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Infinitely Divisible Processes with Interchangeable Increments and Random Measures under Convolution

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1. Introduction

In [3, 4], a unified theory of interchangeability was developed, based on representations in terms of canonical random elements. In the present paper, we continue this study by examining the form of those processes with interchangeable increments which are infinitely divisible. A special case of this problem has been treated previously by Kerstan, Matthes and Mecke [6], pp. 76–77. Here we give, in each of the four fundamental cases of processes with discrete or continuous parameter varying over a bounded or unbounded interval, a characterization in terms of a unique decomposition of Lévy-Hinčin type, which is made clear by the use of shower integrals, the latter generalizing a notion from [6, 7]. Our results in the case of infinite sequences provide the key to an examination of the class of bounded random measures which are infinitely divisible w.r.t. convolution (as opposed to addition, cf. [2, 6, 7]).

It is convenient to include at this point a summary of the canonical representations in [3]. This requires some terminology and notation which we give first. As in [3, 4], $\mathfrak{M}(S)$ denotes for any locally compact Polish space S the class of locally finite measures on S, while $\mathfrak{N}(S)$ denotes the subclass of Z_+-valued measures. Random measures and point processes on S are random elements ξ in these spaces such that ξB is measurable for every Borel set B in S [2, 6]. Integrals are denoted by $\mu f = \int f(s)\,\mu(ds)$, and for any measure μ on R, the measures μ^1 and μ^2 are defined by $\mu^k(dx) \equiv x^k \mu(dx)$. By 1_B we mean the indicator of the set B, and we write for brevity $1_+ = 1_{R_+}$. The symbol δ_s denotes for fixed s the measure satisfying $\delta_s B \equiv 1_B(s)$, while L, g, g_1 and g_2 denote the functions

$$L(x) \equiv x, \qquad g(x) \equiv x/(1+x), \qquad g_k(x) \equiv x^k/(1+x^2)$$

on R. Moreover, $R' = R \setminus \{0\}$ while $D_0[0,1]$ and $D_0(R_+)$ denote the classes of functions on $[0,1]$ and R_+ respectively which are right continuous with left-hand limits, and which start at 0. Finally, $\stackrel{d}{=}$ means equality in distribution.

For any fixed $n \in N$ and any $\pi \in \mathfrak{N}(R)$ with $\pi R = n$, say $\pi = \sum_j \delta_{x_j}$, let V_π be a random vector in R^n whose components are obtained by a random permutation of $x_1, \ldots, x_n$. Such a vector will be called a *sampling vector based on* π. The most general (distribution of a) random vector in R^n with interchangeable components is obtained from V_π by mixing w.r.t. π, i.e. by considering π as a point process on R with $\pi R = n$ a.s. The same approach can be used for infinite sequences, except that we then start with a sequence Y_μ of independent random variables with common distribution μ, such a sequence being referred to below as a *sampling sequence based on* μ, and then mix w.r.t. μ, when regarded as a random measure on R with $\mu R = 1$ a.s.

We next consider a process in $D_0[0,1]$ of the form $X = \alpha L + \sigma B + J_\beta$, where B is a Brownian bridge while

$$J_\beta(t) = \sum_j \beta_j [1_+(t-\tau_j) - t], \qquad t \in [0,1], \tag{1.1}$$

for some i.i.d. variables $\tau_1, \tau_2, \ldots$ which are independent of B and uniformly distributed on $[0,1]$. Here $\alpha \in R$ and $\sigma \in R_+$ while $\beta = \sum_j \delta_{\beta_j}$ is a measure in $\mathfrak{N}(R')$ with $\beta^2 R = \sum \beta_j^2 < \infty$. Such a process X, being the limit of cumulative sampling vectors [3], will be called a *sampling limiting process based on* $(\alpha, \sigma^2, \beta)$. The most general process in $D_0[0,1]$ with interchangeable increments is obtained by mixing w.r.t. $(\alpha, \sigma^2, \beta)$, i.e. by considering α and σ^2 as random variables and β as a point process on R' with $\beta^2 R < \infty$ a.s. (As shown in [3], little is lost in generality by a restriction to D-spaces.)

In the final case of processes in $D_0(R_+)$, we start with an arbitrary process with stationary independent increments $X = \alpha L + \sigma M + Z_\lambda$, where M is a Brownian motion while Z_λ is independent of M with independent increments and satisfies

$$\log \mathsf{E}\, e^{iuZ_\lambda} = t \int \{e^{iux} - 1 - i u\, g_1(x)\}\, \lambda(dx), \qquad u \in R,\ t \in R_+. \tag{1.2}$$

In this case, $\alpha \in R$ and $\sigma \in R_+$ while $\lambda \in \mathfrak{M}(R')$ with $\lambda g_2 < \infty$, and the process X above will be referred to as an *additive process based on* $(\alpha, \sigma^2, \lambda)$. The most general process in $D_0(R_+)$ with interchangeable increments is obtained by mixing w.r.t. $(\alpha, \sigma^2, \lambda)$, i.e. by regarding α and σ^2 as random variables and λ as a random measure on R' with $\lambda g_2 < \infty$ a.s.

The quantities π, μ, $(\alpha, \sigma^2, \beta)$ and $(\alpha, \sigma^2, \lambda)$ above are a.s. unique measurable functions of the corresponding processes X and will be called the *canonical random elements* of X.

The notion of shower integral is best introduced by means of an example. Consider the additive random process Z_λ in (1.2), and let ξ be a Poisson process on R' with intensity λ [2, 6]. If λ is bounded, then ξ has a.s. finitely many unit atoms, say at $\beta_1, \ldots, \beta_\nu$, and it is well known that Z_λ is distributed on $[0,1]$ as the process

$$X(t) = \sum_{j=1}^{\nu} \beta_j\, 1_+(t-\tau_j) - t\, \lambda g_1, \qquad t \in [0,1], \tag{1.3}$$

where the variables τ_j are independent of ξ, mutually independent and uniformly distributed on $[0,1]$. It is suggestive to write (1.3) in the form

$$X(t) = \int \{x\, 1_+(t-\tau')\, \xi(dx) - t\, g_1(x)\, \lambda(dx)\}, \qquad t \in [0,1], \tag{1.4}$$

the prime on τ indicating the assumed independence. From [5] it is seen that the above statement remains true for arbitrary λ with the integral in (1.4) interpreted in the sense of a.s. uniform convergence. By this we mean that, whenever M_t, $t\in R_+$, are measurable sets in R' satisfying $\lambda M_t<\infty$ and $M_t\uparrow R'$, the restrictions of the integral in (1.4) to M_t converge a.s. uniformly to a unique limit X which is distributed like Z_λ. We shall call this common limit an a.s. uniformly convergent *centered Poisson shower* of processes $x\,1_+(t-\tau)$, $t\in[0,1]$. All shower integrals below should be interpreted in this way, with appropriate modifications in each case.

We are now in a position to describe our main results. Under the assumption of infinite divisibility, our random vectors, sequences, processes and measures are distributed as essentially unique sums (or convolutions respectively) of independent random elements of the types listed under the relevant headings below.

1. *Vectors with Interchangeable Components*
a) symmetric Gaussian vector
b) centered Poisson shower of sampling vectors

2. *Infinite Sequences of Interchangeable Variables*
a) common Gaussian translation
b) infinitely divisible sampling sequence
c) centered Poisson shower of sampling sequences

3. *Processes in $D_0[0,1]$ with Interchangeable Increments*
a) linear Gaussian drift
b) Brownian bridge
c) centered Poisson shower of sampling limiting processes

4. *Processes in $D_0(R_+)$ with Interchangeable Increments*
a) linear Gaussian drift
b) additive process
c) centered Poisson shower of additive processes

5. *Bounded Random Measures under Convolution*
a) infinitely divisible probability distribution
b) random deletion
c) Gaussian magnification
d) common Gaussian magnification and translation
e) centered Poisson shower of bounded measures

Precise statements of these results will be found in Theorems 2.1, 2.2, 3.3, 3.4 and 4.3 below.

2. Interchangeable Random Variables

Here and in § 4 we shall use some matrix notation. Thus A' means the transpose of A. Vectors and sequences are interpreted as columns rather than rows (though they are often written as rows for typographical convenience), and in particular $u'x$ denotes the inner product of u and x while $u'\Gamma u$ is the quadratic form with coefficient matrix Γ. For brevity we write $\mathbf{1}=(1,1,\ldots)$ and $\mathbf{g}_1(x_1,x_2,\ldots)=(g_1(x_1),g_1(x_2),\ldots)$.

Theorem 2.1. *For fixed $n\in N$, the relation*

$$X\stackrel{d}{=}a\,\mathbf{1}+s\,G_r+\int\{V'_\pi\,\xi(d\pi)-n^{-1}\,\pi\,g_1\,\mathbf{1}\,\Lambda(d\pi)\}\tag{2.1}$$

defines a unique[1] *correspondence between the distributions of all infinitely divisible random vectors X in* R^n *with interchangeable components and the set of all four-tuples* (a, s, r, Λ) *such that* $a \in R$, $s \in R_+$ *and* $-(n-1)^{-1} \leqq r \leqq 1$, *while* Λ *is a measure on the set* $\{\pi \in \mathfrak{N}(R) \setminus \{n\,\delta_0\} \colon \pi R = n\}$ *satisfying* $\int \pi\, g_2\, \Lambda(d\pi) < \infty$. *The terms on the right of* (2.1) *are assumed to be independent and such that* G_r *is Gaussian in* R^n *with means* 0, *variances* 1 *and mutual covariances* r, *while* ξ *is a Poisson process on* $\mathfrak{N}(R)$ *with intensity* Λ. *The last term in* (2.1) *denotes an a.s. convergent centered Poisson shower of sampling vectors* V_π.

Proof. Suppose that X is infinitely divisible with interchangeable increments. The former property is known to imply that

$$\log \mathsf{E}\, e^{iu'X} = i\,u'\,\alpha - \tfrac{1}{2}\,u'\,\Gamma\,u + \int \{e^{iu'x} - 1 - i\,u'\,\mathbf{g}_1(x)\}\,\lambda(dx) \tag{2.2}$$

for some $\alpha \in R^n$, some non-negative definite matrix Γ and some $\lambda \in \mathfrak{M}(R^n \setminus \{0\})$ satisfying

$$\int g(x'\,x)\,\lambda(dx) < \infty. \tag{2.3}$$

For any permutation operator T on R^n we get by (2.2) and the interchangeability property of X

$$\begin{aligned}\log \mathsf{E}\, e^{iu'X} &= \log \mathsf{E}\, e^{iu'TX} = \log \mathsf{E}\, e^{i(T'u)'X}\\ &= i(T'\,u)'\,\alpha - \tfrac{1}{2}(T'\,u)'\,\Gamma T'\,u + \int \{e^{i(T'u)'x} - 1 - i(T'\,u)'\,\mathbf{g}_1(x)\}\,\lambda(dx)\\ &= i\,u'\,T\alpha - \tfrac{1}{2}\,u'\,T\Gamma T'\,u + \int \{e^{iu'Tx} - 1 - i\,u'\,\mathbf{g}_1(Tx)\}\,\lambda(dx)\\ &= i\,u'\,T\alpha - \tfrac{1}{2}\,u'\,T\Gamma T'\,u + \int \{e^{iu'x} - 1 - i\,u'\,\mathbf{g}_1(x)\}\,\lambda\,T'(dx),\end{aligned}$$

and since α, Γ and λ are unique in (2.2), it follows that

$$T\alpha = \alpha, \qquad T\Gamma T' = \Gamma, \qquad \lambda\,T' = \lambda. \tag{2.4}$$

Since T was arbitrary, we get for α and Γ

$$\alpha = a\,\mathbf{1}, \qquad \Gamma_{ij} \equiv s^2\,\delta_{ij} + s^2\,r(1-\delta_{ij}) \tag{2.5}$$

for some $a \in R$, $s \in R_+$ and $r \in [-1, 1]$. With this choice of elements, Γ is non-negative iff $r \geqq -(n-1)^{-1}$. (To see this, calculate the minimum of $u'\Gamma u$ subject to the restriction $u'u = 1$.)

To determine the form of λ when $\lambda \neq 0$, let λ_0 be defined by

$$\lambda_0(dx) = g(x'\,x)\,\lambda(dx), \qquad x \in R^n, \tag{2.6}$$

and conclude from (2.3) and (2.4) that λ_0 is bounded and satisfies $\lambda_0\,T' = \lambda_0$ for any permutation T. Thus λ_0 agrees after a normalization with the distribution of a random vector with interchangeable components, and it follows from § 1 that

$$\lambda_0 = \int \mathsf{P}\,V_\pi^{-1}\,\Lambda_0(d\pi) \tag{2.7}$$

for some bounded measure Λ_0 on the set $\{\pi \in \mathfrak{N}(R) \colon \pi R = n\}$. By (2.6), $\lambda_0\{0\} = 0$ and hence $\Lambda_0\{n\,\delta_0\} = 0$, so we may define a measure Λ on $\{\pi \in \mathfrak{N}(R) \setminus \{n\,\delta_0\}$:

[1] Except that r is arbitrary when $s = 0$ or $n = 1$.

$\pi R = n\}$ by

$$\Lambda(d\pi) = (g(\pi^2 R))^{-1} \Lambda_0(d\pi), \qquad \pi \in \mathfrak{N}(R). \tag{2.8}$$

Since Λ_0 is bounded and $\pi g_2 \leqq n g(\pi^2 R)$, we obtain $\int \pi g_2 \Lambda(d\pi) < \infty$. Furthermore, we get by (2.6)–(2.8) for $x \in R^n \setminus \{0\}$

$$\begin{aligned}\lambda(dx) &= (g(x'x))^{-1} \lambda_0(dx) = \int (g(x'x))^{-1} \mathsf{P} V_\pi^{-1}(dx) \Lambda_0(d\pi)\\ &= \int \mathsf{P} V_\pi^{-1}(dx)(g(\pi^2 R))^{-1} \Lambda_0(d\pi) = \int \mathsf{P} V_\pi^{-1}(dx) \Lambda(d\pi).\end{aligned}$$

Inserting this and the relation $\alpha = a\mathbf{1}$ into (2.2) yields

$$\begin{aligned}\log \mathsf{E}\, e^{iu'X} &= i a u' \mathbf{1} - \tfrac{1}{2} u' \Gamma u + \int \mathsf{E}\{e^{iu'V_\pi} - 1 - i u' \mathbf{g}_1(V_\pi)\} \Lambda(d\pi)\\ &= i a u' \mathbf{1} - \tfrac{1}{2} u' \Gamma u + \int \{\mathsf{E}(e^{iu'V_\pi} - 1) - i n^{-1} \pi g_1 u' \mathbf{1}\} \Lambda(d\pi).\end{aligned} \tag{2.9}$$

In view of (2.5), the first two terms in (2.9) give rise to the first two terms in (2.1). As for the integral, let us first suppose that Λ is bounded, say with $\Lambda \mathfrak{N}(R) = c > 0$, and put $Y = \int V_\pi' \xi(d\pi)$. Writing $\xi = \sum_{j=1}^{\nu} \delta_{\pi_j}$ and using the fact [2, 6] that ν is Poissonian with mean c while, for given ν, the π_j may be chosen independent with common distribution $c^{-1}\Lambda$, we get

$$\begin{aligned}\mathsf{E}\, e^{iu'Y} &= \mathsf{E}\mathsf{E}\{\mathsf{E}(e^{iu'Y}|\xi)|\nu\} = \mathsf{E}\mathsf{E}\left\{\prod_{j=1}^{\nu} \mathsf{E}(e^{iu'V_{\pi_j}}|\pi_j)|\nu\right\} = \mathsf{E}\{c^{-1}\int \mathsf{E}\, e^{iu'V_\pi} \Lambda(d\pi)\}^\nu\\ &= \exp\{-c[1 - c^{-1}\int \mathsf{E}\, e^{iu'V_\pi} \Lambda(d\pi)]\} = \exp\{\int \mathsf{E}(e^{iu'V_\pi} - 1) \Lambda(d\pi)\},\end{aligned}$$

in agreement with (2.9).

In the case of general Λ, it follows from the continuity theorem for characteristic functions that the integral in (2.1), taken over sets $M_n \subset \mathfrak{N}(R)$ with $\Lambda M_n < \infty$ and $M_n \uparrow \mathfrak{N}(R)$, converges in distribution to a random vector with characteristic function given by the integral in (2.9), and the convergence is even a.s. since the successive differences are independent. If $\{M_n'\}$ is another sequence, the integral in (2.1) taken over the symmetric differences $M_n \triangle M_n'$ converges in probability to zero, which proves that the limiting random vector is a.s. unique. The extension to uncountable families $\{M_t, t \in R_+\}$ is easy, since ξ has only finitely many atoms in sets where Λ is bounded while the measure $\pi g_1 \Lambda(d\pi)$ is absolutely continuous w.r.t. Λ. This completes the proof of (2.1) subject to the stated restrictions on (a, s, r, Λ).

Reversing the above arguments, it is seen that any (a, s, r, Λ) subject to these restrictions can be used to define some X satisfying (2.1). The infinite divisibility of X follows from that of G_r and ξ, while the interchangeability of components is due to the symmetry of $\mathbf{1}$, G_r and V_π. Finally, the uniqueness of (a, s, r, Λ) follows from that of $(\alpha, \Gamma, \lambda)$ in (2.2). □

For the next theorem, we let $N(0,1)$ denote the normalized Gaussian distribution on R and recall the definition of $\mathsf{P} Z_\lambda^{-1}$ in (1.2).

Theorem 2.2. *The relation*

$$X \stackrel{d}{=} (a + c\gamma)\mathbf{1} + sG + U_m + \int \{Y_\mu' \xi(d\mu) - \mu g_1 \mathbf{1} \Lambda(d\mu)\} \tag{2.10}$$

defines a unique correspondence between the distributions of all infinitely divisible sequences X of interchangeable random variables and the set of all five-tulpes (a, c, s, m, Λ) such that $a \in R$, c, $s \in R_+$ and $m \in \mathfrak{M}(R')$ with $m g_2 < \infty$, while Λ is a measure on the set $\{\mu \in \mathfrak{M}(R) \setminus \{\delta_0\}: \mu R = 1\}$ satisfying $\int \mu g_2 \Lambda(d\mu) < \infty$. The terms on the right of (2.10) *are assumed to be independent and such that γ is $N(0,1)$, while G and U_m are sample sequences based on $N(0,1)$ and $\mathsf{P}(Z_m(1))^{-1}$ respectively and ξ is a Poisson process on $\mathfrak{M}(R)$ with intensity Λ. The last term in* (2.10) *denotes an a.s. convergent centered Poisson shower of sampling sequences Y_μ.*

Proof. Suppose that X is infinitely divisible with interchangeable elements. By [7], (2.2) remains true for any $u \in R^\infty$ with finitely many non-zero elements, with $\alpha \in R^\infty$ and non-negative Γ and with λ satisfying

$$\int g_2(x_j)\,\lambda(dx) < \infty, \qquad j \in N. \tag{2.11}$$

Again, (2.4) must hold for any permutation T, and this implies (2.5) for some $a \in R$, $s \in R_+$ and $r \in [0,1]$. Substituting (c^2+s^2, c^2) for $(s^2, s^2 r)$, the first two terms in (2.2) are seen to give rise to the first two terms in (2.10).

We now use the pointwise ergodic theorem in [1], p. 675, to conclude from (2.4) and (2.11) that

$$\frac{1}{n}\sum_{j=1}^{n} g_2(x_j) \to \text{some } h(x), \qquad x \in R^\infty \text{ a.e. } \lambda. \tag{2.12}$$

By Fatou's lemma, it follows from (2.4), (2.11) and (2.12) that

$$\lambda h \leqq \liminf_{n\to\infty} \int \frac{1}{n}\sum_{j=1}^{n} g_2(x_j)\,\lambda(dx) = \int g_2(x_1)\,\lambda(dx) < \infty,$$

so we may define a bounded measure λ_0 on R^∞ by

$$\lambda_0(dx) = h(x)\,\lambda(dx), \qquad x \in R^\infty. \tag{2.13}$$

From (2.4) and (2.12) it is seen that λ_0 is invariant under permutations, so by § 1 there exists some bounded measure Λ_0 on the set $\{\mu \in \mathfrak{M}(R): \mu R = 1\}$ satisfying

$$\lambda_0 = \int \mathsf{P} Y_\mu^{-1} \Lambda_0(d\mu). \tag{2.14}$$

Since $h > 0$ a.e. λ_0, it follows from Theorem 6.1 in [4] that $\mu \neq \delta_0$ a.e. Λ_0, and so we may define a measure Λ on $\{\mu \in \mathfrak{M}(R) \setminus \{\delta_0\}: \mu R = 1\}$ by

$$\Lambda(d\mu) = (\mu g_2)^{-1} \Lambda_0(d\mu), \qquad \mu \in \mathfrak{M}(R). \tag{2.15}$$

Using (2.13)–(2.15) and the law of large numbers, we get for $x \in R^\infty$ with $h(x) > 0$

$$\begin{aligned}\lambda(dx) = (h(x))^{-1}\lambda_0(dx) &= \int \left\{\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} g_2(x_j)\right\}^{-1} \mathsf{P} Y_\mu^{-1}(dx)\Lambda_0(d\mu)\\ &= \int \mathsf{P} Y_\mu^{-1}(dx)(\mu g_2)^{-1}\Lambda_0(d\mu) = \int \mathsf{P} Y_\mu^{-1}(dx)\Lambda(d\mu).\end{aligned}$$

Thus

$$\begin{aligned}\int_{\{h(x)>0\}} \{e^{iu'x} - 1 - iu'\mathbf{g}_1(x)\}\,\lambda(dx) &= \int \mathsf{E}\{e^{iu'Y_\mu} - 1 - iu'\mathbf{g}_1(Y_\mu)\}\,\Lambda(d\mu)\\ &= \int \{\mathsf{E}(e^{iu'Y_\mu} - 1) - i\mu g_1 u'\mathbf{1}\}\,\Lambda(d\mu),\end{aligned}$$

and arguing as in the preceding proof, it is seen that this term gives rise to the shower integral in (2.10). The relation $\int \mu g_2 \Lambda(d\mu) < \infty$ follows from (2.15) and the boundedness of Λ_0.

We next conclude from (2.4), (2.11) and (2.12) that the measure λ_1 on R^∞ defined by

$$\lambda_1(dy) = \int g_2(x_1) 1_{\{h(x)=0, (x_2, x_3, \ldots) \in dy\}}(x) \lambda(dx), \qquad y \in R^\infty,$$

is bounded and invariant under permutations, and hence that $\lambda_1 = \int \mathsf{P} Y_\mu^{-1} \Lambda_1(d\mu)$ for some measure Λ_1 on $\mathfrak{M}(R)$. But since $h=0$ a.e. λ_1 by (2.12), it follows by Theorem 6.1 in [4] that $\mu = \delta_0$ a.e. Λ_1, and so $y=0$ a.e. λ_1. Thus the restriction of λ to the set $\{x: h(x)=0\}$ is supported by the set of sequences in R^∞ with exactly one non-zero element. Defining $m \in \mathfrak{M}(R')$ by

$$m(dz) = \lambda\{x: h(x)=0, x_1 \in dz\}, \qquad z \in R',$$

and using the symmetry implied by (2.4) and (2.12), we thus obtain

$$\int_{\{h(x)=0\}} \{e^{iu'x} - 1 - iu' \mathbf{g}_1(x)\} \lambda(dx) = \sum_{j=1}^{\infty} \int \{e^{iu_j z} - 1 - iu_j g_1(z)\} m(dz).$$

Comparing with (1.2), it is seen that this term gives rise to the third term in (2.10). Since $m g_2 < \infty$ by (2.11), this completes the proof of (2.10) subject to the stated restrictions on (a, c, s, m, Λ). The remainder of the proof is the same as for Theorem 2.1.

3. Processes with Interchangeable Increments

The main results of this section are based on two lemmas of some independent interest.

Lemma 3.1. *Let $X_1, X_2, \ldots$ be independent random processes in $D_0[0,1]$ (or $D_0(R_+)$) with interchangeable increments and canonical random elements $(\alpha_j, \sigma_j^2, \beta_j)$ (or $(\alpha_j, \sigma_j^2, \lambda_j)$), $j \in N$. Then the series $X = \sum X_j$ is a.s. uniformly convergent (on bounded intervals) iff $\alpha = \sum \alpha_j$, $\sigma^2 = \sum \sigma_j^2$ and $\beta = \sum \beta_j$ (or $\lambda = \sum \lambda_j$) are a.s. convergent with $\beta^2 R < \infty$ (or $\lambda g_2 < \infty$) a.s. In this case, X is a process in $D_0[0,1]$ (or $D_0(R_+)$) with interchangeable increments and canonical random element $(\alpha, \sigma^2, \beta)$ (or $(\alpha, \sigma^2, \lambda)$ respectively).*

Proof. Let us first consider the case of processes in $D_0[0,1]$. If all but finitely many processes X_j vanish identically, it is obvious that X is a process in $D_0[0,1]$ with interchangeable increments, and that its canonical random element $(\alpha', \sigma'^2, \beta')$ satisfies $\alpha' = \sum \alpha_j$ and $\beta' = \sum \beta_j$. The relation $\sigma'^2 = \sum \sigma_j^2$ follows in case of non-random σ_j from the fact that $B \stackrel{d}{=} M - M(1)L$ where B is a Brownian bridge and M a Brownian motion on [0, 1], and then in general by conditioning on $\{\sigma_j\}$.

Turning to the case of infinitely many non-zero processes, suppose that $\alpha = \sum \alpha_j$, $\sigma^2 = \sum \sigma_j^2$ and $\beta = \sum \beta_j$ are a.s. convergent and such that $\beta^2 R < \infty$ a.s. From Theorem 2.3 in [3] it is seen that $\sum X_j$ converges in distribution w.r.t. the Skorohod J_1 topology, and since the terms are independent, it follows from [5] that the convergence is even a.s. uniform. The limit X is clearly a process in

$D_0[0, 1]$ with interchangeable increments. To see that its canonical random element $(\alpha', \sigma'^2, \beta')$ equals $(\alpha, \sigma^2, \beta)$, conclude from the first part of the proof that the processes $X'_n = X - \sum_{j=1}^n X_j$ have interchangeable increments with canonical random elements

$$(\alpha'_n, \sigma'^2_n, \beta'_n) = \left(\alpha' - \sum_{j=1}^n \alpha_j, \sigma'^2 - \sum_{j=1}^n \sigma_j^2, \beta' - \sum_{j=1}^n \beta_j\right).$$

Since $X'_n \to 0$ a.s., it follows by [3] that $(\alpha'_n, \sigma'^2_n, \beta'^2_n R) \to 0$ in distribution. On the other hand, this random vector has the a.s. limit $(\alpha' - \alpha, \sigma'^2 - \sigma^2, (\beta' - \beta)^2 R)$, so the latter must be a.s. zero.

Conversely, a.s. convergence of $\sum X_j$ entails by [3] convergence in distribution of $\sum \alpha_j$ and $\sum(\sigma_j^2 \delta_0 + \beta_j^2)$, for the latter sum w.r.t. the weak topology, and again the convergence must be a.s., for the latter sum because the terms are non-negative. But a.s. convergence is clearly impossible unless $\sigma^2 < \infty$ and $\beta^2 R < \infty$ a.s.

For processes in $D_0(R_+)$, the same arguments go through, except for the proof of the finite additivity of $(\alpha, \sigma^2, \lambda)$. For additive processes, however, the additivity is seen from the form of the characteristic functions (cf. (1.2)), and so it follows in general by conditioning on the canonical random elements.

Lemma 3.2. *Let X be a process in $D_0[0, 1]$ or $D_0(R_+)$ with interchangeable increments. Then X and its canonical random element $(\alpha, \sigma^2, \beta)$ or $(\alpha, \sigma^2, \lambda)$ respectively are simultaneously infinitely divisible.*

Proof. The arguments for $D_0[0,1]$ and $D_0(R_+)$ being similar, we may e.g. assume that X is a process in $D_0[0, 1]$. If $(\alpha, \sigma^2, \beta)$ is infinitely divisible, there exist for every fixed $n \in N$ some i.i.d. random elements $(\alpha_j, \sigma_j^2, \beta_j)$, $j = 1, \ldots, n$, in $R \times R_+ \times \mathfrak{N}(R')$ whose sum is distributed like $(\alpha, \sigma^2, \beta)$. Letting $X_1, \ldots, X_n$ be independent processes in $D_0[0, 1]$ with interchangeable increments having these triples as canonical random elements, it is seen from Lemma 3.1 that $\sum X_j$ has interchangeable increments and that its canonical random element equals $\sum_j (\alpha_j, \sigma_j^2, \beta_j) \stackrel{d}{=} (\alpha, \sigma^2, \beta)$, so $\sum X_j \stackrel{d}{=} X$. Since n was arbitrary, this proves that X is infinitely divisible.

Conversely, suppose that X is infinitely divisible. Then there exist for every fixed $n \in N$ some i.i.d. processes $X_1, \ldots, X_n$ on $[0, 1]$ whose sum has the same finite dimensional distributions as X. Since the increments of X are jointly infinitely divisible, their joint characteristic functions can have no zeros, and so the corresponding characteristic functions for $X_1, \ldots, X_n$ are unique. This is easily seen to imply that the X_j have interchangeable increments, and by [3] we may then assume the X_j to lie in $D_0[0, 1]$. The proof may now be completed as above.

Theorem 3.3. *The relation*

$$X \stackrel{d}{=} (a + c\gamma)L + sB + \int \{(\alpha L + \sigma B' + J'_\beta) d\xi(\alpha, \sigma^2, \beta) - g_1(\alpha) L d\Lambda(\alpha, \sigma^2, \beta)\} \tag{3.1}$$

defines a unique correspondence between the distributions of all infinitely divisible random processes X in $D_0[0, 1]$ with interchangeable increments and the set of all four-tuples (a, c, s, Λ) such that $a \in R$ and $c, s \in R_+$, while Λ is a measure on the set $\{(\alpha, \sigma^2, \beta) \in R \times R_+ \times \mathfrak{N}(R') \setminus \{0\} : \beta^2 R < \infty\}$ satisfying

$$\int g(\alpha^2 + \sigma^2 + \beta^2 R) d\Lambda(\alpha, \sigma^2, \beta) < \infty. \tag{3.2}$$

The terms on the right of (3.1) *are assumed to be independent and such that* γ *is* $N(0, 1)$, *while* B *is a Brownian bridge and* ξ *is a Poisson process on* $R \times R_+ \times \mathfrak{N}(R')$ *with intensity* Λ. *The last term in* (3.1) *denotes an a.s. uniformly convergent centered Poisson shower of sampling limiting processes* $\alpha L + \sigma B + J_\beta$.

Proof. Let X be an infinitely divisible process in $D_0[0, 1]$ with interchangeable increments, and note that its canonical random element $(\tilde{\alpha}, \tilde{\sigma}^2, \tilde{\beta})$ is infinitely divisible by Lemma 3.2. We intend to show that this implies the existence of numbers $a \in R$ and c, $s \in R_+$, and of some measure Λ defined on the set $\{(\alpha, \sigma^2, \beta) \in R \times R_+ \times \mathfrak{N}(R') \setminus \{0\} : \beta^2 R < \infty\}$ and satisfying (3.2), such that

$$\log \mathsf{E} e^{iu\tilde{\alpha} - v\tilde{\sigma}^2 - \tilde{\beta}f} = iua - \tfrac{1}{2}u^2 c^2 - vs^2 + \int \{e^{iu\alpha - v\sigma^2 - \beta f} - 1 - iu g_1(\alpha)\} \, d\Lambda(\alpha, \sigma^2, \beta) \tag{3.3}$$

holds for any $u \in R$ and $v \in R_+$, and for any measurable function $f: R' \to R_+$ satisfying $f(x) = O(x^2)$ at $x = 0$. To prove this, consider the representation of $\mathsf{E} e^{-v\tilde{\sigma}^2 - \tilde{\beta}f}$ for $v \in R_+$ and $f: R' \to R_+$ such that $f(x)/x^2$ is simple ([7], pp. 150, 159). By analytic continuation and dominated convergence, this representation remains valid for purely imaginary v and f, and by a comparison with the general representation of $\mathsf{E} e^{iu\tilde{\alpha} - v\tilde{\sigma}^2 - \tilde{\beta}f}$ in this case ([7], p. 153), it is seen that (3.3) must hold for some (a, c, s, Λ) satisfying the stated conditions. Note in particular that $\tilde{\beta}$ can have no constant component since it is infinitely divisible as a point process [2, 6, 7], and furthermore, that only $\tilde{\alpha}$ can contribute to the Gaussian component, that centering in the Poisson integral is only required for the α-component and that the constant component of $\tilde{\sigma}^2$ must be non-negative. By analytic continuation and dominated convergence, (3.3) remains valid for $v \in R_+$ and arbitrary measurable $f: R' \to R_+$ satisfying $f(x) = O(x^2)$ at $x = 0$.

By arguments in the proof of Theorem 2.1, (3.3) is equivalent to

$$(\tilde{\alpha}, \tilde{\sigma}^2, \tilde{\beta}) \stackrel{d}{=} (a + c\gamma, s^2, 0) + \int \{(\alpha, \sigma^2, \beta) d\xi(\alpha, \sigma^2, \beta) - (g_1(\alpha), 0, 0) d\Lambda(\alpha, \sigma^2, \beta)\}, \tag{3.4}$$

where γ is $N(0, 1)$ while ξ is a Poisson process on $R \times R_+ \times \mathfrak{N}(R')$ which is independent of γ with intensity Λ. As usual, the integral is to be interpreted as an a.s. convergent Poisson shower. If Λ is bounded, it follows by Lemma 3.1 that the right-hand side of (3.4) is the canonical random element of the process on the right side of (3.1). For general Λ, we may apply Lemma 3.1 to any countable partition of the domain of integration into sets with finite Λ-measure.

Conversely, any (a, c, s, Λ) with the stated properties can be used to construct a process X satisfying (3.1). The uniqueness assertion follows from the uniqueness in (3.3) and the fact that $(\tilde{\alpha}, \tilde{\sigma}^2, \tilde{\beta})$ is a.s. uniquely determined by X.

Theorem 3.4. *The relation*

$$X \stackrel{d}{=} (a + c\gamma)L + sM + Z_m + \int \{(\alpha L + \sigma M' + Z'_\lambda) d\xi(\alpha, \sigma, \lambda) - g_1(\alpha) L \, d\Lambda(\alpha, \sigma, \lambda)\} \tag{3.5}$$

defines a unique correspondence between the distributions of all infinitely divisible processes X *in* $D_0(R_+)$ *with interchangeable increments and the set of all five-tuples*

(a, c, s, m, Λ) such that $a \in R$, $c, s \in R_+$ and $m \in \mathfrak{M}(R')$ with $m g_2 < \infty$, while Λ is a measure on the set $\{(\alpha, \sigma^2, \lambda) \in R \times R_+ \times \mathfrak{M}(R') \setminus \{0\} : \lambda g_2 < \infty\}$ satisfying $\int g(\alpha^2 + \sigma^2 + \lambda g_2) d\Lambda(\alpha, \sigma^2, \lambda) < \infty$. The terms on the right of (3.5) are assumed to be independent and such that γ is $N(0,1)$, while M is a Brownian motion, Z_m is an additive process based on $(0, 0, m)$ and ξ is a Poisson process on $R \times R_+ \times \mathfrak{M}(R')$ with intensity Λ. The last term in (3.5) denotes a centered Poisson shower of additive processes $\alpha L + \sigma M + Z_\lambda$ which is a.s. uniformly convergent on every bounded interval.

Proof. Assuming X to be an infinitely divisible process in $D_0(R_+)$ with interchangeable increments and canonical random element $(\tilde{\alpha}, \tilde{\sigma}^2, \tilde{\lambda})$ and arguing as in the preceding proof, it is seen that there exists some (a, c, s, m, Λ) with the stated properties satisfying

$$\log \mathsf{E} e^{iu\tilde{\alpha} - v\tilde{\sigma}^2 - \tilde{\lambda} f} = iua - \tfrac{1}{2}u^2 c^2 - v s^2 - mf + \int \{e^{iu\alpha - v\sigma^2 - \lambda f} - 1 - iu g_1(\alpha)\} d\Lambda(\alpha, \sigma^2, \lambda) \tag{3.6}$$

for $u \in R$ and $v \in R_+$, and for any bounded measurable function f: $R' \to R_+$ satisfying $f(x) = O(x^2)$ at $x = 0$. Note in particular that $\tilde{\lambda}$, unlike $\tilde{\beta}$ in the preceding proof, may have a constant component m. Again, (3.6) is seen to be equivalent to a shower representation

$$(\tilde{\alpha}, \tilde{\sigma}^2, \tilde{\lambda}) \overset{d}{=} (a + c\gamma, s^2, m) + \int \{(\alpha, \sigma^2, \lambda) d\xi(\alpha, \sigma^2, \lambda) - (g_1(\alpha), 0, 0) d\Lambda(\alpha, \sigma^2, \lambda)\}, \tag{3.7}$$

where γ is $N(0,1)$ while ξ is a Poisson process on $R \times R_+ \times \mathfrak{M}(R')$ with intensity Λ, and by Lemma 3.1, (3.7) is in turn equivalent to (3.5). The proof may now be completed as for Theorem 3.3.

4. Random Measures under Convolution

The semigroup of distributions of interchangeable variable sequences is equivalent in the following sense to that of normalized random measures under convolution.

Lemma 4.1. *Let $X_1, X_2, \ldots$ be independent sequences of interchangeable random variables with canonical random measures $\mu_1, \mu_2, \ldots$. Then the series $X = \sum X_j$ is a.s. convergent iff the convolution product $\mu = \prod \mu_j$ is a.s. weakly convergent, and in this case, X has interchangeable elements and canonical random measure μ.*

We shall say that a random measure η on R is *infinitely divisible w.r.t. convolution* if there exist for every fixed $n \in N$ some i.i.d. random measures $\eta_1, \ldots, \eta_n$ on R such that $\eta \overset{d}{=} \eta_1 * \cdots * \eta_n$.

Lemma 4.2. *Let X be a sequence of interchangeable random variables with canonical random measure μ. Then X and μ are simultaneously infinitely divisible, the latter w.r.t. convolution.*

The last two lemmas may be proved in the same way as Lemmas 3.1 and 3.2 above. Using these results, it is easy to restate the representation in Theorem 2.2 as one for normalized infinitely divisible random measures. We shall treat the more general case of bounded random measures. Let $\mu_{a,s,m}$ denote the distribution of the value at 1 of an additive process based on (a, s^2, m).

Theorem 4.3. *The relation*

$$\eta \overset{d}{=} \vartheta_p e^{q+r\zeta+b\gamma} \delta_{c\gamma} * \mu_{a,s,m} * \prod \{\mu^{\xi(d\mu)} * e^{-g_1(\log \mu R)\Lambda(d\mu)} \delta_{-(\mu g_1/\mu R)\Lambda(d\mu)}\} \tag{4.1}$$

defines a unique correspondence between the distributions of those bounded random measures $\eta \overset{d}{\neq} 0$ on R which are infinitely divisible w.r.t. convolution, and the set of all nine-tuples $(p, q, a, b, c, r, s, m, \Lambda)$ such that $p \in (0, 1]$, $q, a, b \in R$, $c, r, s \in R_+$ and $m \in \mathfrak{M}(R')$ with $m g_2 < \infty$, while Λ is a measure on the set $\{\mu \in \mathfrak{M}(R) \setminus \{\delta_0\} : \mu R \in (0, \infty)\}$ satisfying

$$\int \{(\mu g_2/\mu R) + g_2(\log \mu R)\} \Lambda(d\mu) < \infty. \tag{4.2}$$

The random elements ϑ_p, ζ, γ and ξ in (4.1) *are independent and such that ϑ_p equals* 1 *and* 0 *with probabilities p and $1-p$ respectively, while ζ and γ are $N(0, 1)$ and ξ is a Poisson process on $\mathfrak{M}(R)$ with intensity Λ. The last factor in* (4.1) *denotes an a.s. weakly convergent centered Poisson shower of bounded measures.*

More explicitly, the last factor in (4.1) equals for bounded Λ and $\xi = \delta_{\mu_1} + \cdots + \delta_{\mu_\nu}$

$$\mu_1 * \cdots * \mu_\nu * e^{-\int g_1(\log \mu R)\Lambda(d\mu)} \delta_{-\int(\mu g_1/\mu R)\Lambda(d\mu)},$$

and is obtained in general as the a.s. weak limit of such expressions calculated for increasing subsets of $\mathfrak{M}(R)$. Note that the random measure η in (4.1) satisfies $\eta R = 1$ a.s. iff $p = 1$ and $q = r = b = 0$ while Λ is confined to the set $\{\mu \in M(R) \setminus \{\delta_0\}$: $\mu R = 1\}$, and that in this case (4.2) reduces to $\int \mu g_2 \Lambda(d\mu) < \infty$.

Proof. Suppose that η is a bounded random measure on R which is infinitely divisible w.r.t. convolution. Then there exist for every fixed $n \in N$ some i.i.d. random measures $\eta_1, \ldots, \eta_n$ such that $\eta \overset{d}{=} \eta_1 * \cdots * \eta_n$, and it is easily verified that $\eta_1, \ldots, \eta_n$ remain i.i.d. conditionally on $\{\eta_1 * \cdots * \eta_n \neq 0\}$. Since n was arbitrary, it follows that the conditional distribution of η, given $\eta \neq 0$, is also infinitely divisible, and we get $\eta \overset{d}{=} \vartheta_p \eta'$ where $p = \mathsf{P}\{\eta \neq 0\}$ while η' is infinitely divisible and independent of ϑ_p. We may thus assume from now on that $\eta \neq 0$ a.s.

Since for bounded measures on R, $\mu = \mu_1 * \mu_2$ iff $\mu R = (\mu_1 R)(\mu_2 R)$ and $\mu/\mu R = (\mu_1/\mu_1 R) * (\mu_2/\mu_2 R)$, it is seen that η and $(\eta R, \eta/\eta R)$ are simultaneously infinitely divisible, the latter w.r.t. multiplication and convolution. For given η, we put $Y = \log \eta R$ and let X be a sampling sequence based on $\eta/\eta R$. Mixing w.r.t. η yields a sequence (Y, X) of random variables (to be interpreted as a column indexed by Z_+ and with Y as element number 0) such that the elements of X are interchangeable with canonical random measure $\eta/\eta R$, and such that $T(Y, X) \overset{d}{=} (Y, X)$ for every permutation T on R^∞ leaving the first element unchanged.

As in Lemma 4.2, it is seen that (Y, X) is infinitely divisible, and so (2.2) holds for some α, Γ and λ with X replaced by (Y, X). Moreover, (2.11) holds with Z_+ in place of N while (2.4) is true for any T as above. In particular, we get $\alpha = (q, a\mathbf{1})$ for some $q, a \in R$, so the first term in (2.2) gives rise to the factor $e^q \mu_{a,0,0}$ in (4.1). From (2.4) it is further seen that

$$\Gamma_{0j} = s_0^2 \delta_{0j} + s_0 s_1 r_0 (1 - \delta_{0j}), \qquad j \in Z_+,$$

$$\Gamma_{jk} = s_1^2 \delta_{jk} + s_1^2 r_1 (1 - \delta_{jk}), \qquad j, k \in N,$$

for some $s_0, s_1 \in R_+$ and $r_0, r_1 \in [-1, 1]$, and elementary calculations show that Γ is non-negative definite iff $r_1 \geqq r_0^2$ (provided $s_0, s_1 \neq 0$; if $s_1 \neq 0$ but $s_0 = 0$, we have instead $r_1 \in [0, 1]$; the final case $s_1 = 0$ is trivial). Letting $\gamma, \zeta, \zeta_1, \zeta_2, \ldots$ be a sampling sequence based on $N(0, 1)$, it is seen that the sum $(r\zeta, s\zeta_1, s\zeta_2, \ldots) + (b\gamma, c\gamma, c\gamma, \ldots)$ has covariance matrix Γ iff

$$s_0^2 = r^2 + b^2, \qquad s_1^2 = s^2 + c^2, \qquad r_0 = bc[(s^2 + c^2)(r^2 + b^2)]^{-1}, \qquad r_1 = c^2(s^2 + c^2)^{-1},$$

and these relations are easily seen to define a unique mapping between the class of all matrixes Γ with the above properties and the set of all four-tuples (r, s, b, c) such that $r, s, c \in R_+$ while $b \in R$. Since the two Gaussian sequences above correspond to the random measures $e^{r\zeta}\mu_{0,s,0}$ and $e^{b\gamma}\delta_{c\gamma}$ (in the same way as (Y, X) corresponds to η), it follows by an obvious extension of Lemma 4.1 that the second term in (2.2) gives rise to the factor $e^{r\zeta + b\gamma}\delta_{c\gamma} * \mu_{0,s,0}$ in (4.1).

We next define the bounded measure λ_0 on R^∞ by

$$\lambda_0(dx) = g_2(x_0)\,\lambda(dx), \qquad x \in R^\infty,$$

and note that $\lambda_0 T' = \lambda_0$ for all permutations T acting on $(x_1, x_2, \ldots)$. By an obvious extension of Theorem 1.1 in [3], there exists some bounded measure Λ_0 on the set $\{\mu \in \mathfrak{M}(R): \mu R \in (0, 1) \cup (1, \infty)\}$ such that

$$\lambda_0 = \int \mathsf{P}(\log \mu R, Y_{\mu/\mu R})^{-1} \Lambda_0(d\mu),$$

and putting

$$\Lambda(d\mu) = (g_2(\log \mu R))^{-1} \Lambda_0(d\mu), \qquad \mu R \neq 1,$$

we obtain for $x \in R^\infty$ with $x_0 \neq 0$

$$\begin{aligned}\lambda(dx) &= (g_2(x_0))^{-1}\lambda_0(dx) = (g_2(x_0))^{-1} \int \mathsf{P}(\log \mu R, Y_{\mu/\mu R})^{-1}(dx)\,\Lambda_0(d\mu)\\ &= \int \mathsf{P}(\log \mu R, Y_{\mu/\mu R})^{-1}(dx)(g_2(\log \mu R))^{-1}\Lambda_0(d\mu)\\ &= \int_{\{\mu R \neq 1\}} \mathsf{P}(\log \mu R, Y_{\mu/\mu R})^{-1}(dx)\,\Lambda(d\mu).\end{aligned}$$

Thus

$$\begin{aligned}&\int_{\{x_0 \neq 0\}} \{e^{iu'x} - 1 - iu'\mathbf{g}_1(x)\}\,\lambda(dx)\\ &= \int_{\{\mu R \neq 1\}} \mathsf{E}\{e^{iu'(\log \mu R, Y_{\mu/\mu R})} - 1 - iu'(g_1(\log \mu R), \mathbf{g}_1(Y_{\mu/\mu R}))\}\,\Lambda(d\mu)\\ &= \int_{\{\mu R \neq 1\}} \{\mathsf{E}[e^{iu'(\log \mu R, Y_{\mu/\mu R})} - 1] - iu'(g_1(\log \mu R), \mathbf{1}\,\mu g_1/\mu R)\}\,\Lambda(d\mu).\end{aligned}$$

But this is seen to be the characteristic function of the a.s. convergent shower integral

$$\int_{\{\mu R \neq 1\}} \{(\log \mu R, Y'_{\mu/\mu R})\,\xi(d\mu) - (g_1(\log \mu R), \mathbf{1}\,\mu g_1/\mu R)\,\Lambda(d\mu)\},$$

where ξ is a Poisson process on $\mathfrak{M}(R)$ with intensity Λ, and by the extended version of Lemma 4.1, the corresponding random measure equals

$$\prod_{\{\mu R \neq 1\}} \{\mu^{\xi(d\mu)} * e^{-g_1(\log \mu R)\Lambda(d\mu)}\,\delta_{-(\mu g_1/\mu R)\Lambda(d\mu)}\}.$$

The remaining term in (2.2),

$$\int_{\{x_0=0\}} \{e^{iu'x} - 1 - iu'\mathbf{g}_1(x)\}\,\lambda(dx),$$

may be treated as in the proof of Theorem 2.2, and this completes the proof of (4.1). The remainder of the proof is standard.

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Note Added in Proof. The random measure case of Theorems 3.3 and 3.4 is treated by partially different methods in [8], § 9.