_{n+1}\in B\mid W_n=x\}=\Pr\{Z_{n+1}\in B\cap\mathscr{X}_2\mid Z_n=x\}\qquad\text{for } x<0,$$

and

$$\Pr\{W_{n+1}\in B\mid W_n=0\}=\tfrac{1}{2}\Pr\{Z_{n+1}\in B\cap\mathscr{X}_2\mid Z_n=0\}+\tfrac{1}{2}\Pr\{X_{n+1}\in B\cap\mathscr{X}_1\mid X_n=0\}.$$

A suitable irreducibility measure for this chain is

$$M(B)=\tfrac{1}{2}M_1(B\cap\mathscr{X}_1)+\tfrac{1}{2}\mu(\mathscr{X}_2)^{-1}\mu(B\cap\mathscr{X}_2).$$

The only possible weakly continuous component for this chain is that part of the transition probabilities due to the $\{Z_n\}$. Even though this is weakly continuous it is not a weakly continuous component of $\{W_n\}$ since it is not irreducible with respect to M: from any $x<0$ it is impossible to reach $B_0=\{1,\frac{1}{2},\frac{1}{3},\ldots\}$ by transitions of $\{Z_n\}$. Consequently, although relatively compact subsets of $\mathscr{X}_2$ have the "correct" R-theoretic properties, one can assert nothing about subsets of B_0 from Theorem 1′ or 2′. Indeed B_0 is recurrent in the classical sense, although we can again make $R>1$ for the chain $\{W_n\}$. So removing the irreducibility of the continuous component makes the conclusions of Theorems 1′ and 2′ false.

Finally we refer the reader to [3] where the ideas of this paper are utilised, and where we give practical examples of chains which are weakly continuous, or which have weakly continuous components.

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