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An L_1 Limit Theorem for Densities of Renewal Measures

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A practical problem of recent interest is that of constructing tests for trend in renewal processes [1, 2]. In [2], we introduced time dependent renewal processes which are generalizations of time dependent Poisson processes [3]. For a class of such processes, it was shown that the problem of construction of tests for trend is asymptotically equivalent to an analogous problem for certain scale alternatives already studied by Hájek [4].

Our proof relies on an L_1 limit theorem for densities of measures of a renewal theoretic type. The theorem in question, and several generalizations, is proved in the present paper. The technique of proof is, at least partially, imitated from Prohorov [5].

Notation. R is the real line. $\mathscr{B}$ is the Borel sets on R. λ is Lebesgue measure on $\{R, \mathscr{B}\}$. $\{\Omega, \mathscr{A}, P\}$ is a probability space that supports a sequence $\{X_n\}$ of i.i.d. real random variables. Let $S_n = \sum_1^n X_k$. On $\{R, \mathscr{B}\}$, define probability measures P_X and P_n as follows:

For $B \in \mathscr{B}$,

$$P_X(B) = P[X_1 \in B]$$

and

$$P_n(B) = \frac{1}{n} \sum_1^n P\left[\frac{S_k}{n} \in B\right].$$

Denote Radon-Nikodym derivatives with respect to Lebesgue measure

$$f = \frac{dP_X}{d\lambda}$$

and

$$p_n = \frac{dP_n}{d\lambda}.$$

Let $p(x) = 1$ if $x \in [0, 1]$ and zero otherwise.

Theorem 1. *Assume* $EX_1 = 1$ *and* $\int f \, d\lambda > 0$, *then* $\int |p_n - p| \, d\lambda \to 0$ *as* $n \to \infty$.

Theorem 2. *Assume*

(1) $g(x)$ *is of bounded variation on* $[0, 1]$

(2) *the characteristic function* $\Phi(t)=E\,e^{itX_1}$ *has a derivative at* $t=0$, $\Phi'(0)=i\mu$ *and take* $\mu>0$.

(3) $\int f\,d\lambda>0$.

Define measures on $\{R,\mathscr{B}\}$ *by*

$$G_n(B)=\frac{1}{n}\sum_{k=1}^{n} g\left(\frac{k}{n}\right) P\left[\frac{S_k}{n}\in B\right] \quad \text{for } B\in\mathscr{B}.$$

Denote $g_n=\frac{dG_n}{d\lambda}$ *and set* $H_n(B)=G_n(B)-\int_B g_n\,d\lambda$ *for* $B\in\mathscr{B}$. *Set* $\tilde{g}(x)=\mu^{-1}g(x/\mu)$ *for* $x\in[0,\mu]$ *and zero otherwise. Then, as* $n\to\infty$, $\int|g_n-\tilde{g}|\,d\lambda\to 0$ *and*

$$\sup\{|H_n(B)|: B\in\mathscr{B}\}\to 0 \qquad as\ n\to\infty.$$

Corollary 1. *Let* $F(x)$, $\Phi(t)$, $g(x)$ *and* $\tilde{g}$ *satisfy the requirement placed on them in Theorem 2. Let* $\{V_n(x)\}$ *be a sequence of uniformly bounded Borel functions on the line. Suppose* $\{V_n(x)\}$ *converges to* $V(x)$. *Then*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} g\left(\frac{k}{n}\right) E\left[V_n\left(\frac{S_k}{n}\right)\right]=\int\tilde{g}(x)\,V(x)\,dx.$$

Theorem 3. *Suppose that* $F(x)$ *has a non vanishing absolutely continuous component, that* $X\geqq 0$, *and that* $EX=\mu$ *is finite and not zero. Let* $T>0$ *be a real number, let*

$$P_T(x)=\frac{\mu}{T}\sum_{n=1}^{\infty}\sum_{k=1}^{n} P\left[\left\{\frac{S_k}{T}\leqq x\right\}\cap\{S_n\leqq T<S_{n+1}\}\right],$$

let $p_T(x)$ *be the derivative of* $P_T(x)$, *and let* $Q_T(x)$ *be the component of* $P_T(x)$ *that is singular with respect to Lebesgue measure. Let* $p(x)=1$ *if* $x\in[0,1]$ *and zero otherwise. Then* $\lim_{T\to\infty} Q_T(\infty)=0$ *and*

$$\lim_{T\to\infty}\int_0^1 |p_T(x)-p(x)|\,dx=0.$$

Corollary 2. *Suppose that* $F(x)$ *has a non vanishing absolutely continuous component, that* $X\geqq 0$, *and that* $EX=\mu$ *is finite and not zero. For any positive real number* $T>0$, *let* $N_T=n$ *if* $S_n\leqq T<S_n$ *for* $n\geqq 1$, *and* $N_T=0$ *if* $S_1>T$. *Let* $h(x)$ *be a bounded Borel function on* $[0,1]$. *Then*

$$\lim_{T\to\infty} E\left[\frac{\mu}{T}\sum_{k=1}^{N_T} h\left(\frac{S_k}{T}\right)\right]=\int_0^1 h(x)\,dx.$$

Two lemmas needed in the proof of Theorem 3 are:

Lemma 1. *Suppose* $g(x)$ *is a function of bounded variation on* $[0,1]$. *Let* $\sup\{|g(x)|: x\in[0,1]\}=K$ *and let* $T(g)$ *denote the total variation of* g *on* $[0,1]$. *Suppose* $\psi(t)$ *is the characteristic function of a real random variable. Then for any* $n\geqq 1$ *and any real* t,

$$\left|\sum_{k=1}^{n} g\left(\frac{k}{n}\right)\psi(t)^k\right|\leqq\frac{2K+T(g)}{|1-\psi(t)|}.$$

Proof. If $\psi(t)=1$, the right side of the inequality is infinite and the left side is less than nK. If $\psi(t)\neq 1$, then since $|\psi(t)|\leqq 1$

$$\begin{aligned}\left|\sum_{k=1}^{n} g\left(\frac{k}{n}\right)\psi(t)^k\right| &= \left|\frac{1-\psi(t)}{1-\psi(t)}\sum_{k=1}^{n} g\left(\frac{k}{n}\right)\psi(t)^k\right| \\ &\leqq \frac{\left|g\left(\frac{1}{n}\right)\right|+|g(1)|+\sum_{k=1}^{n-1}\left|g\left(\frac{k+1}{n}\right)-g\left(\frac{k}{n}\right)\right|}{|1-\psi(t)|} \\ &\leqq \frac{2K+T(g)}{|1-\psi(t)|}.\end{aligned}$$

Lemma 2. *Let $\psi(t)$ be the characteristic function of a real random variable. Suppose $\psi(t)$ has a derivative at $t=0$, $\psi'(0)=i\mu$. Let $h(x)$ be a real function on the line. Suppose $h(x)$ is bounded and almost everywhere continuous on $[0,1]$. For each $n\geqq 1$, let*

$$\gamma_n(t)=\frac{1}{n}\sum_{k=1}^{n} h\left(\frac{k}{n}\right)\psi\left(\frac{t}{n}\right)^k.$$

Then $\{\gamma_n(t)\}$ converges to $\gamma(t)=\int_0^1 e^{i\mu t x}h(x)\,dx$ uniformly for t in any bounded interval.

Proof. Since the function $h(x)$ is Riemann integrable on $[0,1]$, $\tilde{\gamma}_n(t)=\frac{1}{n}\sum_{k=1}^{n} h\left(\frac{k}{n}\right)e^{i\mu t k/n}$ converges to $\gamma(t)$ for all t. There exists a finite positive number K such that $|h(x)|<K$ for all $x\in[0,1]$. So

$$|\gamma_n(t)-\tilde{\gamma}_n(t)|\leqq\frac{K}{n}\sum_{k=1}^{n}\left|\psi\left(\frac{t}{n}\right)^k-e^{\frac{it\mu k}{n}}\right|.$$

Also

$$\left|\psi\left(\frac{t}{n}\right)^k-e^{\frac{it\mu k}{n}}\right|\leqq\left|\psi\left(\frac{t}{n}\right)-e^{\frac{it\mu}{n}}\right|\sum_{j=0}^{k-1}\left|e^{\frac{it\mu j}{n}}\psi\left(\frac{t}{n}\right)^{k-1-j}\right|\leqq k\left|\psi\left(\frac{t}{n}\right)-e^{\frac{it\mu}{n}}\right|.$$

There exists $\varepsilon>0$ such that $\psi(t)=1+i\mu+o(|t|)$ for $|t|<\varepsilon$. So $\left|\psi\left(\frac{t}{n}\right)-e^{it\mu/n}\right|\leqq o\left(\frac{|t|}{n}\right)$ for $|t|<\varepsilon n$. Thus for any fixed t for n large enough

$$|\gamma_n(t)-\tilde{\gamma}_n(t)|\leqq\frac{K}{n}\sum_{k=1}^{n}\frac{k}{n}n\,o\left(\left|\frac{t}{n}\right|\right)=n\,o\left(\frac{1}{n}\right).$$

It follows that for any t that $\{\gamma_n(t)\}$ converges to $\gamma(t)$. Let I be an arbitrary bounded interval on the line. Let $s\in I$ and $t\in I$ and let $b=\sup\{|x|\colon x\in I\}$. For any $n\geqq 1$,

$$\begin{aligned}|\gamma_n(s)-\gamma_n(t)| &\leqq\frac{K}{n}\sum_{k=1}^{n}\left|\psi\left(\frac{s}{n}\right)^k-\psi\left(\frac{t}{n}\right)^k\right| \\ &\leqq\frac{K}{n}\sum_{k=1}^{n}k\left|\psi\left(\frac{s}{n}\right)-\psi\left(\frac{t}{n}\right)\right| \\ &\leqq Kn\left|\psi\left(\frac{s}{n}\right)-\psi\left(\frac{t}{n}\right)\right|.\end{aligned}$$

Since $\psi(t)$ has a derivative at $t=0$, given $\varepsilon>0$, $n_0(b)$ can be found such that for $n \geqq n_0(b)$,

$$Kn\left|\psi\left(\frac{s}{n}\right)-\psi\left(\frac{t}{n}\right)\right| \leqq K\mu|s-t|+\frac{\varepsilon}{2}$$

for $s, t \in I$. Choose $\delta=\varepsilon/(2K\mu)$, then $|\gamma_n(s)-\gamma_n(t)|<\varepsilon$ for $|s-t|<\delta$ for $n \geqq n_0$. That is $\{\gamma_n(t)\colon n \geqq n_0(b)\}$ is equicontinuous on I and since $\{\gamma_n(t)\}$ converges to $\gamma(t)$ for all t, it follows the convergence is uniform on I. The proof is complete.

Proof of Theorem 2. Theorem 2 is a generalization of Theorem 1. For the proof it is enough to take $g(x) \geqq 0$. We follow Prohorov [5] and prove the theorem using the language of distribution functions. Write $F(x)=P_X[-\infty, x]$ and $G_n(x)=G_n[-\infty, x]$. Then

$$G_n(x)=\frac{1}{n}\sum_{k=1}^{n} g\left(\frac{k}{n}\right) F(nx)^{*k}$$

where $F(\cdot)^{*k}$ is the k-fold convolution of $F(\cdot)$.

Since $\int f d\lambda>0$ we can find a finite positive number k so that $\int_{[f \leqq k]} f d\lambda>0$. Set $u(x)=f(x)$ if $f \leqq k$ and zero otherwise. Denote $\alpha=\int u d\lambda$, set $H_1(x)=\alpha^{-1}\int_{[-\infty, x]} u dx$ and write

$$F(x)=\alpha H_1(x)+(1-\alpha) H_2(x).$$

Let $h_1(t)$ and $h_2(t)$ be the characteristic functions of $H_1(x)$ and $H_2(x)$. Since $u(x)$ is bounded and summable $u(x) \in L_2$ and by Plancherel's theorem $\int |h_1(t)|^2$ is finite. Now

$$G_n(x)=\frac{1}{n}\sum_{k=1}^{n} g\left(\frac{k}{n}\right)[\alpha H_1(xn)+(1-\alpha) H_2(nx)]^{*k}.$$

Set

$$R_n(x)=\frac{1}{n}\sum_{k=1}^{n} g\left(\frac{k}{n}\right)(1-\alpha)^k[H_2(xn)]^{*k},$$

let $r_n(x)$ be the derivative of $R_n(x)$, and let $\tilde{R}_n(x)$ be the component of $R_n(x)$ that is singular with respect to Lebesgue measure. Then

$$R_n(\infty)=\int r_n(x)\, dx+\tilde{R}_n(\infty)=\frac{1}{n}\sum_{k=1}^{n} g\left(\frac{k}{n}\right)(1-\alpha)^k$$

from which it follows that $\{r_n(x)\}$ converges to zero in Lebesgue measure and $\lim_{n\to\infty} \tilde{R}_n(\infty)=0$.

Let $K=\sup\{g(x)\colon x \in [0,1]\}$ which is finite by assumption. Consider

$$\begin{aligned}\Sigma_k(t)&=[\alpha h_1(t)+(1-\alpha) h_2(t)]^k-(1-\alpha)^k h_2^k(t)\\&=\sum_{l=1}^{k}\binom{k}{l}\alpha^l(1-\alpha)^{k-l} h_1^l(t)\, h_2^{k-l}.\end{aligned}$$

Then

$$|\Sigma_k(t)| \leqq \sum_{l=1}^{k}\binom{k}{l}\alpha^l(1-\alpha)^{k-l}|h_1(t)|^l.$$

From this inequality we see two things: First, since $|h_1(t)| \leqq 1$, $|\Sigma_k(t)| \leqq |h_1(t)|$ for all t, from which it follows that the characteristic function of $S_n(x) = G_n(x) - R_n(x)$ which we denote $\sigma_n(t)$ is in L_2, that is $|\sigma_n(t)| \in L_2$. Second, since $h_1(t)$ is the characteristic function of an absolutely continuous distribution function, given $\varepsilon > 0$, there exists a positive constant $c < 1$ such that $|h_1(t)| < c < 1$ for all $|t| > \varepsilon$ and so, for $|t| > \varepsilon$,

$$|\Sigma_k(t)| < \frac{|h_1(t)|}{c} [\alpha c + 1 - \alpha]^k = \frac{|h_1(t)|}{c} [1 - \alpha(1-c)]^k.$$

From this inequality we get for $|t| > \varepsilon$

$$|\sigma_n(t)| \leqq \frac{K|h_1(t)|}{cn} \sum_{k=1}^{n} [1 - \alpha(1-c)]^k \leqq \frac{K|h_1(t)|}{n\alpha c}.$$

Since $\tilde{g}(x) \in L_2$, by Plancherel's theorem

$$\int [s_n(x) - \tilde{g}(x)]^2 \, dx = \frac{1}{2\pi} \int |\sigma_n(t) - \gamma(t)|^2 \, dt$$

where $s_n(x)$ is the derivative of $S_n(x)$ and

$$\gamma(t) = \int e^{itx} \tilde{g}(x) \, dx = \int_0^{\mu} e^{itx} g\left(\frac{x}{\mu}\right) \frac{dx}{\mu} = \int_0^1 e^{it\mu x} g(x) \, dx.$$

Since $\Phi(t)$ has a derivative at $t=0$ and $\Phi'(0) = i\mu$, $\lim_{t \to 0} \frac{|1-\Phi(t)|}{|\mu t|} = 1$; given η, such that $0 < \eta < \frac{1}{2}$ say, we can find an $\varepsilon > 0$ such that for $|t| < \varepsilon$, $|1 - \Phi(t)| > (1-\eta)\,|\mu|\,|t|$ which implies $|1 - \Phi(t)|^{-1} < 2\,|\mu t|^{-1}$ for $|t| < \varepsilon$. For such an $\varepsilon > 0$ write

$$\int |\sigma_n(t) - \gamma(t)|^2 \, dt = A_n + B_n$$

where

$$A_n = \int_{|t| \geqq \varepsilon n} |\sigma_n(t) - \gamma(t)|^2 \, dt$$

and

$$B_n = \int_{|t| < \varepsilon n} |\sigma_n(t) - \gamma(t)|^2 \, dt.$$

We have

$$\begin{aligned} A_n &\leqq 2 \int_{|t| \geqq \varepsilon n} |\sigma_n(t)|^2 \, dt + 2 \int_{|t| \geqq \varepsilon n} |\gamma(t)|^2 \, dt \\ &\leqq \frac{2K^2}{n^2 \alpha^2 c^2} \int |h_1(t)|^2 \, dt + 2 \int_{|t| \geqq \varepsilon n} |\gamma(t)|^2 \, dt, \end{aligned}$$

consequently $\lim_{n \to \infty} A_n = 0$. Furthermore,

$$B_n \leqq 2 \int_{|t| < \varepsilon n} |\gamma_n(t) - \gamma(t)|^2 \, dt + 2 \int_{|t| < \varepsilon n} |\rho_n(t)|^2 \, dt$$

where $\rho_n(t)$ is the characteristic function of $R_n(x)$. Let $a > 0$ be arbitrary and write

$$\int_{|t| < \varepsilon n} |\gamma_n(t) - \gamma(t)|^2 \, dt = D_n + E_n$$

where

$$D_n = \int_{|t| \leqq a} |\gamma_n(t) - \gamma(t)|^2 \, dt$$

and

$$E_n = \int_{a < t < \varepsilon n} |\gamma_n(t) - \gamma(t)|^2 \, dt.$$

Then

$$E_n \leqq 2 \int_{a < |t| < \varepsilon n} |\gamma_n(t)|^2 \, dt + 2 \int_{a < |t| < \varepsilon n} |\gamma(t)|^2 \, dt.$$

By Lemma 1,

$$|\gamma_n(t)| \leqq \frac{2K + T(g)}{n \left| 1 - \Phi\left(\frac{t}{n}\right) \right|}$$

where $T(g)$ is the total variation of $g(x)$ on $[0, 1]$. We selected $\varepsilon > 0$ so that

$$|1 - \phi(t)|^{-1} < 2 |\mu t|^{-1} \quad \text{if } |t| < \varepsilon,$$

so

$$|\gamma_n(t)| \leqq \frac{2[2K + T(g)]}{|\mu| \, |t|} \quad \text{for } |t| < \varepsilon n.$$

It follows that

$$E_n \leqq C(a) = \frac{8[2K + T(g)]^2}{\mu^2 a} + 2 \int_{|t| > a} |\gamma(t)|^2 \, dt.$$

Since $\{\gamma_n(t)\}$ converges to $\gamma(t)$ uniformly for $|t| < a$ it follows that $\lim\limits_{n \to \infty} (D_n + E_n) \leqq C(a)$. But $\lim\limits_{a \to \infty} C(a) = 0$ and a was arbitrary; hence $\lim\limits_{n \to \infty} (D_n + E_n) = 0$.

From

$$\rho_n(t) = \frac{1}{n} \sum_{k=1}^{n} g\left(\frac{k}{h}\right) (1 - \alpha)^k [h_2(t)]^k$$

we have $|\rho_n(t)| \leqq K/(n\alpha)$ and

$$\lim_{n \to \infty} 2 \int_{|t| < \varepsilon n} |\rho_n(t)|^2 \, dt \leqq \lim_{n \to \infty} \frac{2\varepsilon K}{\alpha^2 n} = 0.$$

So we get

$$\lim_{n \to \infty} \int |s_n(x) - \tilde{g}(x)|^2 \, dx = 0$$

which implies that $\{s_n(x)\}$ converges in Lebesgue measure to $\tilde{g}(x)$ and since we have already shown that $\{r_n(x)\}$ converges in Lebesgue measure to zero, it follows that $\{g_n(x)\}$, $(g_n(x) = r_n(x) + s_n(x))$, converges in measure to $\tilde{g}(x)$.

Now

$$G_n(\infty) = \int g_n(x) \, dx + \tilde{R}_n(\infty) = \frac{1}{n} \sum_{k=1}^{n} g\left(\frac{k}{n}\right).$$

Since $\lim_{n\to\infty} \tilde{R}_n(\infty)=0$, we have

$$\lim_{n\to\infty}\int g_n(x)\,dx=\int_0^1 g(x)\,dx=\int \tilde{g}(x)\,dx.$$

It follows by a theorem of Vitali that

$$\lim_{n\to\infty}\int |g_n(x)-\tilde{g}(x)|\,dx=0.$$

The theorem is proved.

Proof of Theorem 3. We have

$$\begin{aligned}
P_T(x)&=\frac{\mu}{T}\sum_{n=1}^{\infty}\sum_{k=1}^{n} P[\{S_k\leqq Tx\}\cap\{S_n\leqq T<S_{n+1}\}]\\
&=\frac{\mu}{T}\sum_{k=1}^{\infty}\sum_{n=k}^{\infty} P[\{S_k\leqq Tx\}\cap\{S_n\leqq T<S_{n+1}\}]\\
&=\frac{\mu}{T}\sum_{k=1}^{\infty} P[S_k\leqq Tx] \quad \text{if } 0\leqq x\leqq 1\\
&=\frac{\mu}{T}\sum_{k=1}^{\infty} P[S_k\leqq T] \quad \text{if } x\geqq 1.
\end{aligned}$$

By the renewal theorem $\lim_{T\to\infty} P_T(x)=x$ if $0\leqq x\leqq 1$. Let $\{T_i: i\geqq 1\}$ be an arbitrary sequence of positive real numbers increasing to infinity. If x is a real number, let $[x]$ be the largest integer that is not greater than x. Let $N_i=[T_i/\mu]$ and let $\alpha_i=T_i/(\mu N_i)$; then $\lim_{i\to\infty}\alpha_i=1$ and $\lim_{i\to\infty} N_i=\infty$.

If $0\leqq x\leqq 1$, write

$$P_{T_i}(x)=P_i(x)+Q_i(x)$$

where

$$\begin{aligned}
P_i(x)&=\frac{\mu}{T_i}\sum_{k=1}^{N_i} P[S_k\leqq xT_i]\\
&=\frac{1}{\alpha_i N_i}\sum_{k=1}^{N_i} P[S_k\leqq \mu x\alpha_i N_i]\\
Q_i(x)&=\frac{\mu}{T_i}\sum_{k=N_i+1}^{\infty} P[S_k\leqq xT_i].
\end{aligned}$$

For any real x and integer $n\geqq 1$, write

$$\tilde{P}_n(x)=\frac{1}{n}\sum_{k=1}^{n} P[S_k\leqq \mu x n]$$

and let $p_n(x)$ denote the density of $P_n(x)$. By Theorem 1, $\tilde{p}_n(x)\xrightarrow{L_1} p(x)$ where $p(x)=1$ if $x\in[0,1]$ and zero otherwise. Let $\{\beta_n\}$ be a sequence of real numbers such that $\lim_{n\to\infty}\beta_n=1$. By a theorem of Lebesgue, $\tilde{p}_n(\beta_n x)\xrightarrow{L_1} p(x)$. Let $p_i(x)$ be the derivative of $P_i(x)$; then $p_i(x)=\tilde{p}_{N_i}(\alpha_i x) I_{[0,1]}(x)$. It follows that $p_i(x) \xrightarrow{L_1} p(x)$ on [0, 1]. This result together with the renewal theorem implies that $\lim_{i\to\infty} Q_i(x)=0$. Consequently, if $p_{T_i}(x)$ denotes the derivative of $P_{T_i}(x)$, we have $p_{T_i}(x) \xrightarrow{L_1} p(x)$ on [0, 1]; but the only restrictions on $\{T_i\}$ are $T_i \geqq 0$ and $T_i \uparrow \infty$. So we have

$$\lim_{T\to\infty} \int_0^1 |p_T(x) - p(x)| \, dx = 0.$$

The theorem is proved.

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