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Fine Boundary Minimum Principle and Dual Processes

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Introduction

The boundary minimum principle (BMP) plays an important role in axiomatic potential theory. For instance it is a key argument in the method of solving the Dirichlet problem in harmonic spaces ([1a, 8]) and consequently it becomes an essential hypothesis for the local theory of cones of potentials [11].

Given an open set $\mathscr{U}$ of a strong harmonic space $(\mathscr{H}, E)$, one can state the BMP as follows:

Suppose that f is hyperharmonic in $\mathscr{U}$ and that

$$\liminf_{x\in\mathscr{U},\, x\to y} f(x) \geqq 0 \qquad \text{for every point } y \text{ of the boundary of } \mathscr{U},$$

and moreover that there exists a potential p on E such that $f \geqq -p$ in $\mathscr{U}$. Then f is non negative in $\mathscr{U}$.

If $\mathscr{U}$ is relatively compact the above result reduces to the classical form of the BMP which was first obtained by Brelot [3]. However the most general setting of the BMP is the abstract minimum principle of Bauer ([1a], p. 7), whose proof is essentially based on the compactness property of the state space.

Starting from the case of a strong Brelot harmonic space with the domination principle, Fuglede has recently succeeded in building up a "fine harmonic space" where the underlying topology on E is the fine one. Thus he was lead to a new type of potential theory with results analogous to the usual ones.

Arguments based on the local compactness fail in this fine harmonicity theory; for instance Fuglede's proof of the following fine BMP ([6], IV, 9.1) is mainly based on capacity-theoretic arguments: Suppose that f is finely hyperharmonic in a finely open set $\mathbf{U}$ of E (see Def. 2, § 4) and that $\text{fine-}\liminf_{x\in\mathbf{U},\, x\to y} f(x) \geqq 0$ for *quasi-every* (q.e.) y on the fine boundary of $\mathbf{U}$. Suppose moreover there exists a *semi-bounded* potential p on E such that $f \geqq -p$ in $\mathbf{U}$. Then f is non-negative in $\mathbf{U}$. Using a Hunt process $X = (\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, \zeta, P^x)$ associated with the given harmonic space, a probabilistic proof of the above result was given in ([9b], Th. 6). The proof is reduced to the study of the following inequality:

For q.e. x in E:

$$\begin{aligned} \bar{f}(x) &\geqq \liminf_n E_x(\bar{f}(X_{\eta_n});\ \eta_n < \zeta) \\ &\geqq E_x(\liminf_n \bar{f}(X_{\eta_n});\ \lim_n \eta_n < \zeta) \end{aligned} \tag{1}$$

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where $\bar{f}$ is the function equal to f in $\mathbf{U}$ and 0 in $E \setminus \mathbf{U}$, and $(\eta_n)_{n \in N}$ is an increasing sequence of $\{\mathscr{F}_t\}$-stopping times such that

For q.e. x in E:

$$\lim_n \eta_n = T_{E \setminus \mathbf{U}} \text{ a.s. } P^x \text{ in } \Omega$$

and (2)

$$\bar{f}(x) \geqq E_x\big(\bar{f}(X_{\eta_n});\ \eta_n < \zeta\big) \qquad \text{for } n = 1, 2, \ldots.$$

The proof of (1) and (2) is based on two main arguments:

a) If $\mathbf{U}$ is open for the initial topology of E, such a sequence $(\eta_n)_{n \in N}$ was constructed by Šur in [12], and if the capacity $A \to R_1^A(x)$ $\big(\equiv \inf\{u(x)\colon uX\text{-excessive and } \geqq 1 \text{ on } A\}\big)$ has the Choquet property then one can pass from the initial topology to the fine one. Recall that in a strong Brelot harmonic space the Choquet property of $A \to R_1^A(x)$ is equivalent to the domination principle.

b) For each point x in E such that the supermartingale $\big(p(X_t), \{\mathscr{F}_t\}, P^x\big)$ is of class (D) the Fatou-type lemma in (1) holds for the sequence of random variables $\big(\bar{f}(X\eta_n)\big)_{n \in N}$ at such a point x. Recall that in a strong Brelot harmonic space a potential p is semi-bounded iff the supermartingale $\big(p(X_t), \{\mathscr{F}_t\}, P^x\big)$ is of class (D) for q.e. x in E.

It is natural to ask whether in a) the Choquet property of $A \to R_1^A(x)$ and in b) the semi-boundedness of p could be removed.

Lemma 1 and its Corollary 2 of Section §2 affirm that such an increasing sequence of stopping-times $(\eta_n)_{n \in N}$ could be constructed for any transient standard process with a reference measure. Lemma 4 shows that under the duality hypothesis the Fatou-type lemma in (1) holds for a rather large class of diffusion standard processes and cofinely lower semi-continuous (l.s.c.) functions f.

Section §3 is concerned with the fine BMP under the duality hypothesis of a class of numerical and finely l.s.c. functions in E (Th. 6 and Cor. 7). As far as we know, even in the brownian motion case, the above problems seem to be new in probabilistic potential theory since one used to consider only the cone of excessive functions of a standard process. Note that in the case of a strong harmonic space, the cone of excessive functions of the associated Hunt process is identical to the cone of non negative hyperharmonic functions.

In Section §4 we apply the results of Section §3 to prove the following form of fine BMP in a harmonic space: Given a strong harmonic space $(\mathscr{H}, E)$ (Bauer or Brelot) where:

The state space E has a countable base and the function 1 is hyperharmonic.

A Green function for $(\mathscr{H}, E)$ exists.

Semi-polar sets are polar.

Let f be a finely hyperharmonic function in a finely open subset $\mathbf{U}$ of E. Suppose that

$$\underset{x \in \mathbf{U},\, x \to y}{\text{fine-lim inf}}\, f(x) \geqq 0 \qquad \text{for } every\ x \text{ in the fine boundary of } \mathbf{U},$$

and moreover $f \geqq -p$ in $\mathbf{U}$, where p is a potential in E. Then f is non negative in $\mathbf{U}$.

Let us note that in the given harmonic space $(\mathscr{H}, E)$ every finite potential is semi-bounded [9c].

In the appendix we show that the classical BMP of Brelot [3] could be deduced easily from our method of proving the fine BMP.

Contents

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1. Some Preparations on Dual Processes

The framework of the duality theory of two standard processes is presented in ([2a], Chap. VI). We assume that the reader is familiar with this chapter, so proofs which are readily available in this literature are not repeated. Because we will make use of them later, we introduce here a survey of some new results of Blumenthal and Getoor [2b] on the relation between the fine and cofine topologies and those of Weil [14] on the behaviour of coexcessive functions on paths. The results (1.d) and (1.e) are consequences of the above mentioned ones and seem to have some independent interest.

Duality Hypothesis

Let $(P_t)_{t\geqq 0}$ and $(\hat{P}_t)_{t\geqq 0}$ be two submarkovian, standard semi-groups on the same state space E, $(U_p)_{p\geqq 0}$ and $(\hat{U}_p)_{p\geqq 0}$ be respectively their resolvents and $X=(\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, \zeta, P^x)$, $\hat{X}=(\Omega, \hat{\mathscr{F}}, \hat{\mathscr{F}}_t, \hat{X}_t, \hat{\theta}_t, \hat{\zeta}, P^x)$ be respectively their realizations. We say that $(P_t)_{t\geqq 0}$ and $(\hat{P}_t)_{t\geqq 0}$ (or equivalently (U_p) and $(\hat{U}_p)$) satisfy the hypothesis of duality if:

1) There is a positive radon measure $m(dx)$ on E such that all the measures $\varepsilon_x U_p$, $\hat{U}_p \varepsilon_x$ $(x\in E)$ are absolutely continuous w.r.t. $m(dx)$ and:

$$\int f\, U_p g\, dm = \int f\, \hat{U}_p g\, dm \tag{3}$$

for every $p\geqq 0$ and for every couple f, g of non-negative, measurable functions.

2) The function Uf $(\equiv U_0 f)$ (resp. $f\hat{U}$ $(\equiv f\hat{U}_0)$) is bounded for every f non-negative, bounded, Borel and with compact support, i.e. both X and $\hat{X}$ are transient standard processes.

Recall that by convention $\hat{P}_t(dy, x)$ $(t\geqq 0)$ and also its resolvent $\hat{U}_p(dy, x)$ are cokernels, i.e. a kernel on $\mathscr{B}_E\times E$ which acts on the left on functions and on the right on measures.

If X and $\hat{X}$ satisfy the hypothesis 1) then every couple of α-processes $(\alpha>0)$ constructed from X and $\hat{X}$ is in duality. Under the hypothesis of duality there is a measurable function $u(x, y)$ on $E\times E$ (E is equipped with the σ-algebra of universally measurable (u.m.) sets) such that: $u(\cdot, y)$ is X-excessive for every $y\in E$

and $u(x, \cdot)$ is $\hat{X}$-excessive for every x in E and

$$U(x, dy) = u(x, y)\, m(dy); \qquad \hat{U}(dx, y) = m(dx)\, u(x, y). \tag{4}$$

Terminology and notation associated with the process $\hat{X}$ will be distinguished from those of X by the prefix co-. For instance $u(x, \cdot)$ is coexcessive for every x in E and we have also corresponding notations such as cofine topology, cothin, copolar etc. ...

We say that the fine and cofine topologies differ by semi-polar (or polar) sets provided that the fine and cofine interiors of an arbitrary subset of E differ by a semi-polar (polar) set.

(1.a) Under the duality hypothesis semi-polar sets (polar) are cosemipolar (copolar) and vice-versa, and the fine and cofine topologies differ by semi-polar sets (see [2b], (4.1)).

(1.b) Let $f: E \to [-\infty, +\infty]$ be finely lower semi continuous (l.s.c.), i.e. l.s.c. for the fine topology. Then there exists a $f_*: E \to [-\infty, +\infty]$ cofinely l.s.c. such that $f \geqq f_*$ and $\{f > f_*\}$ is semi-polar ([2b], (4.2)).

(1.c) Let f be α-coexcessive ($\alpha \geqq 0$). Then:

a) f is nearly Borel.

b) For every probability law μ on E, the map $t \to f(X_{t-}(\omega))$ is left continuous and has right limits on $]0, \zeta(\omega)[$ P^μ a.s. (see [14], Th. 6).

(1.d) The U-potential of an u.m., cofinely open non-empty set is non identically null, consequently the complement of a $m(dx)$ null set is cofinely dense.

Proof. Since $m(dx)$ is a reference measure for both processes X and $\hat{X}$, every cofinely open and u.m. (finely open and u.m.) is nearly Borel (n.b.) (see [10], XV, Th. 66). X is transient, for every u.m. and finely open set B:

$$U(x, B) = E_x\left(\int \chi_B(X_t) \,.\, dt\right) > 0 \qquad \text{for } x \in B,$$

where χ_B is the indicator function of the set B. Now let A be u.m. and cofinely open and B be its fine interior. By (1.a) $A \setminus B$ is semi-polar, therefore B is not empty, hence:

$$U(x, A) \geqq U(x, B) > 0 \qquad \text{for } x \in B.$$

Consequently $m(A) > 0$, and the complement of a $m(dx)$ null set is cofinely dense in E. This fact could be seen also directly from the $\hat{U}$-potential of a u.m., cofinely open set.

(1.e) Let μ be a probability law on E.

a) If $A \subset E$ is u.m. and cofinely closed, then

$$\{t: X_{t-}(\omega) \in A;\, 0 < t < \zeta(\omega)\}$$

is closed for the left topology in $]0, \zeta(\omega)[$, a.s. P^μ.

b) Let $f: E \to]-\infty, +\infty]$ be u.m. and cofinely l.s.c. Then the function $t \to f(X_{t-}(\omega))$ is well defined and l.s.c. for the left topology in $]0, \zeta(\omega)[$, P^μ a.s.

Proof. a) Let $\hat{e}_A^\alpha$ ($\alpha > 0$) be the α-coequilibrium potential of A:

$$\hat{e}_A^\alpha(x) = E_x\{\exp(-\alpha\, \hat{T}_A)\}.$$

Then by ([10a], XV, T31) and by [4],

$$\{t\colon X_{t-}(\omega)\in A_n;\ t\in]0,\zeta(\omega)[\}$$

is discrete P^μ a.s., where

$$A_n=\left\{\hat{e}^\alpha_A\leqq 1-\frac{1}{n}\right\}\cap A \qquad (n\in N).$$

On the other hand since $\hat{e}^\alpha_A$ is α-coexcessive, by (1.c) the map

$$t\to \hat{e}^\alpha_A(X_{t-}(\omega)) \text{ is left continuous on }]0,\zeta(\omega)[\tag{5}$$

for a.s. P^μ. We have finally a set Ω' of P^μ-measure 1 such that for every $\omega\in\Omega'$

$$\begin{aligned}&t\to \hat{e}^\alpha_A(X_{t-}(\omega)) \text{ is left continuous in }]0,\zeta(\omega)[\\ &\{t\in]0,\zeta(\omega)[;\ X_{t-}(\omega)\in A_n\} \text{ is discrete}\end{aligned} \tag{6}$$

a) follows then from (6) by an argument analogous to that of ([10a], XV, T 38),

b) the proof of this part follows from part a) and an argument analogous to that of ([10a], XV, T 39). Note that f is not allowed to take the value $-\infty$.

Given a triple $((U_\alpha)_{\alpha\geqq 0}, (\hat{U}_\alpha)_{\alpha\geqq 0}, m(dx))$ with the duality hypothesis, we introduce a regularity condition on the co-resolvent $(\hat{U}_\alpha)_{\alpha\geqq 0}$, namely:

(K-W) The function $f\hat{U}_\alpha$ $(\alpha\geqq 0)$ is bounded and continuous on E for every bounded Borel function f with compact support.

Under this condition, the triple $((U_\alpha)_{\alpha\geqq 0}, (\hat{U}_\alpha)_{\alpha\geqq 0}, m(dx))$ satisfies the so-called Kunita-Watanabe hypothesis (see [10b], Chap. II, 2). We still have to check the condition that $\alpha f\hat{U}_\alpha$ converges pointwise to f as $\alpha\to\infty$ for every continuous f with compact support. But this is automatically satisfied since we suppose that $(\hat{U}_\alpha)_{\alpha\geqq 0}$ is a standard co-resolvent.

(1.f) The triple $((U_\alpha)_{\alpha\geqq 0}, (\hat{U}_\alpha)_{\alpha\geqq 0}, m(dx))$ satisfies the hypothesis of duality and (K-W).

Let $(\mu_n)_{n\in N}$ be a sequence of non-negative measures. Suppose that the sequence of excessive functions

$$u_n(x)=\int u(x,y)\,\mu_n(dy) \qquad (n=1,2,\ldots)$$

has terms bounded by a fixed potential and converges a.s. $m(dx)$ to an excessive function u. Then the sequence of measures (μ_n) converges vaguely to a measure μ and

$$u(x)=\int u(x,y)\,\mu(dy)$$

(see [10b], Chap. III, T 8).

In order to shorten the exposition, let us denote by (B_1) respectively (B_2) the following hypotheses:

(B_1) Duality hypothesis between the standard resolvent $(\hat{U}_\alpha)_{\alpha\geqq 0}$ and the standard co-resolvent $(\hat{U}_\alpha)_{\alpha\geqq 0}$.
Hypothesis (K-W)
The standard process $X=(\Omega,\mathscr{F},\mathscr{F}_t,X_t,\theta_t,\zeta,P^x)$ associated with $(U_\alpha)_{\alpha\geqq 0}$ has continuous paths.

(B_2) All the hypotheses of the case (B_1).
Semi-polar sets are polar.

The above hypotheses are natural in potential theory since we will see in §4 that the Hunt process associated with an harmonic space which has a Green function satisfies always (B_1).

2. Three Lemmas

The first lemma and its corollary are concerned with the approximation of the first exit-time of the union of a countable family of u.m., finely open sets by those of each element of the family. Note that the only hypothesis used in these lemmas is the existence of a reference measure for the transient standard process $X=(\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, \zeta, P^x)$. The remaining lemmas use (B_2) (see §1).

Lemma 3 extends to a abitrary potential a property on balayage known for uniformly integrable potentials. Lemma 4 describes the behaviour of the balayage operator of X on a class of cofinely l.s.c. functions in E. We introduce first some conventions which generalize to arbitrary u.m. functions some integral notations usually defined for bounded u.m. functions.

Let T be a $\{\mathscr{F}_t\}$-stopping time and f be a numerical u.m. function. We will write

$$E_x(f(X_T);\ T<\zeta)=\int^*_{\{T<\zeta\}} f(X_T(\omega)).P^x(d\omega) \tag{7}$$

provided that

$$\int_{\{T<\zeta\}} f^-(X_T(\omega)).P^x(d\omega)<+\infty,$$

where $f^-=\sup(-f,0)$. Hence $E_x(f(X_T);\ T<\zeta)$ is finite if and only if $f(X_T)$ is P^x-integrable on $\{T<\zeta\}$, otherwise it is equal to $+\infty$.

A numerical function defined on E is always assumed to have value null at the point Δ. In the rest of the paper we denote by T_A the hitting-time of the process X for a n.b. set A of $E_\Delta(=E\cup\{\Delta\})$, and by τ_A the first exit-time of X from A, i.e., the hitting-time of the complement of A.

Lemma 1. *Let $X=(\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, \zeta, P^x)$ be a transient standard process with a reference measure. Let A_1 and A_2 be two n.b. subsets of E_Δ. Then there exist an increasing sequence (ξ_n) of $\{\mathscr{F}_t\}$-stopping times and a semi-polar set e of E such that*

a) *For every x in $E\setminus e$, one has*

$$\xi_\infty\equiv\lim_n \xi_n \quad \text{a.s. } P^x. \tag{8}$$

b) *If the function f is l.s.c. and bounded from below on E_Δ with $f(\Delta)=0$ and if*

$$f(x)\geqq E_x(f(X_{T_{A_i}});\ T_{A_i}<\zeta)\ x\in E \qquad (i=1,2).$$

Then

$$\begin{aligned} f(x)&\geqq E_x(f(X\xi_n);\ \xi_n<\zeta)\\ &\geqq E_x(f(X_{T_{A_1\cap A_2}});\ T_{A_1\cap A_2}<\zeta) \qquad \text{for every } x \text{ in } E\setminus e \text{ and } n\in N. \end{aligned} \tag{9}$$

Proof. If x is a regular point of $A_1\cap A_2$, it suffices to define $\xi_n=T_{A_1\cap A_2}$ ($n=1,2,\dots$) and (9) is trivial.

In the general case we construct an increasing sequence (ξ_n^1) of $\{\mathscr{F}_t\}$-stopping times, setting

$$\begin{aligned}&\xi_1^1 = T_{A_1}, \quad \xi_2^1 = \xi_1^1 + T_{A_2}(\theta_{\xi_1^1}), \ldots, \xi_{2k+1}^1 = \xi_{2k}^1 + T_{A_1}(\theta_{\xi_{2k}^1})\\ &\xi_\infty^1 = \lim \xi_n^1.\end{aligned} \tag{10}$$

If x is irregular for $A_1 \cap A_2$ and if $P^x\{\xi_\infty^1 < T_{A_1 \cap A_2}\} = 0$, then the sequence (ξ_n^1) satisfies (8) at this point. Otherwise, we construct again an increasing sequence (ξ_n^2) of $\{\mathscr{F}_t\}$-stopping times, setting

$$\begin{aligned}&\xi_1^2(\omega) = \begin{cases} \xi_\infty^1(\omega) + T_{A_1}(\theta_{\xi_\infty^1}) & \text{if } \xi_\infty^1(\omega) < T_{A_1 \cap A_2}(\omega) \\ T_{A_1 \cap A_2}(\omega) & \text{if } \xi_\infty^1(\omega) = T_{A_1 \cap A_2}(\omega) \end{cases}\\ &\xi_2^2 = \xi_1^2 + T_{A_2}(\theta_{\xi_1^2}), \ldots, \xi_{2k+1}^2 = \xi_{2k}^2 + T_{A_1}(\theta_{\xi_{2k}^2}), \ldots.\end{aligned} \tag{11}$$

By a transfinite induction argument, namely the Tukey Lemma, there exists a filter $(\xi_\infty^i)_{i \in I}$ of stopping times such that

$$\inf_{i \in I} P^x\{\xi_\infty^i < T_{A_1 \cap A_2}\} = 0. \tag{12}$$

I claim that every stopping time ξ_∞^i $(i \in I)$ satisfies the condition

$$t + \xi_\infty^i(\theta_t) \geqq \xi_\infty^i, \; P^x \quad \text{a.s. for every } t > 0.$$

In fact, since $\xi_1^1 = T_{A_1}$ by definition, ξ_1^1 satisfies the above properties. By induction it suffices to verify those properties for ξ_2^1 and ξ_∞^1. We have

$$\begin{aligned} t + \xi_2^1(\theta_t) &= t + \xi_1^1(\theta_t) + T_{A_2}\big(\theta_{\xi_1^1(\theta_t)}\big) \\ &= t + \xi_1^1(\theta_t) + T_{A_2}(\theta_{t + \xi_1^1(\theta_t)}). \end{aligned}$$

Now, since for each fixed $\omega \in \Omega$ the function $s \to s + T_{A_2}(\theta_s(\omega))$ $(\equiv \inf\{t > s : X_t(\omega) \in A_1\})$ is an increasing function of s and since $t + \xi_1^1(\theta_t) \geqq \xi_1^1$ a.s. P^x, we have $t + \xi_2^1(\theta_t) \geqq \xi_2^1$ a.s. $(x \in E)$.

Since every term of the increasing sequence of stopping times $(\xi_n^1)_n$ satisfies the required properties, it is clear that $\xi_\infty^1 = \sup_n \xi_n^1$ satisfied the required property.

Since the potential kernel of X is transient, for each $i \in I$ the function

$$u^i(x) = P^x\{\xi_\infty^i < T_{A_1 \cap A_2}\}$$

is an $(X - M)$ supermedian function w.r.t. the exact multiplicative functional

$$M_t = 1_{]0, T_{A_1 \cap A_2}}(t)$$

(see the proof in [9a], Lemma 2.2). We denote also by $\hat{u}^i$ the $(X - M)$ excessive regularization of u^i.

The Cartan-Meyer convergence applied to the M-subprocess ([2a], (V, 1.6)) implies from (12) the existence of a decreasing sequence $(\hat{u}^{n_j})_{j \in N}$ of the filter $(\hat{u}^i)_{i \in I}$ such that

$$\inf_j \hat{u}^{n_j}(x) = 0 \quad \text{for } x \in E_{M - e_0},$$

where E_M is the set of permanent points of (M_t) which is nothing else than the set of irregular points of $A_1 \cap A_2$, and where $e_0 \subset E_M$ is semi-polar. Thus the designed semi-polar set e is $e_0 \cup \left(\bigcup_{j=1}^{\infty} \{\hat{u}_{n_j} < u_{n_j}\} \right)$ and the designed sequence of stopping times (ξ_n) is:

$$\xi_1^{n_1}, \ldots, \xi_k^{n_1}, \ldots, \xi_\infty^{n_1}, \xi_2^{n_2}, \ldots, \xi_\infty^{n_k}, \xi_1^{n_{k+1}}, \ldots. \tag{13}$$

(b) Consider first the sequence (ξ_n^1). By hypothesis b):

$$f(x) \geqq E_x f(X_{\xi_1^1});\ \xi_1^1 < \zeta).$$

By the definition of $(\xi_i^1)_i$ in (10) and by the strong Markov property of X, we have

$$f(x) \geqq E_x\big(f(X_{\xi_i^1});\ \xi_i^1 < \xi\big) \qquad (i \in N).$$

The process X is quasi-left-continuous, hence

$$f(x) \geqq E_x\big(\liminf_n f(X_{\xi_n^1});\ \xi_\infty^1 < \zeta\big) + E_x\big(\liminf_n f(X_{\xi_n^1});\ \xi_n^1 < \zeta;\ \xi_\infty^1 = \zeta\big)$$

by the Fatou lemma and the lower-semi-continuity of f. Since f is l.s.c. in E_Δ and $f(\Delta)=0$, we have

$$f(x) \geqq E_x\big(f(X_{\xi_\infty^1});\ \xi_\infty^1 < \zeta\big). \tag{14}$$

By recurrence, the inequality (14) is true for every term of the sequence (13), hence part (a) implies

$$f(x) \geqq E_x\big(f(X_{T_{A_1 \cap A_2}});\ T_{A_1 \cap A_2} < \zeta\big)$$

for x in $E - e$.

Remarks. 1) Lemma 1 generalizes a result of Šur ([12], Lemma 2). Sur treated the case of two closed sets for the given topology and showed that for every x in E, $\xi_\infty^1 = T_{A_1 \cap A_2}$ a.s. P^x. However this fact is true if and only if x belongs to the complement of $A_1^r \cap A_2^r \setminus (A_1 \cap A_2)^r$ as we pointed out in the "note added in the proof" of [9 b]. It is an interesting question to ask whether the semi-polar set e in Lemma 1 is exactly the set $A_1^r \cap A_2^r \setminus (A_1 \cap A_2)^r$.

2) Part b) of the Lemma 1 is presented here to illustrate the approximation procedure of part a). The next Lemma 4 is a much deeper result of this kind and is a key argument of the next section.

Corollary 2. *Assume the same assumption as in Lemma* 1. *Let* $(A_i)_{i \in N}$ *be a sequence of n.b. subsets of E. Then there are a sequence* (η_n) *of* $\{\mathscr{F}_t\}$*-stopping times and a semi-polar set e such that for every* $x \in E \setminus e$,

$$\lim_n \eta_n = T_{\bigcap_{i=1}^{\infty} A_i} \quad a.s. \quad P^x.$$

Proof. 1) For the case of k finely closed, n.b. sets $A_1, A_2, \ldots, A_k$, it is enough to define first the increasing sequence of $\{\mathscr{F}_t\}$-stopping times (η_k^i), defined by:

$$\begin{aligned}
&\eta_1^1 = T_1, && \eta_1^2 = \eta_1^1 + T_2(\theta_{\eta_1^1}), \ldots, \eta_1^k = \eta_1^{k-1} + T_k(\theta_{\eta_1^{k-1}})\\
&\eta_2^1 = \eta_1^k + T_1(\theta_{\eta_1^k}), && \eta_2^2 = \eta_2^1 + T_2(\theta_{\eta_2^1}), \ldots, \eta_2^k = \eta_2^{k-1} + T_k(\theta_{\eta_2^{k-1}})\\
&\vdots \\
&\eta_j^1 = \eta_{j-1}^k + T_1(\theta_{\eta_{j-1}^k}), && \eta_j^2 = \eta_j^1 + T_2(\theta_{\eta_j^1}), \ldots, \eta_j^i = \eta_j^{i-1} + T_j(\theta_{\eta_j^{i-1}}),
\end{aligned} \tag{15}$$

where $T_i = T_{A_i}$ $(1 \leqq i \leqq k)$.

Then for each fixed $1 \leqq j_0 \leqq k$ we can apply the arguments of part a) of Lemma 1 to the sequence $(\eta_n^{j_0})_{n\in N}$ and get finally a semi-polar set e_{j_0} and a sequence $(\xi_n^{j_0})$ of $\{\mathscr{F}_t\}$-stopping times such that $\lim_n \xi_n^{\tau_j} = T_{\cap A_i}$ a.s. P^x for $x \notin e_{j_0}$.

2) Setting $\Pi \equiv \lim_k T_{\bigcap_{i=1}^{\infty} A_i}$, we have $\Pi \leqq T_{\bigcap_{i=1}^{\infty} A_i}$. Consider the increasing sequence of stopping times

$$\Pi_1^1 = \Pi + T_1(\theta_\Pi);\ \Pi_2^1 = \Pi_1^1 + T_2(\theta_{\Pi_1^1});\ \ldots;\ \Pi_n^1 = \Pi_{n-1}^1 + T_n(\theta_{\Pi_{n-1}^1}). \tag{16}$$

Now (η_n) can be constructed in the same way as (ξ_n) in the proof of Lemma 1.

Let $X = (\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, \zeta, P^x)$ be a standard process on the state space E. Let $(K_n)_{n\in N}$ be an increasing sequence of compact sets of E such that

$$K_n \subset \mathring{K}_{n+1} \qquad (n=1,2\ldots) \quad \text{and} \quad \bigcup_{n=1}^{\infty} \mathring{K}_n = E \tag{17}$$

and write $T_{E \setminus \mathring{K}_n} = \tau_n$.

Definition 1. Assume that a reference measure $m(dx)$ for X exists.

a) A X-excessive function p is a *pseudo-potential* if there exists a sequence (K_n) with the property (17) such that

$$\lim_n p(X_{\tau_n}) = 0, \qquad \text{a.s. } P^x \text{ a.s. } m(dx). \tag{18}$$

b) p is a *potential* provided that p is finite a.s. $m(dx)$ and $\lim_n E_x(p(X_{\tau_n})) = 0$ a.s. $m(dx)$.

c) p is an *uniformely integrable potential* if

$$\lim_n E_x(p(X_{\eta_n})) = 0$$

for all x outside of a semi-polar set of E and for every increasing sequence $\{\eta_n\}$ of $\{\mathscr{F}_t\}$-stopping times such that $\lim_n \eta_n = \zeta$ a.s.

Remarks. 1) Since for every x in $(p < +\infty)$ $E(p(X_{\tau_n}))_{n\in N}$ is a non-negative super-martingale adapted to $(\Omega, \mathscr{F}_{\tau_n}, P^x)$ a potential is a pseudo-potential. The converse is no more true. For instance, let X be the brownian motion on $E = R^3 \setminus \{0\}$ and let $p(x) = |x|^{-1}$. Then p is a pseudo-potential but not a potential.

2) If X is the brownian motion on R^3 then $p(x) = |x|^{-1}$ is a potential but not a uniformly integrable one.

3) If p is a pseudo-potential then

$$\lim_n p(X_{R_n}) = 0 \qquad \text{a.s. } P^x \text{ on } \{R = \zeta\} \qquad (\text{see [2a], IV, 5}),$$

whenever (R_n) is an increasing sequence of stopping times with limit R.

4) Our Definition 1 differs from the usual ones ([2a], IV, 5) by the fact that p is allowed to have infinite values.

Lemma 3. *Assume* (B_2). *Let* A_1 *and* A_2 *be two n.b. subsets of* E. *Let* $p(x) = \int u(x,y)\, r(dy)$ *be a potential which is finite on the cofine closure of* $E \setminus A_1 \cap A_2$. *Let* $(\xi_n)_{n\in N}$ *be the increasing sequence of* $\{\mathscr{F}_t\}$*-stopping times:*

$$\xi_1 = T_{A_1};\ \xi_2 = \xi_1 + T_{A_2}(\theta_{\xi_1});\ \ldots;\ \xi_{2K+1} = \xi_{2K} + T_{A_1}(\theta_{\xi_{2K}});\ \ldots. \tag{19}$$

Then there exists a polar set $e \subset E$ *such that:*

For every $x \in (p < +\infty) \setminus e$, *the supermartingale* $(p(X_{\xi_n}) X_{\{\xi_n < \zeta\}})_{n \in N}$ *is uniformly integrable and:*

$$\lim_n E_x(p(X_{\xi_n});\ \xi_n < \zeta) = E_x(p(X_{\xi_\infty});\ \xi_\infty < \zeta) \tag{20}$$

where $\xi_\infty = \lim_n \xi_n$.

Furthermore of A_1 *and* A_2 *are finely closed, then* (20) *holds for every* x *in* $E \setminus A_1 \cap A_2$.

Proof. We are familiar with the sequence (ξ_n) in the proof of Lemma 1. We construct now an increasing sequence of $\{\hat{\mathscr{F}}_t\}$-stopping times, setting

$$\hat{\xi}_1 = \hat{T}_{A_2};\ \hat{\xi}_2 = \hat{\xi}_1 + \hat{T}_{A_1}(\hat{\theta}_{\hat{\xi}_1});\ \ldots;\ \hat{\xi}_{2k+1} = \hat{\xi}_{2k} + \hat{T}_{A_2}(\hat{\theta}_{\hat{\xi}_{2k}})$$

and

$$\hat{\xi}_\infty = \lim_n \hat{\xi}_n.$$

Then it follows immediately from $P_{A_1} u(x, y) = u \hat{P}_{A_1}(x, y)$ (see [2a], (VI, 1.16)) that

$$(P_{A_2} \cdot P_{A_1})\, u(x, y) = u(\hat{P}_{A_1} \cdot \hat{P}_{A_1})(x, y).$$

Hence it follows from an induction argument that

$$P_{\xi_{2n}} u(x, y) = u \hat{P}_{\hat{\xi}_{2n}}(x, y) \qquad (n \geqq 2).$$

Consider now two arbitrary u.m. non-negative functions f and g. Since $Uf(X_{\xi_{2n}}) \to Uf(X_{\xi_\infty})$ a.s. and $g\,\hat{U}(\hat{X}_{\hat{\xi}_{2n}}) \to g\,\hat{U}(\hat{X}_{\hat{\xi}_\infty})$ a.s. as $n \to +\infty$, we have

$$\iint g(x) \cdot P_{\xi_\infty} u(x, y) \cdot f(y) \cdot m(dx)\, m(dy) = E_\nu(Uf(X_{\xi_\infty})) = \lim_n E_\nu(Uf(X_{\xi_{2n}}))$$

where $\nu(dx) = g(x) \cdot m(dx)$, and

$$\iint g(x) \cdot u \hat{P}_{\xi_\infty}(x, y) \cdot f(y) \cdot m(dx) \cdot (dy) = \hat{E}_\mu(g\,\hat{U}(\hat{X}_{\hat{\xi}_\infty})) = \lim_n \hat{E}_\mu(g\,\hat{U}(\hat{X}_{\hat{\xi}_{2n}}))$$

where $\mu(dy) = f(y) \cdot m(dy)$. Hence

$$u \hat{P}_{\hat{\xi}_\infty}(x, y) = P_{\xi_\infty} u(x, y) \qquad \text{a.s. } m(dx) \times m(dy) \text{ on } E \times E.$$

But on the one hand for a fixed x in E the coexcessive function $y \to P_{\xi_\infty} u(x, y)$ is the $\hat{X}$-excessive regularization of the co-supermedian function $y \to u \hat{P}_{\hat{\xi}_\infty}(x, y)$ and on the other hand $u \hat{P}_{\hat{\xi}_\infty}(x, y) = u(x, y)$ for y belongs to the co-fine interior of $A_1 \cap A_2$, we have:

For every x fixed in E:

$$\{y \mid u \hat{P}_{\xi_\infty}(x, y) > P_{\xi_\infty} u(x, y)\}$$

is a polar subset of the cofine closure of $E \setminus A_1 \cap A_2$.

Suppose first that $r(dy)$ is a *bounded Radon* measure on E. The decreasing of excessive functions:

$$u_n(x) = E_x(p(X_{\xi_n});\ \xi_n < \zeta) \qquad (n = 1, 2, \ldots)$$

converges a.s. $m(dx)$ to the excessive regularization $\hat{u}$ of $\inf_n u_n$ and furthermore:

$$\begin{aligned} P_{\xi_n} p(x) &= \int r(dz) \cdot P_{\xi_n} u(x, z) \\ &= \int u(x, y) \cdot \hat{P}_{\hat{\xi}_n} r(dy) \qquad (n = 1, 2, \ldots). \end{aligned}$$

We claim that the sequence of Radon measure $r_n = \hat{P}_{\xi_n} . r$ $(n=1,2,\ldots)$ converges vaguely to the measure $\hat{P}_{\xi_\infty} . r$. In fact, for a continuous function f with compact support we have

$$\int f(y) . r_n(dy) = \int \hat{E}_z(f(\hat{X}_{\xi_n}); \ \xi_n < \zeta) . r(dz).$$

Since the process $\hat{X}$ is quasi-left continuous, the sequence of bounded functions $(z \to \hat{E}_z(f(\hat{X}_{\xi_n})))_{n \in N}$ converges pointwise to the bounded function $(z \to \hat{E}_z(f(\hat{X}_{\xi_\infty})))$. Thus we have:

$$\begin{aligned} \lim_n \int f(y) . r_n(dy) &= \int \hat{E}_z(f(\hat{X}_{\xi_\infty}) . r(dz) \\ &= \int f(y) . \hat{P}_{\xi_\infty} r(dy). \end{aligned}$$

By assumption $p(x)$ is finite on the cofine closure of $E \setminus A_1 \cap A_2$, hence $r(dy)$ doesn't change polar subsets of this set ([9 c]), by (1.g) § 1 it turns out that

$$\begin{aligned} \hat{u}(x) &= \int u(x,y) . \hat{P}_{\xi_\infty} r(dy) \\ &= \int r(dy) \int u \hat{P}_{\xi_\infty}(x,y) \\ &= \int r(dy) \int P_{\xi_\infty} u(x,y) \\ &= P_{\xi_\infty} p(x). \end{aligned}$$

Now let e be the polar set $\{\inf_n u_n > \hat{u}\}$. For a point x of $(p < +\infty) \setminus e$, $(p(X_{\xi_n}) X_{\{\xi_n < \zeta\}})_{n \in N}$ is a non-negative supermartingale adapted to $(\Omega, (\mathscr{F}_{\xi_n})_{n \in N}, P^x)$, and furthermore since $\lim_n p(X_{\xi_n}) = p(X_{\xi_\infty})$ a.s. P^x on $\{\xi_\infty < \zeta\}$, hence (20) is proved for every x in $(p < +\infty) \setminus e$. For every x in $(p < +\infty)$ and every $q \geqq 0$, we have by strong Markov property:

$$\begin{aligned} q U_q \hat{u}(x) &= q U_q \inf_n u_n(x) = \inf_n q U_q u_n(x) \\ &= \inf_n E_x \left(\int_0^\infty q e^{-qt} . u_n(X_t) . dt \right. \\ &\geqq \inf_n \int_0^\infty q e^{-qt} . E_x(p(X_{\xi_n}) . \chi_{\{\xi_n > t\}}) . dt. \end{aligned}$$

If x belongs to $(E \setminus A_1 \cap A_2)$ then $p(x) < +\infty$ and $\xi_n > 0$ a.s. P^x for $n \geqq 2$. Hence, if we let q increase to $+\infty$, the last right hand side term will converge to $\inf_n u_n(x)$ while the left hand term increases to $\hat{u}(x)$. This implies $\hat{u}(x) = \inf_n u_n(x)$. Since every non-negative Radon measure on E is the sum of a sequence of non-negative, bounded Radon measures, the case of an abitrary measure $r(dy)$ could be deduce easily from the above case.

Lemma 4. *Assume* (B_2). *Let f be a numerical function defined on E and A_1, A_2 two u.m. subsets of E such that:*

1) *A_1 and A_2 are finely closed.*

2) *f is $> -\infty$ and cofinely l.s.c. on E.*

3) *For $i = 1, 2$:*

$$f(x) \geqq E_x(f(X_{T_{A_i}}); \ T_{A_i} < \zeta) \quad \textit{quasi-everywhere (q.e.) in } E.$$

4) *$f(x) \geqq -p(x)$ q.e. in E, where p is a potential finite on the cofine closure of $E \setminus A_1 \cap A_2$.*

Then there exists a polar set $\bar{e} \subset E$ *such that:*

$$f(x) \geqq E_x\big(f(X_{T_{A_1 \cap A_2}});\ T_{A_1 \cap A_2} < \zeta\big) \tag{21}$$

for every $x \in \big(E \setminus (A_1 \cap A_2) \cup \bar{e}\big) \cap (f < +\infty)$.

Proof. By "q.e." we always mean "for x outside of a polar set". Since $f^-(x) = \sup\big(-f(x), 0\big) \leqq p(x)$ q.e. by 4), we have

$$E_x\big(f^-(X_{T_{A_i}});\ T_{A_i} < \zeta\big) \leqq p(x) < +\infty \qquad \text{for } x \in \{p < +\infty\}.$$

Due to assumptions 1), 3) and the convention (7), there are two polar sets e_1 and e_2 such that

$$\begin{aligned} &E_x\big(f(X_{T_{A_i}});\ T_{A_i} < \zeta\big) && (i=1,2) \text{ is well-defined for } x \text{ in} \\ & && (E \setminus e_1 \cap e_2) \cup \{p < +\infty\} \\ &f(x) \geqq E_x\big(f(X_{T_{A_i}});\ T_{A_i} < \zeta\big) && \text{for every } x \text{ in } E \setminus e_i \ (i=1,2). \end{aligned} \tag{22}$$

By Lemma 1 there is a polar set $\bar{e}_1 \subset E \setminus (A_1 \cap A_2)^r$ and an increasing sequence of $\{\mathscr{F}_t\}$-stopping times (η_n) such that for every x in $E \setminus \bar{e}_1$

$$\lim_n \eta_n = T_{A_1 \cap A_2} \qquad \text{a.s. } P^x \text{ on } \{T_{A_1 \cap A_2} < \zeta\}.$$

Note that for $f(x) = +\infty$. (21) is trivially true provided that the integral in the second member of this inequality is well defined, i.e., at least if the point x belongs also to $(E \setminus e_1 \cap e_2) \cup \{p < +\infty\}$.

1) As the first step, we will now show

For every $x \in (E \setminus A_1 \cap A_2) \cap (p < +\infty) \cap \big((f < +\infty) \setminus (e_1 \cup e_2)\big)$ $\big(f(X\eta_n)\big)_{n \in N}$ is a supermartingale adapted to

$$\big(\Omega, (\mathscr{F}_{\eta_n})_{n \in N}, P^x\big). \tag{23}$$

Consider first the increasing sequence of stopping times $(\xi_k^1)_{k \in N}$ defined by (13)

$$\begin{aligned} f(x) &\geqq E_x\big(f(X_{T_{A_1}});\ T_{A_1} < \zeta\big) && \text{for } x \in E \setminus e_1 \\ &= E_x\big(f(X_{\xi_1^1});\ \xi_1^1 < \zeta\big) && \text{for } x \in E \setminus e_1. \end{aligned}$$

Set

$$g_k(\omega) = f(X_{\xi_k^1})\, \chi_{\{\xi_1^1 < \zeta\}}(\omega) \qquad (k = 1, 2 \ldots). \tag{24}$$

Then, again by 3), we have a.s. P^x on $\{\xi_1^1 < \zeta\}$:

$$\begin{aligned} &\text{If } \ X_{\xi_1^1}(\omega) \in A_2 \ \text{ then } \ E_{X_{\xi_1^1}(\omega)}\big(f(X_{T_{A_2}})\, \chi_{\{T_{A_2} < \zeta\}}\big) = f\big(X_{\xi_1^1}(\omega)\big) \\ &\text{If } \ X_{\xi_1^1}(\omega) \in A_2 \ \text{ then } \ E_{X_{\xi_1^1}(\omega)}\big(f(X_{T_{A_2}})\, \chi_{\{T_{A_2} < \zeta\}}\big) \leqq f\big(X_{\xi_1^1}(\omega)\big). \end{aligned} \tag{25}$$

For every $x \in (E \setminus e_1 \cup e_2) \cap \{p < +\infty\} \cap \{f < +\infty\}$, $f(X_{T_{A_2}})$ is P^x-integrable and furthermore, since ξ is a strong terminal time, we have by the strong Markov property of X

$$\begin{aligned} E_x\big(g_2(\omega) \mid \mathscr{F}_{\xi_1^1}\big) &= E_x\big(f(X_{\xi_2^1})\, \chi_{\{\xi_2^1 < \zeta\}} \mid \mathscr{F}_{\xi_1^1}\big) \\ &= E_x\big(f(X_{T_{A_2}})\, \chi_{\{T_{A_2} < \zeta\}}(\theta_{\xi_1^1}) \mid \mathscr{F}_{\xi_1^1}\big) \\ &= E_{X_{\xi_1^1(\omega)}}\big(f(X_{T_{A_2}})\, \chi_{\{T_{A_2} < \zeta\}}\big). \end{aligned}$$

Therefore (25) implies

$$E_x(g_2(\omega)|\mathscr{F}_{\xi_1^1}) \leqq g_1(\omega) \quad \text{a.s. } P^x.$$

$(g_k(\omega))_{k\in N}$ is a supermartingale adapted to $(\Omega, \{\mathscr{F}_{\xi_k^1}\}, P^x)$. But

$$\begin{aligned}\sup_k E_x(g_k^-) &= \sup_k E_x(f^-(X_{\xi_k^1});\ \xi_k^1<\zeta) \\ &\leqq \sup_k E_x(p(X_{\xi_k^1});\ \xi_k^1<\zeta) \leqq p(x) < +\infty.\end{aligned}$$

Hence there exists a P^x-integrable randon variable $Z(\omega)$ such that

$$\lim_k g_k(\omega) = Z(\omega) \quad \text{a.s. } P^x$$

and

$$\begin{aligned} f(x) &\geqq E_x(g_k(\omega)) \quad (k=1,2\dots) \\ &\geqq \lim_k E_x(g_k(\omega)). \end{aligned}$$

Since f is finely l.s.c. and since X has continuous paths, we have by (1.e), § 1

$$\begin{aligned} Z(\omega) &= \lim_k g_k(\omega) \\ &= \lim_k f(X_{\xi_k^1}(\omega)) \cdot \chi_{\{\xi_k^1<\zeta\}} \geqq (\lim_k X_{\xi_k^1}(\omega)) = f(X_{\xi_\infty^1}(\omega)) \end{aligned}$$

a.s. P^x in $\{\xi_\infty^1<\zeta\}$.

Now, since by assumption A_1 and A_2 are finely closed and p is finite on the co-fine closure of $E\setminus A_1\cap A_2$, it follows from Lemma 3 that for every $x\in(E\setminus A_1\cap A_2)$ the supermartingale $(p(X_{\xi_n^1}))_{n\in N}$ is uniformely integrable and converges to $p(X_{\xi_\infty^1})$.

Let A be a fixed element of $\mathscr{F}_{\xi_n^1}$. Then, by 4) and by Fatou's lemma

$$f(X_{\xi_n^1}) + p(X_{\xi_n^1} \geqq 0 \quad \text{a.s. } (n=1,2,\dots)$$

and

$$\begin{aligned} \liminf_n E_x(f(X_{\xi_n^1});\ A) + \liminf_n E_x(p(X_{\xi_n^1});\ A) \\ \geqq E_x(Z(\omega);\ A) + E_x(\lim_n p(X_{\xi_n^1});\ A). \end{aligned} \tag{26}$$

Therefore

$$\begin{aligned} E_x(f(X_{\xi_n^1});\ A) &\geqq \liminf_n E_x(f(X_{\xi_n^1});\ A) \\ &\geqq E_x(Z(\omega);\ A) \\ &\geqq E_x(f(X_{\xi_\infty^1});\ A) \end{aligned} \tag{27}$$

which shows that $(f(X_{\xi_k^1}))_{k\in N\cup\{\infty\}}$ is a supermartingale adapted to $(\Omega, (\mathscr{F}_{\xi_k^1})_{k\in N\cup\{\infty\}}, P^x)$. Now consider the increasing sequence of $\{\mathscr{F}_t\}$-stopping times $(\xi_k^2)_{k\in N}$ defined by (11). Then, since for y fixed $P_{\xi_\infty} u(x,y) = u\hat{P}_{\xi_\infty}(x,y)$ for q.e. $x\in E$ similar arguments as those of the proof of Lemma 3 show that the sequence of random variables $(p(X_{\xi_\infty^1}), (p(X_{\xi_k^2}))_{k\in N})$ is a uniformly integrable supermartingale for every $x\in(E\setminus A_1\cap A_2)$. Therefore it turns out that $(f(X_{\xi_\infty^1}), (f(X_{\xi_\infty^2}))_{k\in N\cup\{\infty\}})$ is a supermartingale. And so we can apply the above arguments step by step to every increasing sequence of stopping times $(\xi_k^i)_{k\in N}\,(i\in I)$. Since $(\eta_n)_{n\in N}$ is defined by (13), (23) is proved.

2) From the proof of the 1st step, we have

$$P_{\eta_n} u(x,y) = u\hat{P}_{\hat{\eta}_n} u(x,y)$$

where $(\hat{\eta}_n)_{n\in N}$ is an increasing sequence of $\{\hat{\mathscr{F}}_t\}$-stopping times, constructed step by step from $(\eta_n)_{n\in N}$ as in the proof of Lemma 3. The proof of Lemma 3 applies without change to the sequence (η_n) and so the supermartingale $(p(X_{\eta_n}))_{n\in N}$ is uniformly integrable for every $x\in(E\setminus A_1\cap A_2)$. Using (23) and the arguments of the proof of (27), we have

$$f(x)\geqq E_x(f(X_{\eta_\infty});\ \eta_\infty<\zeta) \qquad \eta_\infty=\lim_n \eta_n \tag{28}$$

for every $x\in(E\setminus A_1\cap A_2)\cap((f<+\infty)\setminus e_1\cup e_2)$. This implies (21) with $\bar{e}=e_1\cup e_2\cup\bar{e}_1$.

Remarks. 1) To show that $(g_k)_{k\in N}$ is a supermartingale adapted to $(\Omega, (\mathscr{F}_{\xi_k^1})_{k\in N}, P^x)$ for each x in $(E\setminus e_1\cup e_2)\cap\{\mathfrak{h}<+\infty\}$, we don't have to use the continuity of the paths of the process X. Hence, if $\xi_\infty^1=T_{A_1\cap A_2}$ a.s. P^x on $\{T_{A_1\cap A_2}<\zeta\}$ and if a.s. P^x, $\exists\, k(\omega)\in N$ such that $\xi_k^1(\omega)=T_{A_1\cap A2}(\omega)$, then

$$f(x)\geqq E_x(f(X_{T_{A_1\cap A_2}});\ T_{A_1\cap A_2}<\zeta), \tag{29}$$

whenever f is u.m., non-negative and satisfies 3) of Lemma 4. In [12], Šur showed that if A_1 and A_2 are closed for the initial topology, a sequence of stopping times with the above mentioned properties can be constructed and (29) is true for every x in E. But in general (29) does not seem to be true for x belonging to the semi-polar set $A_1\cap A_2\setminus(A_1\cap A_2)^r$, which is nothing else than the semi-polar set $\bar{e}$ of our Lemma 4.

2) The difficulty of the proof comes from the fact that, although $g_k(\omega)$ converges to an integrable random variable $Z(\omega)$, we don't have in general the inequality

$$\lim_k E_x(g_k(\omega))\geqq E_x(Z(\omega)),$$

since the supermartingale $(g_k(\omega))_{k\in N}$ has signed values. If, however, the potential p is *uniformely integrable* (in particular if $p\equiv 0$), then the above inequality is true for every point x such that $(p(X_t), P^x)$ is a supermartingale of class (D). This special case is treated in our previous work [9b].

3. The Fine Boundary Minimum Principle

In this section, we describe the fine BMP for some class of finely l.s.c. functions in E under hypothesis (B_2). The main results are Theorem 6 and its Corollary 7.

The following proposition is proved without the duality hypothesis and seems to have some independent interest.

Proposition 5. *Let $X=(\Omega,\mathscr{F},\mathscr{F}_t,X_t,\theta_t,\zeta,P^x)$ be a transient standard process, A, a n.b. finely closed subset of E and f a numerical u.m. function in E. Then*

a) *the set*

$$\mathrm{U}=(E\setminus A)\cap\{x\,|\,E_x(f^-(X_{T_A});\ T_A<\zeta)<+\infty\} \tag{30}$$

is finely open, and

b) *the function*

$$x\to F(x)=E_x(f(X_{T_A});\ T_A<\zeta) \tag{31}$$

is well defined and finely u.s.c. in U.

Proof. By our convention (7) the function $F(x)$ is well defined in U; it may have the value $+\infty$.

1) Suppose first that f is non-negative and let $B \subset \mathbf{U}(\equiv E \setminus A)$ be finely open with the fine closure contained in U. It suffices to prove that f is finely u.s.c. in B. I claim that for every n.b. set $C \subset B$ and for $x \in C^r \cap B$

$$F(x) \geqq \inf \{F(y):\ y \in C\}. \tag{32}$$

In fact, let $K \subset B$ be compact and denote by I_c the infinimum on the right side of (32). Then

$$E_x(F(X_{T_K});\ T_K < T_A) \geqq I_c \,.\, P^x(T_K < T_A)$$

since $X_{T_K}(\omega) \in K \subset C$ a.s. P^x in $\{T_K < \zeta\}$. On the other hand

$$\begin{aligned} F(X_{T_K}) &= E_{X_{T_K}}(f(X_{T_A});\ T_A < \zeta) \\ &= E_x(f(X_{T_A}) \cdot \chi_{\{T_A<\zeta\}} \circ \theta_{T_K} | \mathscr{F}_{T_K}). \end{aligned}$$

Hence

$$\begin{aligned} E_x(F(X_{T_K});\ T_K < T_A) &= E_x(f(X_{T_K + T_A(\theta_{T_K})}) \cdot \chi_{\{T_K + T_A(\theta_{T_K})\}};\ T_K < T_A < \zeta) \\ &= E_x(f(X_{T_A});\ T_K < T_A < \zeta) \\ &\leqq F(x) \end{aligned}$$

so that

$$F(x) \geqq I_c \,.\, P^x(T_K < T_A). \tag{33}$$

Now, if $x \in C^r \subset$ fine closure of $B \subset C$, then $P^x(T_C < T_A) = 1$. By a theorem of Hunt ([2a], (I.10.19)), there is an increasing sequence $\{K_n\}$ of compact subsets of C such that $T_{K_n} \downarrow T_C = 0$ a.s. P^x. Hence $\lim_n P^x(T_{K_n < T_A}) \uparrow 1$, and (32) follows from (33). Now let J_c be the interval $(-\infty, c) \subset R$ and define $W = F^{-1}(J_c) \cap B$. If $x \in W$ then by (32) x is irregular for $B \setminus W$, and so W is finely open.

2) It follows from the 1st step that for an arbitrary f the set $\mathbf{U}$ defined by (30) is finely open. Furthermore the proposition is true for every function f bounded from below on E. Let f be arbitrary and define $f_n = f \vee (-n)$. Then

$$F_n(x) = E_x(f \vee (-n)\,(X_{T_A});\ T_A < \zeta)$$

is finely u.s.c. on $E \setminus A$, and it is not difficult to see that $F_n(x) \downarrow F(x)$ for x in $\mathbf{U}$.

Remark. The integral $E_x(f(X_{T_A});\ T_A < \zeta)$ makes also sense if we suppose $E_x(f^+(X_{T_A});\ T_A < \zeta) < +\infty$ and adopt the convention that $E_x(f(X_{T_A});\ T_A < \zeta)$ is $-\infty$ if $E_x(f^-(X_{T_A});\ T_A < \zeta) = -\infty$. In this case F is finely l.s.c. on the finely open set $(E \setminus A) \cap \{x | E_x(f^+(X_{T_A});\ T_A < \zeta) < +\infty\}$ and consequently F is finite and finely continuous in $(E \setminus A) \cap \{x | F(x) \text{ finite}\}$.

Theorem 6. *Assume* (B_2). *Let* $(A_i)_{i \in N}$ *be a sequence of u.m. finely closed subsets of E. Let f be a numerical function on E such that*

1) $f > -\infty$, *cofinely l.s.c. in E.*

2) $f \geqq -p$ *q.e. in E where p is a potential, finite on the cofine closure of* $E \setminus A$, *where* $A \equiv \bigcap_{i=1}^{\infty} A_i$.

3) For $i = 1, 2, \ldots$:

$$f(x) \geqq E_x(f(X_{T_{A_i}});\ T_{A_i} < \zeta) \quad q.e. \tag{34}$$

Then:

a) *There is a semi-polar set* $\bar{e} \subset E$ *such that*

$$f(x) \geqq E_x\big(f(X_{T_A});\ T_A < \zeta\big) \qquad \text{for } x \in \big(E \setminus (A \cup \bar{e})\big) \cap (f < +\infty). \tag{35}$$

b) *Furthermore*

$$f(x) \geqq E_x\big(f(X_{T_A});\ T_A < \zeta\big) \quad \text{for } x \in (E \setminus A) \cap \big\{x \mid E_x\big(f(X_{T_A});\ T_A < \zeta\big) < +\infty\big\}. \tag{36}$$

Proof. Consider first k sets $A_1, A_2, \dots, A_k$. By 3) there are then k polar sets $e_1, \dots, e_k$ such that

$$f(x) \geqq E_x\big(f(X_{T_{A_i}});\ T_{A_i} < \zeta\big) \quad \text{for } x \in E \setminus e_k \ (i = 1, \dots, k).$$

By Corollary 2 there are a polar set $\bar{e}_k$ and a sequence of stopping times $(\eta_n^k)_{n \in N}$ such that $\lim_n \eta_n^k = T_k$ a.s. P^x for $x \in E \setminus \bar{e}_k$, where $T_k = T_{\bigcap_{i=1}^k A_i}$. Lemma 4, applied to this finite case, shows that: $\big(f(X_{\eta_n^k})\big)_{n \in N}$ is a supermartingale adapted to $\big(\Omega, (\mathscr{F}_{\eta_n^k})_{n \in N}, P^x\big)$ and that

$$f(x) \geqq E_x\big(f(X_{T_k});\ T_k < \zeta\big)$$

for every x in $\left(E \setminus \left(\bigcap_{i=1}^k A_i\right) \cup \bar{e}_k \cup \left(\bigcup_{i=1}^k e_i\right)\right) \cap (f < +\infty)$. It is not difficult to see that $\big(f(X_{T_k}) \cdot \chi_{\{T_k < \zeta\}}\big)_{k \in N}$ is a supermartingale adapted to $\big(\Omega, (\mathscr{F}_{T_k}), P^x\big)$ and that $f(X_\Pi) \cdot \chi_{\{\Pi < \zeta\}}$ is P^x-integrable, where $\Pi = \lim_k T_k$, for every x in

$$\left(E \setminus \bigcup_{k=1}^\infty (e_k \cup \bar{e}_k)\right) \cap \{p < +\infty\} \cap \{f < +\infty\}.$$

On the other hand, since

$$P_{T_k} u(x, y) = u P_{\hat{T}_k}(x, y), \qquad \text{where } \hat{T}_k = \hat{T}_{\bigcap_{i=1}^k A_i} \qquad (k = 1, 2, \dots)$$

it can be shown as in the proof of Lemma 3 that:

For every x fixed in E:

$$P_\Pi(x, y) = u P_{\hat{\Pi}}(x, y) \qquad \text{for } \forall\, y \in \text{cofine interior of } \bigcap_{i=1}^\infty A_i$$

$$P_\Pi u(x, y) = u P_{\hat{\Pi}}(x, y) \ (\text{q.e.}), \qquad \text{where } \hat{\Pi} = \lim_k \hat{T}_k.$$

Hence the proof of Lemma 3 applies without change to the sequence of stopping times (T_k), and it turns out that the supermartingale $\big((p(X_{T_k}))_{k \in N}, P^x\big)$ is uniformly integrable for every x in $(E \setminus A)$. Therefore the sequence of random variables $\big((f(X_{T_k}))_{k \in N}, f(X_\Pi)\big)$ forms a supermartingale adapted to $\big(\Omega, (\mathscr{F}_{T_k})_{k \in N}, \mathscr{F}_\Pi, P^x\big)$ for any such x. Now we repeat the same arguments with the increasing sequence of $\{\mathscr{F}_t\}$-stopping times $(\Pi_k^1)_{k \in N}$ defined by (16), and so on. Finally, we get: For every x in $\left(E \setminus \bigcap_{i=1}^\infty A_i\right)$, $\big(p(X_{\eta_n})\big)_{n \in N}$ is an uniformely integrable supermartingale, where (η_n) is the sequence of stopping times defined in Corollary 2, and for every x in

$$\left(E \setminus \left(\bigcap_{i=1}^\infty A_i\right) \cup \left(\bigcup_{i=1}^\infty e_k \cup \bar{e}_k\right)\right) \cap (f < +\infty)$$

$$\big(f(X_{\eta_n})\big)_{n \in N} \text{ is a supermartingale adapted to } \big(\Omega, (\eta_n)_{n \in N}, P^x\big). \tag{37}$$

But by Corollary 2 there is a polar set $\bar{e}' \subset E$ such that

$$\lim_n \eta_n = T_A \quad \text{a.s. } P^x \text{ on } \{T_A < \zeta\}.$$

Setting $\bar{e} \equiv \bar{e}' \cup \left(\bigcup_{k=1}^{\infty} e_k \cup \bar{e}_k\right)$, the proof for (35) is exactly the same as for (27), and we have finally for every x in $(E \setminus A \cup \bar{e}) \cap (f < +\infty)$

$$f(x) \geqq \liminf_n E_x(f(X_{\eta_n}); \eta_n < \zeta)$$

and

$$f(x) \geqq E_x(f(X_{T_A}); T_A < \zeta).$$

To prove (36), consider first the function

$$F(x) = E_x(f(X_{T_A}); T_A < \zeta)$$

which by Proposition 5 is well defined and finely u.s.c. on the finely open set $(E \setminus A) \cap \{x | E_x(f^-(X_{T_A}); T_A < \zeta) < +\infty\}$. By (1.b), § 1, there is a finely l.s.c. function f^* such that $f \geqq f^*$ and $\{f > f^*\}$ is polar. Hence on the finely open set:

$$\mathbf{U} = (E \setminus A) \cap \{p < +\infty\} \cap \{x | F(x) < +\infty\}$$

the function $u(x) = f^*(x) - F(x)$ is well defined and finely l.s.c. But for $x \in U \cap (E \setminus \bar{e})$, $u(x)$ is non negative by (35). Since $\bar{e}$ is polar, $u(x)$ is non negative for any x in $\mathbf{U}$.

Remark. For every x in $\left(E \setminus A \cup \bigcup_{i=1}^{\infty} e_i\right) \cap \{p < +\infty\}$, the supermartingale $f(X\eta_n)\chi_{\{\eta_n < \zeta\}}$ converges a.s. P^x to a P^x-integrable randon variable $Y(\omega)$. We know from the proof of the theorem that $E_x(Y(\omega)) \geqq E_x(f(X_{T_A}); T_A < \zeta)$ for x in $(E \setminus \bar{e}) \cap \{p < +\infty\}$, and therefore $f(X_{T_A})\chi_{\{T_A < \zeta\}}$ is P^x-integrable for such x. Consequently the set $(E \setminus A \cup \bar{e})$ is contained in $\{x | F(x) < +\infty\}$.

Corollary 7 (the fine BMP). *Assume* (B_2). *Let U be a.u.m. finely open subset of E. Let f be a numerical function defined on U such that:*

1) *$f > -\infty$, finely l.s.c.*

2) *$f \geqq -p$ in U, where p is a potential, finite on the cofine closure of U.*

3) *There is a base $(V_\alpha)_{\alpha \in I}$ of the fine topology in U, consisting of finely open sets V_α ($\alpha \in I$) such that for every $\alpha \in I$, the fine closure of V_α is contained in U and:*

$$f(x) \geqq E_x(f(X_{\tau_\alpha}); \tau_\alpha < \zeta) \quad \textit{for in } x \textit{ in } V_\alpha$$

where $\tau_\alpha = T_{E \setminus V_\alpha}$.

4) $\text{fine-}\liminf_{x \in U, x \to y} f(x) \geqq 0$ *for every point y of the fine boundary of U.*

Then for every finely open subset V of U we have:

$$f(x) \geqq E_x(f(X_{\tau_V}); \tau_V < \tau_U) \quad \textit{for q.e. } x \textit{ in } V \cap (f < +\infty). \tag{38}$$

Furthermore f is non-negative in U.

Proof. Define the function $\bar{f} = E \to]-\infty, +\infty]$ as follows:

$$\bar{f} \equiv \begin{cases} f \text{ in } U \\ 0 \text{ in } E \setminus U. \end{cases}$$

Then $\bar{f}$ is finely l.s.c. and $> -\infty$ by 4).

Due to (1.b), §1, there is a co-finely l.s.c. function f^* defined on E such that $\bar{f} \geqq f^*$ and $(\bar{f} > f^*)$ is semi-polar, hence polar. Consequently $f^* > -\infty$ since $E \setminus \{\bar{f} > f^*\}$ is cofinely dense in E. Clearly $f^* \geqq -p$ q.e. in E by 2).

Let V be a finely open subset of U. By 3) there exists an open covering of V by elements of the base $(V_\alpha)_{\alpha \in I}$; Since the fine topology is quasi-Lindelöf ([5], Theo. 8.1), there is a sequence $(V_i)_{i \in N}$ of elements of $(V_\alpha)_{\alpha \in I}$ such that:

$$\chi \bigcup_{n=1}^{\infty} V_n = \chi_V \qquad \text{q.e.}$$

p has finite values on the cofine closure of $\bigcup_{n=1}^{\infty} V_n$ since this set is contained in the cofine closure of U.

Setting $A_i = E \setminus V_i$ $(i \in N)$, the function f^* and the sequence of finely closed sets $(A_i)_{i \in N}$ satisfy all the assumptions of Theorem 6. Hence (35) implies:

$$f^*(x) \geqq E_x(f^*(X_{T_A}); T_A < \zeta) \qquad \text{for q.e. } x \text{ in } V \cap (f^* < +\infty)$$

where $A = \bigcup_{i=1}^{\infty} A_i$.

Since f^* is equal to f q.e. in U and equal to 0 q.e. in $E \setminus U$, (38) follows immediately from the above inequality. Furthermore if we put $V = U$ in the above equality, then:

For q.e. x in $U \cap (f < +\infty)$:

$$\begin{aligned} f(x) &= f^*(x) \\ &= E_x(f^*(X_{\tau_U}); \tau_U < \zeta) \\ &= E_x(\bar{f}(X_{\tau_U}); \tau_U < \zeta). \end{aligned}$$

Since $X_{\tau_U}(\omega) \in E \setminus U$ a.s. P^x, it follows that $f(x) \geqq 0$ for q.e. x in U, hence f is non-negative everywhere in U.

4. Applications to Axiomatic Potential Theory

As we mentioned in the introduction, one can construct from a Brelot harmonic space satisfying the domination principle a sheaf of finely harmonic functions on the class of finely open sets. The most crucial and basic steps in developing Fuglede's theory of fine harmonicity (c.f. [6]) consist in establishing *a fine BMP* for finely hyperharmonic functions and in asserting the continuity for the fine topology of these functions. These properties have been proved by probabilistic methods in [9b]. Applying the results of the last section, we will give in this section a more general form of the fine BMP, which holds both for elliptic and parabolic harmonic spaces provided a Green function exists.

(4.a) With a strong harmonic space $(\mathcal{H}, E)$ (Brelot or Bauer), where E has countable base and the constant 1 is hyperharmonic, one can associate a Hunt process $X=(\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, \zeta, P^x)$ with continuous paths such that:

The cone of X-excessive functions is identical to the cone of non-negative hyperharmonic functions.

The potential kernel $V(x, dy)$ of the process X maps the function 1 into a *strict* potential of $(\mathcal{H}, E)$ and is called an *admissible kernel.*

All the potential-theoretic notions of $(\mathcal{H}, E)$ can be interpreted by those of X (see [7] and [1b]).

(4.b) Suppose we have a strong Brelot or Bauer harmonic space $(\mathcal{H}, E)$ where the state space E has countable base and the function 1 is hyperharmonic. A function $u(x, y)$ defined in $E\times E$ is called a *Green function* if:

1) $u(x, y)$ is l.s.c. in $E\times E$ and continuous off the diagonal.

2) For each fixed y in E, $x\to u(x, y)$ is a potential in E with carrier $\{y\}$.

3) Each potential p in E can be represented in a unique way as

$$p(x)=\int u(x, y)\, m_p(dy) \tag{39}$$

where $m_p(dy)$ is a non-negative Radon measure on E.

4) The map $y\to u(\cdot, y)$ from E into the cone $\mathcal{P}$ of potentials on E is continuous for the T-topology of this cone. One can define the relative T-topology on the set $\mathcal{P}_0$ of potentials on E with one point carrier as follows: Take the weakest topology on this set such that the map $p\to$ carrier (p) from $\mathcal{P}_0$ onto E is continuous, and the map $p\to p(x)$ of elements of $\mathcal{P}_0$ with carrier $\neq\{x\}$ into R^+ is also continuous for every fixed x in E.

For a Brelot harmonic space such that for every point y in E all the potentials in E with the same singleton carrier $\{y\}$ are proportional (the so-called *case of unicity*), a classical result of Hervé showed that a Green function for the given harmonic space always exists ([8], Chap. III).

(4.c) Recently Taylor ([13], Th. (5.4)) proved the following result: Given a strong harmonic space $(\mathcal{H}, E)$ (Brelot or Bauer) where the state space E has countable base, 1 is hyperharmonic and a Green function exists, there is a Green function $G(x, y)$ of $(\mathcal{H}, E)$ and a positive Radon measure $m(dy)$ on E such that:

1) $V(x, dy)=G(x, y)\,.\,m(dy)$ is an admissible kernel of $(\mathcal{H}, E)$.

We denote by $(P_t)_{t\geqq 0}$ and by $X=(\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, \zeta, P^x)$ the corresponding semi-group and the Hunt process associated with $(\mathcal{H}, E)$.

2) $V^*(y, dx)=G(x, y)\,.\,m(dx)$ is the potential kernel of a transient Feller semi-group $(\hat{P}_t)_{t\geqq 0}$.

3) $(P_t)_{t\geqq 0}$ and $(\hat{P}_t)_{t\geqq 0}$ satisfy the duality hypothesis.

It is proved furthermore in [13] that $(P_t)_{t\geqq 0}$ and $(\hat{P}_t)_{t\geqq 0}$ are in duality in the sense of Kunita-Watanabe i.e., the hypothesis (K-W) (§ 1, (1.f)) is satisfied.

Definition 2 ([6]). Let **U** be a finely open subset of a strong harmonic space $(\mathcal{H}, E)$ (Brelot or Bauer). Suppose that E has countable base, the function 1 is hyperharmonic and finely open sets are nearly Borel. A numerical function f defined in **U** is called *finely hyperharmonic* if:

1) $f > -\infty$ and finely l.s.c. in $\mathbf{U}$.

2) There is a base $(V_\alpha)_{\alpha \in I}$ of the fine topology in $\mathbf{U}$ such that for every $\alpha \in I$ the fine closure of V_α is contained in $\mathbf{U}$ and

$$f(x) \geqq \hat{R}_f^{E \setminus V_\alpha}(x) \quad \text{for } x \text{ in } V_\alpha. \tag{40}$$

Using the Hunt process X associated with $(\mathscr{H}, E)$ (see (4.a)), (40) can be expressed as follows:

$$f(x) \geqq E_x\big(f(X_{T_{E \setminus V_\alpha}});\, T_{E \setminus V_\alpha} < \zeta\big) \quad \text{for } x \text{ in } V_\alpha.$$

We can now state the fine BMP for finely hyperharmonic functions:

Theorem 8. *Suppose we have an harmonic space $(\mathscr{H}, E)$ (Brelot or Bauer) where the state space E has countable base, 1 is hyperharmonic and a Green function exists. Assume furthermore that semi-polar sets are polar. Let f be a finely hyperharmonic function in the finely open set* $\mathbf{U}$ *such that:*

a) $\underset{x \in U,\, x \to y}{\text{fine-lim inf}} f(x) \geqq 0$ *for every y in the fine boundary of* $\mathbf{U}$,

b) $f \geqq -p$ *in* $\mathbf{U}$ *where p is a potential in E.*

Then f is non-negative in $\mathbf{U}$.

Proof. By (4.c) we are actually in the case (B_2) where the process X is the Hunt process associated with the given harmonic space $(\mathscr{H}, E)$ and $\hat{X}$ is a strong Feller process with semi-group $(\hat{P}_t)_{t \geqq 0}$. Furthermore p is a potential of $(\mathscr{H}, E)$ in E hence it is a X-potential in the sense of Definition 1.b). Define $\bar{f}$ to be the function equal to f in U and 0 outside of U and let f^- be $\sup(-\bar{f}, 0)$. It can be proved from assumptions a) and b) that $p - f^-$ is a non-negative finely hyperharmonic function on E, hence by ([9 b], Theo. 7) this function is actually an hyperharmonic function on E. Since $0 \leqq p - f^- \leqq p$, $p - f^-$ is even a potential in E which we call q. Then:

$$f^- = p - q. \tag{41}$$

Let $e = \{p = +\infty\}$, then e is a closed, polar set. Since $f(x) > -\infty$ for every x in U, it follows from (41) that $\bar{f}$ is locally bounded from below for the initial topology at each point of e.

1) Suppose first that U is relatively compact, i.e., there is an open subset ω of E such that $U \subset \omega \subset \bar{\omega} \subset E$. Since $e_0 = \bar{\omega} \cap e$ is a compact set, there exists an open set ω_1 containing e_0 such that f is bounded from below on ω_1. Now let:

$$\omega_n = \omega_1 \cap (p > n) \quad (n = 1, 2, \ldots). \tag{42}$$

We will show that for every finely open set $W \subset U \setminus e$ we have:

$$\bar{f}(x) \geqq E_x\big(\bar{f}(X_{\tau_W});\, \tau_W = \zeta\big) \quad \text{for q.e. } x \in W \cap (\bar{f} < +\infty).$$

For $n = 1, 2, \ldots$, setting $W = (W \setminus \omega_n) \cup (W \cap \omega_n)$. For each $n \in N$, $p(x)$ is finite on the cofine closure of $W - \omega_n$ since this set if contained in the closed set $\bar{\omega} - \omega_n$, hence by Corollary 7:

$$\bar{f}(x) \geqq E_x\big(\bar{f}(X_{\tau_{W \setminus \omega_n}});\, \tau_{W \setminus \omega_n} < \zeta\big) \quad \text{q.e.} \quad x \text{ in } (W \setminus \omega_n) \cap (\bar{f} < +\infty). \tag{43}$$

On the other hand for each x in W $p(x)$ is finite hence it follows from (42) that:

$$\lim_n T_{\omega_n} = +\infty \quad \text{a.s. } P^x.$$

Hence:

$$\lim_n E_x(\bar{f}(X_{\tau_{W\setminus\omega_n}});\ \tau_{W_n\setminus\omega_n}=\tau_W<\zeta)=E_x(\bar{f}(X_{\tau_W});\ \tau_W<\zeta)$$

and

$$\liminf_n E_x(\bar{f}(X_{\tau_{\omega_n}});\ \tau_{\omega_n}<\tau_\omega)=0$$

since f is bounded from below on ω_1.

It follows from (43) that:

$$\begin{aligned}\bar{f}(x)&\geqq \lim_n E_x(\bar{f}(X_{\tau_{W\setminus\omega_n}});\ \tau_{W\setminus\omega_n}=\tau_W<\zeta)\\&\quad+\liminf_n E_x(\bar{f}(X_{T_\omega});\ T_\omega<\tau_W)\\&=E_x(\bar{f}(X_{\tau_W});\ \tau_W<\zeta)\qquad\text{for q.e. } x \text{ in } W\cap(\bar{f}<+\infty).\end{aligned}$$

If we put $W=U\setminus e$ since $X_{\tau(\omega)}(\omega)\in E\setminus U$ a.s. P^x, we get finally:

$$\begin{aligned}\bar{f}(x)&=f(x)\\&\geqq E_x(\bar{f}(X_{\tau_U});\ \tau_U<\zeta)\\&=0\qquad\text{for q.e. } x \text{ in } U\cap(f<+\infty).\end{aligned}$$

$f(x)$ is non-negative q.e. in U, hence everywhere.

2) Suppose now that U is an arbitrary u.m. finely open subset of E. By definition of a potential (Def. 1.b)), there exists an increasing sequence of compact sets $(K_n)_{n\in N}$ of E such that $K_n\subset\mathring{K}_{n+1}$, $\bigcup_{n=1}^{\infty}\mathring{K}_n=E$ and

$$\lim_n E_x(p(X_{\tau_{\mathring{K}_n}});\ \tau_{\mathring{K}_n}<\zeta)=0\qquad\text{a.s. } m(dx).$$

For fixed m define $U_m=U\cap\mathring{K}_m$. Then we have

$$\begin{aligned}(\tau_{U_m}<\zeta)&=(\tau_{U_m}<\tau_{\mathring{K}_m})\cup(\tau_{U_m}\geqq\tau_{\mathring{K}_m})\\&=(\tau_U=\tau_{U_m};\ \tau_{U_m}<\tau_{\mathring{K}_m})\cup(\tau_{U_m}\geqq\tau_{\mathring{K}_m}).\end{aligned}$$

Hence it follows from 1) that

$$\begin{aligned}f^*(x)&\geqq E_x(f^*(X_{\tau_{U_m}});\ \tau_{U_m}<\zeta)\\&\equiv E_x(f^*(X_{\tau_{U_m}});\ \tau_{U_m}=\tau_U;\ \tau_{U_m}<\tau_{\mathring{K}_n})+E_x(f^*(X_{\tau_{U_m}});\ \tau_{U_m}\geqq\tau_{\mathring{K}_m})\end{aligned}$$

for $x\in U_m\cap(p<+\infty)\cap(x\,|\,E_x(f^*(X_{\tau_{U_m}})<+\infty))$.

But on the other hand, since $X_{\tau_U}\in E\setminus U$ a.s. on $\{\tau_U<\zeta\}$, by the definition of $\bar{f}$ and f^* the first integral on the right hand side is equal to zero. Furthermore, by 2) the second integral can be estimated as follows:

$$E_x(f^*(X_{\tau_{U_m}});\ \tau_{U_m}\geqq\tau_{\mathring{K}_m})\geqq -E_x(p(X_{\tau_{\mathring{K}_m}});\ \tau_{U_m}\geqq\tau_{\mathring{K}_m}).$$

Letting $m\to+\infty$, we set that $f^*(x)$ and hence $f(x)$ is non-negative. $x\in U\cap(p<+\infty)$, a.s. $m(dx)$. Therefore f is non-negative in $U\cap(p<+\infty)$ since the above mentioned set is finely dense in $U\cap(p<+\infty)$.

Furthermore the set $\{x\,|\,p(x)=+\infty\}$ is polar, thus $U\cap(p<+\infty)$ is finely dense in U and by the fine l.s. continuity of f^* we obtain

$$f(x)\geqq f^*(x)\geqq 0\qquad\text{for } x \text{ in } U.$$

Remarks. 1) Suppose that $(\mathscr{H}, E)$ is a Brelot harmonic space and that we have the case of unicity. Since semi-polar sets are polar it turns out that the convergence axiom is satisfied, and hence by ([8], Th. 25.3) we have the domination principle.

In this case there is no difference in supposing that $(\mathscr{H}, E)$ is elliptic or parabolic (see Revue Roum. Math. Pures et Appl. 12, 1489–1502 (1967)). On the other hand Lemma 1 could be proved in a much easier way by using the fact that the capacity $A \to R_1^A(x)$ has the Choquet property (see [9b]). Under the duality hypothesis and the assumption that X-excessive functions are l.s.c., Blumenthal and Getoor proved in [2b] that the hypothesis "semi-polar sets are polar" is equivalent to the maximum principle, i.e.:

Let μ be a non negative Radon measure with compact support K in E, then the potential of μ, $G\mu(x) = \int u(x, y) \,.\, \mu(dy)$, attains its maximum on K.

For an arbitrary strong harmonic space $(\mathscr{H}, E)$ the maximum principle is clearly weaker than the domination principle. However, if a Green function for $(\mathscr{H}, E)$ exists, by (4.c) we are in the case (B_1) and hence the hypothesis "semi-polar sets are polar" is equivalent to the maximum principle. But in our previous work [9c] we showed that in the case (B_2) every finite potential is semi-bounded. If also the domination principles holds for $(\mathscr{H}, E)$ then this fact is equivalent to the following:

A finite potential p on E (say of compact carrier $S(p)$) is continuous in E if p is continuous on $S(p)$ (Th. 10.15, [6]).

2) In [9b] we supposed that p is semi-bounded, i.e., is X-uniformly integrable in the sense of Definition 1.c). Then the hypothesis a) of Theorem 8 can be weakened to allow that y belongs q.e. to the fine boundary of U. However, if p is an arbitrary potential on E, this condition a) cannot be weakened, even in the classical case as shown by the following counter-example:

$E \equiv R^3$; $(\mathscr{H}, E) \equiv$ Newtonian potential theory.

$U \equiv$ Open unit ball minus the center and $f(x) = -|x|^{-1} + 1$.

Note that in the case (B_2), we can suppose in Theorem 8 that y belongs q.e. to the fine boundary of U and that p is finite.

3) For a strong Brelot harmonic space with the domination principle, Fuglede ([6], Lemma 10.14) proved that the fine BMP in the form of Theorem 8 is equivalent to the following statement:

For every potential p the relation $\hat{R}_p^S = p$ holds for every finely closed set S such that p is finely harmonic in $E \setminus S$ (i.e. both p and $-p$ are finely hyperharmonic on this set) and finite on the fine boundary of S.

Appendix

Given a strong harmonic space (Brelot or Bauer), where the state space E has countable base and the function 1 is hyperharmonic, we will show that our proof of the fine BMP gives again in a rather simple way the classical BMP where the initial topology on E is concerned.

Let $\mathscr{U}$ be an open relatively compact subset of E and let f be an hyperharmonic function in $\mathscr{U}$. Suppose that

$$\liminf_{x\in\mathscr{U},\, x\to y} f(x) \geqq 0 \qquad \text{for every } y \text{ on the boundary of } \ .$$

Then f is non-negative in $\mathscr{U}$.

By the definition of hyperharmonicity of f in $\mathscr{U}$, there exists a sequence $(V_n)_{n\in N}$ of open sets with closure in $\mathscr{U}$ such that:

1) $f(x) \geqq E_x\big(f(X_{T_{E\setminus V_n}}); \ T_{E\setminus V_n} < \zeta\big)$ for x in E $(n=1,2,\ldots)$,

2) $\bigcup_{n=1}^{\infty} V_n = \mathscr{U}$

where $X=(\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \theta_t, \zeta, P^x)$ is a Hunt process associated with the given harmonic space. Now let $\bar{f}$ be the function equal to f in and equal to zero in $E\setminus\mathscr{U}$, and let T_i be the hitting-time of X for the closed set $E\setminus V_i$ $(i=1,2,\ldots)$. Consider the increasing sequence of stopping-times $(\eta_n^k)_{n\in N}$ constructed from $(T_i;\ i=1,2,\ldots,k)$ by formula (15). By the quasi-left continuity of the process X it is not difficult to see that

$$\lim_n \eta_n^k = \tau_k \qquad \text{a.s. } P^x \text{ for every } x \text{ outside of } \bigcap_{i=1}^{k}(E\setminus V_i) \setminus \Big(\bigcap_{i=1}^{k}(E\setminus V_i)\Big)^r,$$

where τ_k is the hitting-time of X for $\bigcap_{i=1}^{k}(E\setminus V_i)$ (see for example the proof of ([12], Lemma 2) and Remark 1 of Lemma 1, § 1).

Now, since $\bar{f}$ satisfies all the hypothesis of Lemma 1.b), § 1, we have

$$\bar{f}(x) \geqq E_x\big(\bar{f}(X_{\tau_k}); \ \tau_k < \zeta\big)$$

for x outside of a semi-polar set and $k=1,2,\ldots$.

Again by the quasi-left continuity of X we have $\lim_k \tau_k = T_{E\setminus\mathscr{U}}$ a.s. P^x for x outside of a semi-polar set. For such points we have

$$\bar{f}(x) \geqq E_x\big(\bar{f}(X_{T_{E\setminus\mathscr{U}}}); \ T_{E\setminus\mathscr{U}} < \zeta\big).$$

Since $X_{T_{E\setminus\mathscr{U}}}(\omega)$ belong a.s. P^x on $\{T_{E\setminus\mathscr{U}} < \zeta\}$ to the boundary of $\mathscr{U}$, we have $\bar{f}(x) = f(x) \geqq 0$ for x in $\mathscr{U}$ outside of semi-polar set. Hence f is non-negative in $\mathscr{U}$.

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