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On the Central Limit Problem for Sums with Random Coefficients

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1. Introduction

One classical form of the central limit problem considers the asymptotic distribution of weighted sums of independent, identically distributed random variables (r. vs). In this paper we consider the asymptotic distribution of such sums where the constant weights are replaced by a sequence of non-negative r. vs, i. e. we consider the sum of weighted, independent, identically distributed r. vs, where the weights are themselves r. vs.

Let $X_1, X_2, \ldots,$ be independent, identically distributed r. vs and for each $n=1, 2, \ldots,$ let $\alpha_{n1}, \ldots, \alpha_{nn}$ be non-negative r. vs which are independent of the $\{X_n\}$. We consider sums of the form

$$S_n = \sum_{k=1}^{n} \alpha_{nk} X_k, \qquad n=1, 2, \ldots.$$

Such sums can arise in various applications; one instance is in Brown and Fisher [2], where the r. vs $\alpha_{n1}, \ldots, \alpha_{nn}$ are proportions which are produced by a sampling process.

When the $\{\alpha_{nk}\}$ are constants, Jamison, Orey and Pruitt [4] have derived a sufficient condition for S_n to converge in probability to a constant. In Section 2 their condition is generalized to allow the $\{\alpha_{nk}\}$ to be r. vs. The generalized condition is obtained by "randomizing" their condition, i.e. by taking their condition on the $\{\alpha_{nk}\}$ and insisting that it hold in probability.

In Section 3 we consider the central limit problem for the $\{S_n\}$ and show how the martingale results of Brown and Eagleson [1] can be applied to obtain two sets of sufficient conditions for $\{S_n\}$ to converge in law to normality. One of these sets of conditions (Theorem 3) corresponds to the randomization of the known results when the $\{\alpha_{nk}\}$ are constants (see, for example Theorem V, 1.2, p. 153 of Hájek and Šidák [3]). We also point out how the assumption that the $\{X_n\}$ and the $\{\alpha_{nk}\}$ are independent can be weakened.

Finally, in Section 4 some examples are given and some remarks made.

2. The Weak Law

Theorem 1. *Let $X_1, X_2, \ldots$ be independent, identically distributed r. vs, and for each $n=1, 2, \ldots,$ let $\alpha_{n1}, \ldots, \alpha_{nn}$ be non-negative r. vs which are independent of*

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the $\{X_n\}$. *For* $y>0$, *let*

$$f(y)=y\,P(|X_1|\geqq y)$$

and

$$g(y)=\int_{[|X_1|<y]} X_1\,dP,$$

and assume that

$$\lim_{y\to\infty} f(y)=0, \tag{1}$$

and that

$$\lim_{y\to\infty} g(y) \quad \textit{exists and is equal to } \mu. \tag{2}$$

Assume further that

$$\max_{k\leqq n} \alpha_{nk} \to^{P} 0 \quad \textit{as} \quad n\to\infty, \tag{3}$$

and that there exists a finite constant C *for which*

$$\lim_{n\to\infty} P\left(\sum_{k=1}^{n} \alpha_{nk} > C\right)=0. \tag{4}$$

Then

$$\sum_{k=1}^{n} \alpha_{nk}(X_k-\mu) \to^{P} 0 \quad \textit{as} \quad n\to\infty.$$

Proof. Our proof is based on that of Theorem 1 of Jamison, Orey and Pruitt [4]. Assume without loss of generality that $\mu=0$ (or alternatively, consider the r.vs $\{X_n-\mu\}$ instead of $\{X_n\}$), and for $y>0$, let

$$h(y)=y^{-1}\int_{[|X_1|<y]} X_1^2\,dP.$$

Now both $f(y)\to 0$ and $g(y)\to 0$ as $y\to\infty$, by hypothesis, and $h(y)\to 0$, as shown in the proof of Theorem 1 of Jamison, Orey and Pruitt [4]. Thus if

$$f^*(y)=\sup_{z\geqq y} f(z),$$

$$g^*(y)=\sup_{z\geqq y} g(z),$$

and

$$h^*(y)=\sup_{z\geqq y} h(z),$$

then $f^*(y)$, $g^*(y)$, and $h^*(y)$ each tend to zero monotonically as $y\to\infty$.

For fixed $\varepsilon>0$, let $A=[\max_{k\leqq n} \alpha_{nk}>\varepsilon]$, and $B=\left[\sum_{k=1}^{n} \alpha_{nk}>C\right]$, and choose n so large that both $PA\leqq\varepsilon$ and $PB\leqq\varepsilon$.

Define $W_{nk}=\alpha_{nk} X_k I(|\alpha_{nk} X_k|<1)$, where $I(D)$ denotes the indicator function of the set D and let $\mathscr{C}_n=\mathscr{B}(\alpha_{n1},\ldots,\alpha_{nn})$ be the σ-field generated by the r.vs $\alpha_{n1},\ldots,\alpha_{nn}$. Then

$$\begin{aligned}
&P(W_{nk}\neq\alpha_{nk} X_k \text{ for some } k=1,2,\ldots,n)\\
&\quad\leqq \sum_{k=1}^{n} P\big(A^c B^c I(|\alpha_{nk} X_k|\geqq 1)\big)+PA+PB\\
&\quad= \sum_{k=1}^{n} E\big(I(A^c B^c)\, P(|\alpha_{nk} X_k|\geqq 1|\mathscr{C}_n)\big)+PA+PB\\
&\quad= \sum_{k=1}^{n} E\big(I(A^c B^c)\,\alpha_{nk} f(\alpha_{nk}^{-1})\big)+PA+PB\\
&\quad\leqq C f^*(\varepsilon^{-1})+\varepsilon+\varepsilon,
\end{aligned}$$

which is made arbitrarily small by taking ε small.

Thus it suffices to show that

$$T_n=\sum_{k=1}^{n} W_{nk} \to^{P} 0 \quad \text{as} \quad n\to\infty.$$

Now
$$E(T_n|\mathscr{C}_n)=\sum_{k=1}^n E\big(\alpha_{nk} X_k I(|\alpha_{nk} X_k|<1)|\mathscr{C}_n\big)$$
$$=\sum_{k=1}^n \alpha_{nk}\, g(\alpha_{nk}^{-1})$$
and thus
$$\begin{aligned}P(|T_n|>\delta|\mathscr{C}_n)&\leqq\delta^{-2} E(T_n^2|\mathscr{C}_n)\\ &=\delta^{-2}\{E((T_n-E(T_n|\mathscr{C}_n))^2|\mathscr{C}_n)+(E(T_n|\mathscr{C}_n))^2\}\\ &=\delta^{-2}\{\textstyle\sum_{k=1}^n E((W_{nk}-E(W_{nk}|\mathscr{C}_n))^2|\mathscr{C}_n)+(E(T_n|\mathscr{C}_n))^2\}\\ &\leqq\delta^{-2}\{\textstyle\sum_{k=1}^n E(W_{nk}^2|\mathscr{C}_n)+(E(T_n|\mathscr{C}_n))^2\}\\ &=\delta^{-2}\{\textstyle\sum_{k=1}^n \alpha_{nk}\, h(\alpha_{nk}^{-1})+(\sum_{k=1}^n \alpha_{nk}\, g(\alpha_{nk}^{-1}))^2\}\end{aligned}$$
so that, with a computation similar to a previous one,
$$\begin{aligned}P(|T_n|>\delta)&\leqq E\big(I(A^c B^c)\,P(|T_n|>\delta|\mathscr{C}_n)\big)+PA+PB\\ &\leqq\delta^{-2}\big(C.h^*(\varepsilon^{-1})+(C.g^*(\varepsilon^{-1}))^2\big)+\varepsilon+\varepsilon,\end{aligned}$$
and $T_n\to^p 0$ as $n\to\infty$. The proof is complete.

Corollary. *Let* (1), (2) *and* (3) *hold. If we also have*
$$\sum_{k=1}^n \alpha_{nk}\to^p \theta, \textit{ constant}, \quad \textit{as} \quad n\to\infty,$$
then
$$\sum_{k=1}^n \alpha_{nk} X_k\to^p \mu\,\theta \qquad \textit{as} \quad n\to\infty.$$

Proof. The proof follows immediately from the Theorem.

3. The Central Limit Theorem

In some practical situations one would observe S_n, knowing the value of n, whereas the r.vs $\{X_k\}$ and $\{\alpha_{nk}\}$ may be unobservable. Thus we need to consider the limiting distribution of $a_n S_n$ as $n\to\infty$, where $\{a_n\}$ is some sequence of norming constants, and to determine conditions under which the limit distribution is normal. However, to make the notation simpler, we shall absorb the norming constants into the $\{\alpha_{nk}\}$ and consider the convergence to normality of $S_n=\sum_{k=1}^n \alpha_{nk} X_k$.

One approach to the problem would be to randomize classical conditions as was done in Theorem 1 of Section 2. That is, take the known conditions on the $\{\alpha_{nk}\}$ for S_n to converge to normality, when the $\{\alpha_{nk}\}$ are constants, and make the convergences in these conditions hold in probability. By conditioning on the values taken by the $\{\alpha_{nk}\}$, the classical central limit theorem can be applied. This approach can certainly be used, but it will only work when the $\{\alpha_{nk}\}$ and the $\{X_k\}$ are independent. However, this assumption of independence can be weakened if we first note that, for each n, S_n is a sum of martingale differences; and then use the following result of Brown and Eagleson [1] (Corollary 1 to Theorem 1), restated here as a Theorem for convenience:

Consider a double array of random variables, which we take without loss of generality to be a triangular array, whose rows are martingale difference sequences. That is, for each $n=1,2,\ldots$ we have r.vs $X_{n1},\ldots,X_{nn}$ on a probability space $(\Omega,\mathscr{F},P)$, with sub σ-fields $\mathscr{F}_{n0}\subset\mathscr{F}_{n1}\subset\cdots\subset\mathscr{F}_{nn}$ of $\mathscr{F}$ such that X_{nk} is $\mathscr{F}_{nk}$-

measurable and $E(X_{nk}|\mathscr{F}_{n,k-1})=0$ a.s. for $k=1,2,\ldots,n$. Such an array is called a martingale triangular array. Let $S_n=X_{n1}+\cdots+X_{nn}$, $n=1,2,\ldots$.

Theorem 2. *If for a martingale triangular array* $(X_{nk},\mathscr{F}_{nk},1\leqq k\leqq n)$, $n=1,2,\ldots$,

$$\max_{k\leqq n}E(X_{nk}^2|\mathscr{F}_{n,k-1})\to^p 0 \quad \textit{as} \quad n\to\infty, \tag{5}$$

$$\textstyle\sum_{k=1}^n E(X_{nk}^2|\mathscr{F}_{n,k-1})\to^p \sigma^2 \ (\textit{constant}) \quad \textit{as} \quad n\to\infty, \tag{6}$$

and

$$\textstyle\sum_{k=1}^n E\big(X_{nk}^2\, I(|X_{nk}|\geqq\varepsilon)|\mathscr{F}_{n,k-1}\big)\to^p 0 \quad \textit{as} \quad n\to\infty \textit{ for all } \varepsilon>0, \tag{7}$$

then S_n *converges in law as* $n\to\infty$ *to* $N(0,\sigma^2)$.

Since for any $\varepsilon>0$,

$$\max_{k\leqq n}E(X_{nk}^2|\mathscr{F}_{n,k-1})\leqq\varepsilon^2+\textstyle\sum_{k=1}^n E\big(X_{nk}^2\, I(|X_{nk}|\geqq\varepsilon)|\mathscr{F}_{n,k-1}\big),$$

condition (7) implies condition (5). So in order to apply the Theorem, only (6) and (7) need be checked.

In what follows we will consider, for simplicity, only the case when the $\{\alpha_{nk}\}$ are non-negative and independent of the $\{X_k\}$. However, an advantage of obtaining our results through applications of Theorem 2 is that more general situations could be considered. In particular, all our results still hold when the $\{\alpha_{nk}\}$ are not necessarily non-negative, and when $\{\alpha_{n1},\ldots,\alpha_{nk}\}$ are independent of $\{X_k, X_{k+1},\ldots\}$, for each k. This will be the case, for instance, when α_{nk} depends only on $X_1,\ldots,X_{k-1}$; see for example Theorem 7.

In applying Theorem 2, for the same $\{X_{nk}\}$ one may choose different sequences of σ-fields $\{\mathscr{F}_{nk}\}$ and thus obtain different sufficient conditions. For the S_n we are considering there are two obvious choices of $\{\mathscr{F}_{nk}\}$, leading to the following two theorems:

Theorem 3. *Let* $X_1,X_2,\ldots$ *be independent, identically distributed r.vs with zero mean and unit variance and for each* $n=1,2,\ldots$, *let* $\alpha_{n1},\ldots,\alpha_{nn}$ *be non-negative r.vs which are independent of the* $\{X_n\}$. *In order that the sum*

$$S_n=\textstyle\sum_{k=1}^n \alpha_{nk}X_k$$

should converge in distribution to the standardized normal distribution, it is sufficient that

$$\max_{k\leqq n}\alpha_{nk}^2\to^p 0 \quad \textit{as} \quad n\to\infty, \tag{8}$$

and

$$\textstyle\sum_{k=1}^n \alpha_{nk}^2\to^p 1 \quad \textit{as} \quad n\to\infty. \tag{9}$$

Proof. Let $\mathscr{F}_{n0}=\mathscr{B}(\alpha_{n1},\ldots,\alpha_{nn})$ and $\mathscr{F}_{nk}=\mathscr{B}(\alpha_{n1},\ldots,\alpha_{nn},X_1,\ldots,X_k)$, $k=1,2,\ldots,n$. Clearly $\mathscr{F}_{n0}\subset\mathscr{F}_{n1}\subset\cdots\subset\mathscr{F}_{nn}$ and $\alpha_{nk}X_k$ is $\mathscr{F}_{nk}$-measurable. Further, because of the independence assumptions,

$$\begin{aligned}E(\alpha_{nk}X_k|\mathscr{F}_{n,k-1})&=\alpha_{nk}E(X_k|\mathscr{F}_{n,k-1})\\&=\alpha_{nk}E(X_k)=0.\end{aligned}$$

Thus for each n, $\{\alpha_{nk}X_k,\mathscr{F}_{nk},1\leqq k\leqq n\}$ is a sequence of martingale differences and we may apply Theorem 2.

For this sequence of martingale differences

$$E\big((\alpha_{nk} X_k)^2 | \mathscr{F}_{n,k-1}\big) = \alpha_{nk}^2 E(X_k^2 | \mathscr{F}_{n,k-1}) = \alpha_{nk}^2.$$

Thus (9) immediately implies (6) and it only remains to check condition (7).

Let $k(y) = \int_{[|X_1| \geqq y]} X_1^2 \, dP$. Then $k(0) = 1$ and $k(y)$ is a monotonically decreasing function of y which tends to zero as $y \to \infty$. Since $\max_{k \leqq n} \alpha_{nk} \to^P 0$ as $n \to \infty$,

$$k\big(\varepsilon (\max_{k \leqq n} \alpha_{nk})^{-1}\big) \to^P 0 \quad \text{as} \quad n \to \infty,$$

for all $\varepsilon > 0$. Hence

$$\begin{aligned}
&\textstyle\sum_{k=1}^n E\big((\alpha_{nk} X_k)^2 I(|\alpha_{nk} X_k| \geqq \varepsilon) | \mathscr{F}_{n,k-1}\big) \\
&\quad = \textstyle\sum_{k=1}^n \alpha_{nk}^2 E\big(X_k^2 I(|X_k| \geqq \varepsilon \alpha_{nk}^{-1})\big) \\
&\quad \leqq \textstyle\sum_{k=1}^n \alpha_{nk}^2 E\big(X_k^2 I(|X_k| \geqq \varepsilon (\max_{k \leqq n} \alpha_{nk})^{-1})\big) \\
&\quad = k\big(\varepsilon (\max_{k \leqq n} \alpha_{nk})^{-1}\big) \textstyle\sum_{k=1}^n \alpha_{nk}^2 \to^P 0 \quad \text{as} \quad n \to \infty
\end{aligned}$$

since $\sum_{k=1}^n \alpha_{nk}^2 \to^P 1$ as $n \to \infty$. The theorem is proved.

Making another choice of the σ-fields $\{\mathscr{F}_{nk}\}$ we obtain:

Theorem 4. *Let $X_1, X_2, \ldots$ be independent, identically distributed r.vs with zero mean and unit variance and for each $n = 1, 2, \ldots$, let $\alpha_{n1}, \ldots, \alpha_{nn}$ be non-negative r.vs which are independent of the $\{X_n\}$. In order that the sum*

$$S_n = \textstyle\sum_{k=1}^n \alpha_{nk} X_k$$

should converge in distribution to the standardized normal distribution, it is sufficient that

$$\max_{k \leqq n} \alpha_{nk} \to^P 0 \quad \text{as} \quad n \to \infty, \tag{10}$$

and

$$\textstyle\sum_{k=1}^n E(\alpha_{nk}^2 | \mathscr{H}_{n,k-1}) \to^P 1 \quad \text{as} \quad n \to \infty, \tag{11}$$

where $\mathscr{H}_{n0} = \{\phi, \Omega\}$ and $\mathscr{H}_{nk} = \mathscr{B}(\alpha_{n1}, \ldots, \alpha_{nk})$, $1 \leqq k \leqq n$, $n = 1, 2, \ldots$.

Proof. Let $\mathscr{F}_{n0}$ be the trivial σ-field and let $\mathscr{F}_{nk} = \mathscr{B}(\alpha_{n1}, \ldots, \alpha_{nk}, X_1, \ldots, X_k)$, $k = 1, \ldots, n$. Clearly $\mathscr{F}_{n0} \subset \mathscr{F}_{n1} \subset \cdots \subset \mathscr{F}_{nn}$ and $\alpha_{nk} X_k$ is $\mathscr{F}_{nk}$-measurable. Let $\mathscr{G}_{nk} = \mathscr{B}(\alpha_{n1}, \ldots, \alpha_{n,k+1}, X_1, \ldots, X_k)$, $k = 0, \ldots, n-1$, and $\mathscr{G}_{nn} = \mathscr{F}_{nn} = \mathscr{B}(\alpha_{n1}, \ldots, \alpha_{nn}, X_1, \ldots, X_n)$. Then $\mathscr{F}_{nk} \subset \mathscr{G}_{nk} \subset \mathscr{F}_{n,k+1} \subset \mathscr{G}_{n,k+1}$, $k = 0, \ldots, n-1$. Thus

$$\begin{aligned}
E(\alpha_{nk} X_k | \mathscr{F}_{n,k-1}) &= E\big(E(\alpha_{nk} X_k | \mathscr{G}_{n,k-1}) | \mathscr{F}_{n,k-1}\big) \\
&= E\big(\alpha_{nk} E(X_k | \mathscr{G}_{n,k-1}) | \mathscr{F}_{n,k-1}\big) = 0,
\end{aligned}$$

as

$$E(X_k | \mathscr{G}_{n,k-1}) = E X_k = 0.$$

Thus for each n, $\{\alpha_{nk} X_k, \mathscr{F}_{nk}, 1 \leqq k \leqq n\}$ is a sequence of martingale differences and we may again apply Theorem 2.

For this sequence of martingale differences,

$$\begin{aligned}
E\big((\alpha_{nk} X_k)^2 | \mathscr{F}_{n,k-1}\big) &= E(X_k^2) E(\alpha_{nk}^2 | \mathscr{F}_{n,k-1}) \\
&= E(\alpha_{nk}^2 | \mathscr{H}_{n,k-1}).
\end{aligned}$$

So (11) implies (6) and it only remains to check condition (7). Let the function $k(.)$ be as in the proof of Theorem 3, and let $\{\delta_n\}$ be a positive real sequence which tends monotonically to zero slowly enough so that if

$$A_n = [\max_{k \leqq n} \alpha_{nk} < \delta_n],$$

then

$$\lim_{n\to\infty} PA_n = 1. \tag{12}$$

Thus

$$\begin{aligned}\textstyle\sum_{k=1}^{n} & E\big((\alpha_{nk} X_k)^2 I(|\alpha_{nk} X_k| \geqq \varepsilon) | \mathscr{F}_{n,k-1}\big)\\ &= \textstyle\sum_{k=1}^{n} E\big(E(\alpha_{nk}^2 X_k^2 I(|\alpha_{nk} X_k| \geqq \varepsilon) | \mathscr{G}_{n,k-1}) | \mathscr{F}_{n,k-1}\big)\\ &= \textstyle\sum_{k=1}^{n} E\big(\alpha_{nk}^2 k(\varepsilon . \alpha_{nk}^{-1}) | \mathscr{H}_{n,k-1}\big).\end{aligned}$$

On the set A_n, this r.v. is less than

$$k(\varepsilon \delta_n^{-1}) \textstyle\sum_{k=1}^{n} E(\alpha_{nk}^2 | \mathscr{H}_{n,k-1}) \to^p 0 \quad \text{as} \quad n \to \infty$$

because of (11) and the fact that $k(\varepsilon \delta_n^{-1}) \downarrow 0$ as $n \to \infty$. From (12) the exceptional set A_n^c has small probability for large n, so it follows that condition (7) is satisfied. The Theorem is proved.

Remark. Under the extra condition that $\lim_{n\to\infty} \sum_{j=1}^{n} E\,\alpha_{nj}^2 = 1$ and using the methods of Scott [5], the conditions of Theorem 3 and 4 can easily be shown to be equivalent.

4. The Weighting Random Variables

In this section, we introduce two examples of weighting r.vs $\{\alpha_{nk}\}$ and consider when the conditions of some of the previous theorems are satisfied.

Example 1. Let $Y_1, Y_2, \ldots$ be independent, identically distributed non-negative r.vs which are not degenerate at zero, and set

$$\alpha_{nk} = Y_k (Y_1 + \cdots + Y_n)^{-1}. \tag{13}$$

If the $\{Y_k\}$ are gamma variables, then for fixed n, $\{\alpha_{nk}, 1 \leqq k \leqq n\}$ have a joint multivariate beta distribution. If the $\{Y_k\}$ are degenerate, then $\alpha_{nk} = n^{-1}$ for all k, $1 \leqq k \leqq n$.

The conditions of Theorem 1 are closely related to those of the classical weak law of large numbers, and in this connection, a corresponding condition on the $\{Y_k\}$ ensures that conditions (3) and (4) both hold, as follows.

Lemma 1. *Let $\{\alpha_{nk}\}$ be given by (15) and let*

$$\lim_{y\to\infty} y\, P(Y_1 \geqq y) = 0. \tag{14}$$

Then

$$\max_{k \leqq n} \alpha_{nk} \to^p 0 \quad \textit{as} \quad n \to \infty. \tag{15}$$

Proof. Let $W_j = Y_j I(Y_j \leqq C)$ for all j, where C is a positive, finite constant, chosen so that $EW_1 > 0$. Then $EW_j = EW_1 < \infty$, for all j, and

$$\begin{aligned}\limsup_{n\to\infty} & n(Y_1 + \cdots + Y_n)^{-1}\\ &\leqq \lim_{n\to\infty} n(W_1 + \cdots + W_n)^{-1}\\ &= (EW_1)^{-1}.\end{aligned} \tag{16}$$

But

$$\max_{k \leqq n} \alpha_{nk} = (\max_{k \leqq n} Y_k)(Y_1 + \cdots + Y_n)^{-1}$$

so to prove (15) it will suffice, from (16) to show that

$$n^{-1} \max_{k \leqq n} Y_k \to^p 0 \quad \text{as} \quad n \to \infty. \tag{17}$$

We have

$$\begin{aligned} P(n^{-1} \max_{k \leqq n} Y_k \leqq \varepsilon) &= \big(P(Y_1 \leqq n\,\varepsilon)\big)^n \\ &= \big(1 - P(Y_1 > n\,\varepsilon)\big)^n \to 1 \quad \text{as} \quad n \to \infty, \end{aligned}$$

from (14), thus establishing (15) and completing the proof.

Since $\sum_{k=1}^n \alpha_{nk} = 1$ a.s. when the $\{\alpha_{nk}\}$ are given by (13), we immediately have, upon combining Lemma 1 with Theorem 1,

Theorem 5. *Let $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ be two independent sequences of independent, identically distributed r.vs, the $\{Y_j\}$ being non-negative and not degenerate at zero, with*

$$\lim_{y \to \infty} y\, P(|X_1| \geqq y) = 0,$$
$$\lim_{y \to \infty} \int_{[|X_1| < y]} X_1\, dP = \mu, \quad \textit{finite},$$

and

$$\lim_{y \to \infty} y\, P(Y_1 \geqq y) = 0.$$

Then

$$(Y_1 + \cdots + Y_n)^{-1} \sum_{k=1}^n Y_k (X_k - \mu) \to^p 0 \quad \textit{as} \quad n \to \infty.$$

If attention is now turned to the central limit theorem, the $\{\alpha_{nk}\}$ must be adjusted with appropriate norming constants, and so we set

$$\alpha_{nk} = n^{\frac{1}{2}}\, Y_k (Y_1 + \cdots + Y_n)^{-1}. \tag{18}$$

Lemma 2. *Let the $\{\alpha_{nk}\}$ be given by* (18) *and let*

$$E Y_1^2 = \sigma^2 + \mu^2 < \infty, \tag{19}$$

with

$$E Y_1 = \mu.$$

Then

$$\max_{k \leqq n} \alpha_{nk} \to^p 0 \quad \textit{as} \quad n \to \infty \tag{20}$$

and

$$\lim_{n \to \infty} \sum_{k=1}^n \alpha_{nk}^2 = 1 + \sigma^2/\mu^2 \quad \textit{a.s.} \tag{21}$$

Proof. The proof of (20) corresponds exactly to that of Lemma 1 since we know from (19) that

$$\lim_{y \to \infty} y\, P(Y_1 > y^{\frac{1}{2}}) = 0.$$

Eq. (21) follows from (19) and the strong law of large numbers.

We can now combine Lemma 2 with Theorem 3 to yield

Theorem 6. *Let $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ be two independent sequences of independent, identically distributed r.vs with the $\{Y_k\}$ a.s. positive and*

$$E X_1 = 0, \qquad E X_1^2 = 1,$$
$$E Y_1 = \mu \quad \textit{and} \quad E Y_1^2 = \sigma^2 + \mu^2 < \infty.$$

Then

$$n^{\frac{1}{2}}(Y_1+\cdots+Y_n)^{-1}\sum_{k=1}^{n} Y_k X_k$$

converges in law as $n\to\infty$ *to a* $N(0, 1+\sigma^2/\mu^2)$ *distribution.*

Example 2. For an example where the $\{\alpha_{nk}\}$ are neither non-negative nor independent of the $\{X_k\}$, consider a case of *serial correlation*, namely

$$\alpha_{nk}=n^{\frac{1}{2}} X_{k-1},$$

with $X_0=1$. Here, α_{nk} depends only on n and $X_1,\dots,X_{k-1}$ so that S_n will be asymptotically normally distributed provided that the conditions of Theorem 3 are satisfied. For these α_{nk}, if $EX_1^2=\sigma^2<\infty$, then

$$\lim_{n\to\infty}\sum_{k=1}^{n}\alpha_{nk}^2=\sigma^2 \quad \text{a.s.}$$

by the strong law of large numbers, and $\max_{k\leq n}\alpha_{nk}^2\to^{p} 0$ as $n\to\infty$, by similar reasoning to that in the proof of Lemma 1. Then we have the following (well-known)

Theorem 7. *Let* $X_1, X_2, \dots$ *be a sequence of independent, identically distributed r.vs with* $EX_1=0$, $EX_1^2=\sigma^2<\infty$ *and let* $X_0\equiv 1$. *Then*

$$n^{\frac{1}{2}}\sum_{k=1}^{n} X_{k-1} X_k$$

converges in law as $n\to\infty$ *to a* $N(0,\sigma^2)$ *distribution.*

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