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Comultiplicative Functionals and the Birthing of a Markov Process

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1. Introduction

It has been known for many years that a multiplicative functional, m, may be used to kill a Markov process. A special case of this construction is when m is given by a terminal time T as $m_t = 1_{[0,T)}(t)$. More recently Meyer, Smythe, and Walsh [8] introduced the notion of a coterminal time and showed how a coterminal time may be used to "birth" a Markov process. In view of this it is natural to ask if there is a process – naturally called a comultiplicative functional – which has the same relationship to coterminal times as a multiplicative functional has to terminal times.

After some preliminaries in Section 2 we introduce the notion of a comultiplicative functional in Section 3, Definition (3.1). In Sections 3 and 5 we show that there is a complete duality between comultiplicative functionals and an appropriate class of multiplicative functionals. In the case of coterminal times this reduces to the duality with terminal times given in [8]. In Section 4 we show how a comultiplicative functional may be used to birth a Markov process in a manner that is dual to that by which a multiplicative functional is used to kill a process. Sections 4 and 5 are independent of each other and may be read in either order.

What we develop here might be called an "algebraic" theory since we assume the exceptional sets in our definitions are empty. In light of the recent work of Walsh [9] and Meyer [7] this causes no problems in dealing with multiplicative functionals. However, the σ-algebras that we use are motivated by the results of Meyer [7]. Obviously there are dual perfection properties for comultiplicative functionals, but we do not discuss this here. I hope to return to it in a future publication. The duality developed in Sections 3 and 5 does not involve a Markov process. It may be viewed as a chapter in the duality between shift (Is birth better?) operators (θ_t) and killing operators (k_t) developed by various authors in recent years. See, especially, Azema [1]. The Markov process, itself, enters only in Section 4.

In [8] Meyer, Smythe, and Walsh also showed that a cooptional time may be used to kill a process. There is an analogous result here for cooptional functionals – that is, a functional n satisfying only condition (3.1)(i) of Definition (3.1). How-

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ever, as in [8], this killing with a cooptional functional is much simpler than birthing with a comultiplicative functional, and we leave the construction as an exercise for the interested reader.

2. Notation

We begin by describing the set-up that we shall use throughout this paper. Let $E_\Delta = E \cup \{\Delta\}$ be a separable metric space and let W be the space of all right continuous paths ω from $\mathbb{R}^+ = [0, \infty)$ to E_Δ that admit Δ as a cemetery. That is, $\omega\colon \mathbb{R}^+ \to E_\Delta$ is right continuous and $\omega(t) = \Delta$ for all $t \geqq s$ if $\omega(s) = \Delta$. As usual, $\zeta(\omega) = \inf\{t\colon \omega(t) = \Delta\}$ denotes the lifetime of ω, and by right continuity $\omega(\zeta(\omega)) = \Delta$ if $\zeta(\omega) < \infty$. By convention the infimum of the empty set is $+\infty$, and we extend each ω to $\mathbb{R}^+ \cup \{\infty\}$ by $\omega(\infty) = \Delta$. We let $[\Delta]$ denote the path that is identically equal Δ. Thus $\zeta(\omega) = 0$ if and only if $\omega = [\Delta]$. For each $t \in \mathbb{R}^+$ we define the *shift operator* θ_t and the *killing operator* k_t on W as follows:

$$\theta_t \omega(s) = \omega(s+t), \tag{2.1}$$

$$\begin{aligned} k_t \omega(s) &= \omega(s) \quad &&\text{if } s < t \\ &= \Delta &&\text{if } s \geqq t. \end{aligned} \tag{2.2}$$

Observe that $k_0 \omega = [\Delta]$ and that $\theta_t k_t \omega = [\Delta]$ for all ω. Note also the following identities valid for $s, t \in \mathbb{R}^+$:

$$\begin{aligned} &\text{(i)} \quad \theta_t \theta_s = \theta_{t+s}, \\ &\text{(ii)} \quad k_t k_s = k_{t \wedge s};\ t \wedge s = \min(t, s), \\ &\text{(iii)} \quad \theta_t k_{t+s} = k_s \theta_t. \end{aligned} \tag{2.3}$$

We now fix a subset Ω of W that is closed under the action of θ_t and k_t for each $t \in \mathbb{R}^+$. We let $X_t\colon \Omega \to E_\Delta$ be the coordinate maps, $X_t(\omega) = \omega(t)$ and define the canonical σ-algebras $\mathscr{F}_t^0 = \sigma(X_s\colon s \leqq t)$, $\mathscr{F}^0 = \sigma(X_s, s \geqq 0)$ where each X_s is regarded as a map from Ω to $(E_\Delta, \mathscr{E}_\Delta)$. Here $\mathscr{E}_\Delta$ (resp. $\mathscr{E}$) is the σ-algebra of Borel subsets of E_Δ (resp. E). Also let $\mathscr{E}_\Delta^*$ and $\mathscr{E}^*$ be the universal completions of $\mathscr{E}_\Delta$ and $\mathscr{E}$. Define $\mathscr{F}^*$ and $\mathscr{F}_t^*$ to be the *universal* completions of $\mathscr{F}^0$ and $\mathscr{F}_t^0$ respectively. It is easy to see that k_t is $\mathscr{F}_t^0 | \mathscr{F}^0$ and also $\mathscr{F}_t^* | \mathscr{F}^*$ measurable. Moreover, if $H \in \mathscr{F}^0$ (resp. $H \in \mathscr{F}^*$), then $H \in \mathscr{F}_{t+}^0$ (resp. $\mathscr{F}_{t+}^*$) if and only if $H \circ k_s = H$ for all $s > t$. Finally one easily checks that

$$\mathscr{F}_{t+}^0 = \bigcap\nolimits_{s>t} k_s^{-1} \mathscr{F}^0; \qquad \mathscr{F}_{t+}^* = \bigcap\nolimits_{s>t} k_s^{-1} \mathscr{F}^*. \tag{2.4}$$

We shall use this notation consistently in the sequel. However, we shall have to impose additional assumptions on the separable metric space E_Δ in Section 4. The need for introducing the σ-algebras $\mathscr{F}^*$ and $\mathscr{F}_t^*$ is best understood by looking at the "perfection" properties of multiplicative functionals described on pp. 181–185 of Meyer [7]. Notice that we have not, as yet, introduced any probabilities on Ω.

3. Comultiplicative Functionals

(3.1) *Definition. An $\mathscr{F}^*$ measurable process $n = (n_t)_{t \geqq 0}$ is a comultiplicative functional, abbreviated comf, provided $t \to n_t$ is left continuous on* $(0, \infty)$, $n_0 = 0$, $0 \leqq n_t \leqq 1$

for all $t \in \mathbb{R}^+$, *and* n *satisfies:*

(i) $n_t \circ \theta_s = n_{t+s}$; $t>0$, $s \geqq 0$.

(ii) $n_t = n_t \circ k_s n_s$; $0<t\leqq s$

(iii) $n_t \circ k_s = 1$; $t>s$.

It is assumed that the properties in Definition (3.1) hold *identically* in ω. Note that the normalization $n_0=0$ implies that (3.1)(ii) holds for $0 \leqq t \leqq s$, and that (3.1)(ii) implies that $t \to n_t$ is increasing. Consequently n has right limits on $\mathbb{R}^+$. Of course, (3.1)(i) is just the statement that the left continuous process n is cooptional. See [1]. Note that if $t, s \in \mathbb{R}^+$, then $n_{t \wedge s} = n_t \circ k_s n_s$.

Recall from Meyer-Smythe-Walsh [8], that a positive random variable L on $(\Omega, \mathscr{F}^*)$ is a *coterminal time* provided:

(i) $L \circ \theta_t = (L-t)^+$, $t \geqq 0$,

(ii) $L \circ k_t = L$ on $\{L<t\}$,

(iii) $L \circ k_t \leqq t$, $t \geqq 0$.

It is immediate that a positive random variable L on $(\Omega, \mathscr{F}^*)$ is a coterminal time if and only if the process $n_t = 1_{(L,\infty)}(t)$ is a comf. Thus a comf is "the process version" of a coterminal time.

We collect some elementary properties of a comf in the following:

(3.2) **Proposition.** *Let* n *be a comf. Then:*

(i) $n_t \circ k_r = n_t \circ k_s n_s \circ k_r$ *if* $0 \leqq t \leqq s \leqq r$.

(ii) *If* $t \leqq s$, *then* $n_r \leqq n_r \circ k_s \leqq n_r \circ k_t$ *for all* r.

(iii) $\bar{n}_t = \inf_{s>0} n_t \circ k_s = \lim_{s \to \infty} n_t \circ k_s$ *defines a comf satisfying* $n_t \leqq \bar{n}_t$ *and* $n_t \circ k_s = \bar{n}_t \circ k_s$; $t, s \in \mathbb{R}^+$.

Proof. Composing (3.1)(ii) with k_r yields (i). Fix $t \leqq s$. If $r>s$, then (3.1)(iii) implies $n_r \leqq n_r \circ k_s$, while the same inequality follows if $r \leqq s$ from (3.1)(ii). Replacing s by t in this inequality and composing with k_s gives $n_r \circ k_s \leqq n_r \circ k_t$ if $t \leqq s$. Thus for each fixed $t \geqq 0$, $s \to n_t \circ k_s$ is decreasing and so

$$\bar{n}_t = \inf_{s>0} n_t \circ k_s = \lim_{s \to \infty} n_t \circ k_s \geqq n_t.$$

Clearly $\bar{n}_0 = 0$ and $\bar{n}$ is $\mathscr{F}^*$ measurable. If $t \geqq 0$, then

$$\bar{n}_t \circ \theta_s = \lim_{u \to \infty} n_t \circ k_u \circ \theta_s = \lim_{u \to \infty} n_t \circ \theta_s \circ k_{u-s} = \lim_{u \to \infty} n_{t+s} \circ k_{u-s} = \bar{n}_{t+s},$$

and so $\bar{n}$ satisfies (3.1)(i). It is immediate that $\bar{n}_t \circ k_s = n_t \circ k_s$, and so $\bar{n}$ satisfies (3.1)(iii). Composing (3.1)(ii) with k_r, letting $r \to \infty$, and using the above fact we see that $\bar{n}$ satisfies (3.1)(ii). Finally if $t<s$, $\bar{n}_t = n_t \circ k_s \bar{n}_s$ shows that $t \to \bar{n}_t$ is left continuous on $(0, s)$, and since $s>0$ is arbitrary, $\bar{n}$ is left continuous. This completes the proof of (3.2).

(3.3) *Definition. The comf,* $\bar{n}$, *defined in* (3.2) *is called the exact regularization of* n. *A comf,* n, *is exact if* $n=\bar{n}$, *that is, if* $n_t = \lim_{s \to \infty} n_t \circ k_s$ *for each* $t>0$.

We are going to construct a multiplicative functional (*mf*) from a given comf. However, let us first be precise about what we shall mean by a *mf* in this paper.

(3.4) *Definition. An* $\mathscr{F}^*$ *measurable process* $m=(m_t)_{t\geq 0}$ *is a multiplicative functional provided* $t\to m_t$ *is right continuous on* $\mathbb{R}^+$, $0\leqq m_t\leqq 1$ *for all* $t\in\mathbb{R}^+$, *and* m *satisfies:*

(i) m_t *is* $\mathscr{F}_{t+}^*$ *measurable.*

(ii) $m_{t+s}=m_t m_s\circ\theta_t$; $t,s\in\mathbb{R}^+$.

(iii) $m_t([\Delta])=1$ *for all* $t\geqq 0$.

It is assumed that the above properties hold identically in ω. Clearly (3.4)(ii) implies that $t\to m_t$ is decreasing on $\mathbb{R}^+$. Also observe that (3.4)(i) is equivalent to $m_t\circ k_s=m_t$ for $s>t$. If $t\geqq\zeta(\omega)$, $\theta_t\omega=[\Delta]$ and so (3.4)(ii) and (3.4)(iii) imply that $m_t(\omega)=m_{\zeta(\omega)}(\omega)$ if $t\geqq\zeta(\omega)$. Recall that a *mf*, m, is *exact* if $\lim_{s\downarrow 0} m_{t-s}\circ\theta_s=m_t$ for each $t>0$, and that if m is exact, then $s\to m_{t-s}\circ\theta_s$ is right continuous on $[0,t)$. Here and in the sequel $\lim_{s\downarrow r}$ stands for $\lim_{s\to r, s>r}$ with a similar convention for $\lim_{s\uparrow r}$. If $m^*=(m_t^*)$ is a *mf*, then it is well known that $m_t=\lim_{s\downarrow 0} m_{t-s}^*\circ\theta_s$ if $t>0$ and $m_0=\lim_{t\downarrow 0} m_t$ defines an exact *mf* called the *exact regularization* of m^* and that $m_t\geqq m_t^*$. Moreover, $m_t(\omega)=m_t^*(\omega)$ for all $t\in\mathbb{R}^+$ if $m_0^*(\omega)=1$; in particular, $m_t(\omega)=m_t^*(\omega)$ if $m_t^*(\omega)>0$. See, for example, [9]. However, the situation described here is much simpler since we are dealing with ordinary limits rather than essential limits.

We now fix a comf, n, and proceed to construct an associated *mf*. Firstly define for $t>0$

$$m_t^{\#}=\lim_{s\downarrow 0} n_s\circ k_t=n_{0+}\circ k_t.$$

Since $t\to n_s\circ k_t$ is decreasing for each fixed s according to (3.2)(ii), it follows that $t\to m_t^{\#}$ is decreasing on $(0,\infty)$. Therefore we may define for $t\geqq 0$

$$m_t^*=m_{t+}^{\#}=\lim_{u\downarrow t} n_{0+}\circ k_u. \tag{3.5}$$

Clearly $t\to m_t^*$ is right continuous and decreasing on $\mathbb{R}^+$ with $0\leqq m_t^*\leqq 1$. Moreover m_t^* is $\mathscr{F}_{t+}^*$ measurable because it is clearly $\mathscr{F}^*$ measurable and $m_t^*\circ k_s=m_t^*$ if $s>t$. From (3.2)(i) we have $n_r\circ k_s=n_t\circ k_s n_r\circ k_t$ for $0<r\leqq t\leqq s$. Letting $r\downarrow 0$ and then taking right limits in t we find

$$m_s^{\#}=n_{t+}\circ k_s m_t^*,\qquad 0\leqq t<s. \tag{3.6}$$

Let $n^*(t,s)=n_{t+}\circ k_s$ if $0\leqq t<s$ and note that $s\to n^*(t,s)$ is decreasing on (t,∞). Taking right limits on s in (3.6) yields

$$m_s^*=n^*(t,s+)m_t^*,\qquad 0\leqq t\leqq s \tag{3.7}$$

where, of course, $n^*(t,s+)=\lim_{u\downarrow s} n^*(t,u)$ is defined for all $s\geqq t$. Next observe that

$$m_u^{\#}\circ\theta_v=n_{0+}\circ k_u\circ\theta_v=n_{0+}\circ\theta_v\circ k_{u+v}=n_{v+}\circ k_{u+v}=n^*(v,u+v),$$

and taking right limits on u we see that $m_u^*\circ\theta_v=n^*(v,(u+v)+)$ for all $u,v\geqq 0$. Combining this with (3.7) gives

$$m_s^*=m_{s-t}^*\circ\theta_t m_t^*,\qquad 0\leqq t\leqq s. \tag{3.8}$$

Since $k_u[\Delta]=[\Delta]$, (3.1)(iii) implies that $n_t([\Delta])=1$ for all $t>0$. This and (3.5) show that $m_t^*([\Delta])=1$ for all $t\geqq 0$. Consequently $m^*=(m_t^*)_{t\geqq 0}$ is a *mf* as defined in (3.4). Finally define m to be exact regularization of m^*, that is,

$$\begin{aligned} m_t &= \lim_{s\downarrow 0} m^*_{t-s}\circ\theta_s, \qquad t>0 \\ m_0 &= \lim_{t\downarrow 0} m_t. \end{aligned} \tag{3.9}$$

Let us express m directly in terms of n. From (3.9), for $t>0$ we have

$$\begin{aligned} m_t &= \lim_{s\downarrow 0} m^*_{t-s}\circ\theta_s = \lim_{s\downarrow 0}\lim_{u\downarrow(t-s)} m^\#_u\circ\theta_s = \lim_{s\downarrow 0}\lim_{u\downarrow(t-s)} n_{0+}\circ k_u\circ\theta_s \\ &= \lim_{s\downarrow 0}\lim_{u\downarrow(t-s)} n_{s+}\circ k_{u+s} = \lim_{s\downarrow 0}\lim_{u\downarrow t} n_{s+}\circ k_u. \end{aligned}$$

But using the fact that $s\to n_s$ is increasing this readily yields

$$m_t = \lim_{s\downarrow 0}\lim_{u\downarrow t} n_s\circ k_u, \qquad t>0. \tag{3.10}$$

A comparison of (3.10) and (3.5) is of interest.

We summarize what we have proved so far and collect some additional facts about m in the following:

(3.11) **Theorem.** *Let $m=(m_t)$ be defined by (3.10) if $t>0$ and $m_0=\lim_{t\downarrow 0} m_t$. Then m is an exact multiplicative functional satisfying:*

(i) *$m_t\circ k_t \geqq m_t$ for each $t\geqq 0$.*

(ii) *For each $t>0$, $\lim_{s\uparrow t} m_{t-s}\circ\theta_s\circ k_t$ is either zero or one.*

(iii) *If n is exact, then $m_\infty = n_{0+}$ where, of course, $m_\infty = \lim_{t\to\infty} m_t$.*

Proof. In view of the above construction we need only establish properties (i), (ii), and (iii). Since $k_0\omega=[\Delta]$ for all ω and $m_0([\Delta])=1$, we need only prove (i) for $t>0$. But then from (3.10), $m_t\circ k_t = n_{0+}\circ k_t = m_t^\#$. On the other hand $u\to n_s\circ k_u$ is decreasing and so (3.10) implies $m_t \leqq m_t^\#$, establishing (i). If $0\leqq s<t$, then from (3.10)

$$\begin{aligned} m_{t-s}\circ\theta_s &= \lim_{r\downarrow 0}\lim_{u\downarrow(t-s)} n_r\circ k_u\circ\theta_s = \lim_{r\downarrow 0}\lim_{u\downarrow(t-s)} n_r\circ\theta_s\circ k_{u+s} \\ &= \lim_{r\downarrow 0}\lim_{v\downarrow t} n_{r+s}\circ k_v, \end{aligned}$$

and so

$$m_{t-s}\circ\theta_s\circ k_t = n_{s+}\circ k_t; \qquad 0\leqq s<t. \tag{3.12}$$

Letting $s\uparrow t$, this gives $\lim_{s\uparrow t} m_{t-s}\circ\theta_s\circ k_t = n_t\circ k_t$. But composing (3.1)(ii) with k_s and letting $t\uparrow s$ shows that $n_t\circ k_t$ is either zero or one for each $t>0$, proving (ii).

Finally $m_\infty = \lim_{t\to\infty}\lim_{s\downarrow 0}\lim_{u\downarrow t} n_s\circ k_u$, and since the limits on t and s are really infima, they may be interchanged to obtain

$$m_\infty = \lim_{s\downarrow 0}\lim_{t\to\infty}\lim_{u\downarrow t} n_s\circ k_u = \lim_{s\downarrow 0}\lim_{u\to\infty} n_s\circ k_u.$$

The last equality follows since $u\to n_s\circ k_u$ is decreasing. If n is exact, $n_s\circ k_u \to n_s$ as $u\to\infty$, and so $m_\infty = n_{0+}$ proving (iii).

The following corollary will be used in the next section.

(3.13) **Corollary.** *Let n be exact. Then* $n_{t+}=m_\infty\circ\theta_t$ *for each* $t\geqq 0$ *and* $n_{t+}-n_{s+}=m_\infty\circ\theta_t(1-m_{t-s}\circ\theta_s)$ *for* $0\leqq s<t$.

Proof. The first statement is an immediate consequence of (3.11)(iii). For the second observe that $m_\infty=m_{t-s}m_\infty\circ\theta_{t-s}$, and hence $m_\infty\circ\theta_s=m_{t-s}\circ\theta_s m_\infty\circ\theta_t$.

(3.14) *Remark.* If n is a comf, then it is readily verified that

$$R=\sup\{t\colon n_t=0\}=\inf\{t\colon n_t>0\}$$

defines a coterminal time.

4. Birthing a Markov Process

In this section we shall show how to use a comf to birth a Markov process in a manner that is dual to that by which a *mf* is used to kill a Markov process. See for, example, Section III-3 of [2]. Also the reader should compare our result with that given for coterminal times in [8].

Throughout this section we assume that E_Δ is a U-space, that is, it is homeomorphic to a *universally measurable* subspace of a compact metric space F. In addition we assume that for each initial (probability) measure μ on $(E_\Delta, \mathscr{E}_\Delta)$ there exists a probability measure P^μ on $(\Omega, \mathscr{F}^*)$ such that the coordinate maps (X_t) form a *right process* as defined in [4] under P^μ. As usual we write $\mathscr{F}^\mu$ for the P^μ completion of $\mathscr{F}^0$ (this also equals the P^μ completion of $\mathscr{F}^*$) and $\mathscr{F}_t^\mu$ for the σ-algebra generated by $\mathscr{F}_t^0$ (equivalently $\mathscr{F}_t^*$) and all P^μ null subsets in $\mathscr{F}^\mu$. Our hypotheses imply that the family $(\mathscr{F}_t^\mu)$ is right continuous for each μ and $\mathscr{F}_{t+}^0\subset\mathscr{F}_{t+}^*\subset\mathscr{F}_t^\mu$ for each $t\geqq 0$. As usual we write (P_t) and (U^α) for the semigroup and resolvent of this right process and recall that one of the basic assumptions is that $t\to f(X_t)$ is almost surely right continuous if f is α-excessive for the semigroup (P_t). We refer the reader to [4] for the basic properties of right processes.

We now fix for the remainder of this section an *exact* comf, n, and let m be the exact *mf* constructed from n in Section 3. We write (Q_t) and (V^α) for the semigroup and resolvent generated by m, that is, for $t\in\mathbb{R}^+$ and $\alpha>0$

$$Q_t f(x)=E^x[f(X_t)m_t];\qquad V^\alpha f(x)=E^x\int_0^\infty e^{-\alpha t}f(X_t)m_t\,dt \tag{4.1}$$

for $f\in b\mathscr{E}^*$ (the bounded universally measurable functions on E). We adopt the familiar convention that any function f defined on E is extended to E_Δ by $f(\Delta)=0$. Recall that $X_\infty(\omega)=\Delta$ for all $\omega\in\Omega$. It is standard to check that under our assumptions (Q_t) is a semigroup of subMarkov kernels on $(E,\mathscr{E}^*)$ and that (V^α) is its resolvent. Let

$$S=\inf\{t\colon m_t=0\}. \tag{4.2}$$

Since m is exact it is well known and easy to check that almost surely $t\to V^\alpha f(X_t)$ is right continuous and has left limits for $f\in b\mathscr{E}^*$. In addition, if h is α-m-excessive, that is, α-excessive relative to the semigroup (Q_t), then almost surely $t\to h(X_t)$ is right continuous and has left limits on $[0,S)$. Actually slightly more is true: Almost surely $t\to m_t h(X_t)$ is right continuous and has left limits on $\mathbb{R}^+$. See [2] for these facts.

Recall from (3.11) that $n_{0+} = m_\infty$ and define

$$g(x) = E^x(m_\infty) = E^x(n_{0+}); \quad x \in E. \tag{4.3}$$

Of course, $g(\Delta)=0$. Clearly $0 \leqq g \leqq 1$, and the following calculation shows that g is m-excessive:

$$Q_t g(x) = E^x\{g(X_t) m_t\} = E^x\{m_\infty \circ \theta_t m_t;\ t<\zeta\} = E^x\{m_\infty;\ t<\zeta\} \uparrow g(x)$$

as $t \downarrow 0$ for $x \in E$. Let $E_m = \{x \in E\colon P^x[m_0=1]=1\}$ be the set of permanent points for m and let $E_g = \{x \in E\colon g(x)>0\}$. Then E_m and E_g are universally measurable and $E_g \subset E_m \subset E$. (Actually E_m and E_g are nearly Ray Borel (see [4], but we shall not need this fact.) Finally define

$$R = \sup\{t\colon n_t = 0\} = \inf\{t\colon n_t > 0\}. \tag{4.4}$$

As remarked in (3.14), R is a coterminal time. Since $n_{0+} = m_\infty$, if $g(x)>0$, then $P^x(R=0)>0$ and $P^x(S=\infty)>0$.

We now come to the first fact that we shall need.

(4.5) **Proposition.** *Let f be α-m-excessive and define $h(x)=f(x)/g(x)$ if $x \in E_g$ and $h(x)=0$ if $g(x)=0$. Then $t \to h(X_t)$ is almost surely right continuous on the interval (R, ∞).*

Proof. It suffices to show for each rational r that almost surely on $\{R<r\}$, $t \to h(X_t)$ is right continuous on (r, ∞). If $R<r$, then by (3.11), $m_\infty \circ \theta_r = n_{0+} \circ \theta_r = n_{r+} > 0$, and so $S \circ \theta_r = \infty$. Since f is α-m-excessive, this implies that almost surely on $\{R<r\}$, $t \to f(X_t) \circ \theta_r$ is right continuous on $\mathbb{R}^+$, or equivalently, $t \to f(X_t)$ is right continuous on $[r, \infty)$. But g is m-excessive and so to complete the proof of (4.5) it suffices to show that almost surely on $\{R<r\}$, $t \to g(X_t)$ never vanishes on (r, ∞). Let $T = \inf\{t\colon g(X_t)=0\}$. Then T is an $(\mathscr{F}_t^\mu)$ stopping time for each μ since g is well measurable (see [4]). Since $m_\infty \circ \theta_r > 0$ on $\{R<r\}$ the desired result will follow if we show $P^\mu[\Lambda]=0$ for all μ where

$$\Lambda = \{m_\infty \circ \theta_r > 0, \quad r + T \circ \theta_r < \infty\}.$$

Now $u \to m_\infty \circ \theta_u = n_{u+}$ is increasing and $r \leqq r + T \circ \theta_r = T_r$. Therefore

$$\{m_\infty \circ \theta_r > 0\} \subset \{m_\infty \circ \theta_{T_r} > 0\} = \{m_\infty \circ \theta_T \circ \theta_r > 0\}.$$

Also almost surely on $\{R<r\}$, $g(X_{T_r})=0$ because of the right continuity of $t \to g(X_t)$ on $[r, \infty)$. As a result

$$P^\mu(\Lambda) \leqq E^\mu\{P^{X(r)}[m_\infty \circ \theta_T > 0, g(X_T)=0,\ T<\infty]\}.$$

But for each x in E,

$$\begin{aligned} &P^x[m_\infty \circ \theta_T > 0, g(X_T)=0,\ T<\infty] \\ &\quad = E^x\{P^{X(T)}[m_\infty > 0];\ E^{X(T)}(m_\infty)=0,\ T<\infty\} = 0, \end{aligned}$$

completing the proof of (4.5).

We now define the "conditioned" semigroup

$$\begin{aligned} K_t(x, dy) = Q_t^g(x, dy) &= [g(x)]^{-1} Q_t(x, dy)\, g(y); \quad x \in E_g \\ &= \varepsilon_\Delta(dy); \quad g(x)=0. \end{aligned} \tag{4.6}$$

It is well known and easy to check that $\{K_t\}_{t\geq 0}$ is a subMarkov semigroup and that $K_t(x,\cdot)$ is carried by E_g if $x\in E_g$. Since $t\to m_t g(X_t)$ is almost surely right continuous, it is clear that $t\to K_t f(x)$ is right continuous whenever f is a bounded continuous function on E. Moreover the resolvent $(W^\alpha)_{\alpha>0}$ of (K_t) is given by

$$\begin{aligned} W^\alpha(x,dy) &= [g(x)]^{-1}\, V^\alpha(x,dy) g(y); \qquad x\in E_g \\ &= \alpha^{-1}\varepsilon_\Delta(dy); \qquad g(x)=0. \end{aligned} \tag{4.7}$$

Observe that (4.5) implies that $t\to W^\alpha f(X_t)$ is almost surely right continuous on (R,∞) if $f\in b\mathscr{E}^*$. Also the inequalities $Q_t g\leq g$ and $\alpha V^\alpha g\leq g$ imply that $Q_t(fg)=g K_t f$ and $V^\alpha(fg)=g\, W^\alpha f$.

We shall say that a numerical process $Y=Y_t(\omega)$ defined on $\mathbb{R}^+\times\Omega$ is $(\mathscr{F}^*_{t+})$ *well measurable* provided it is measurable with respect to the σ-algebra on $\mathbb{R}^+\times\Omega$ generated by all bounded $\mathscr{B}^+\otimes\mathscr{F}^*$ measurable processes that are adapted to $(\mathscr{F}^*_{t+})$ and which are right continuous and have left limits. Here $\mathscr{B}^+$ is the σ-algebra of Borel subsets of $\mathbb{R}^+$. The next proposition contains the basic calculation that we shall need.

(4.8) **Proposition.** *Fix μ on E. Let f be a bounded continuous function on E and let Y be a bounded $(\mathscr{F}^*_{t+})$ well measurable process. Then*

$$E^\mu \int f(X_{t+s+\lambda})\, Y_{s+\lambda}\, dn(\lambda) = E^\mu \int K_t f(X_{s+\lambda})\, Y_{s+\lambda}\, dn(\lambda) \tag{4.9}$$

for each $t\geq 0$ and $s>0$. Here we have written $n(\lambda)$ for n_λ for typographical convenience.

Proof. It suffices to prove (4.9) when Y is bounded, right continuous, and $(\mathscr{F}^*_{t+})$ adapted, and so in the remainder of the proof these properties are assumed to hold for Y. Fix $s>0$. Then both sides of (4.9) are right continuous in t and so it suffices to show that they have the same Laplace transform. Let $\varphi(t)$ denote the left hand side of (4.9). For notational convenience write $n^+(u)=n_{u+}$ for $u\geq 0$ and (j,k) for the dyadic rational $j/2^k$; $j=0,\cdots,k=1,2,\cdots$. Also let $\Delta n(j,k)=n^+[(j,k)]-n^+[(j-1,k)]$. Since $\lambda\to f(X_{t+s+\lambda})\, Y_{s+\lambda}$ is right continuous we may write

$$\begin{aligned} \varphi(t) &= E^\mu \int f(X_{t+s+\lambda})\, Y_{s+\lambda}\, dn(\lambda) \\ &= E^\mu\{f(X_{t+s})\, Y_s n^+(0)\} + \lim_{k\to\infty}\sum_{j\geq 1} E^\mu\{f(X_{t+s+(j,k)})\, Y_{s+(j,k)}\,\Delta n(j,k)\}. \end{aligned}$$

Let $A(j)$ denote the j-th term in this summation and let

$$M(j,k)=1-m_{(1,k)}\circ\theta_{(j-1,k)}.$$

Then from (3.13), $\Delta n(j,k)=m_\infty\circ\theta_{(j,k)} M(j,k)$, while

$$m_\infty\circ\theta_{(j,k)} = m_s\circ\theta_{(j,k)}\, m_t\circ\theta_{s+(j,k)}\, m_\infty\circ\theta_{t+s+(j,k)}.$$

Because $\mathscr{F}^*_{r+}\subset\mathscr{F}^\mu_r$ for each $r\geq 0$, if we use the Markov property first at the instant $t+s+(j,k)$ and then at $s+(j,k)$ we find

$$\begin{aligned} A(j) &= E^\mu\{fg(X_{t+s+(j,k)})\, Y_{s+(j,k)}\, m_s\circ\theta_{(j,k)}\, m_t\circ\theta_{s+(j,k)}\, M(j,k)\} \\ &= E^\mu\{Q_t fg(X_{s+(j,k)})\, Y_{s+(j,k)}\, m_s\circ\theta_{(j,k)}\, M(j,k)\}. \end{aligned}$$

Substituting $Q_t fg = g K_t f$ into this expression, using the Markov property at $s+(j,k)$ once again, and then reversing the above steps one finds

$$\begin{aligned} A(j) &= E^\mu\{K_t f(X_{s+(j,k)})\, Y_{s+(j,k)} m_\infty \circ \theta_{s+(j,k)} m_s \circ \theta_{(j,k)} M(j,k)\} \\ &= E^\mu\{K_t f(X_{s+(j,k)})\, Y_{s+(j,k)} \Delta n(j,k)\}. \end{aligned}$$

A similar but simpler argument shows that

$$E^\mu\{f(X_{t+s})\, Y_s n^+(0)\} = E^\mu\{K_t f(X_s)\, Y_s n^+(0)\}.$$

Therefore the Laplace transform of the left hand side of (4.9) may be written

$$E^\mu\{W^\alpha f(X_s)\, Y_s n^+(0)\} + \lim_{k\to\infty} \sum_{j\geq 1} E^\mu\{W^\alpha f(X_{s+(j,k)})\, Y_{s+(j,k)} \Delta n(j,k)\}. \tag{4.10}$$

On the other hand the Laplace transform of the right hand side of (4.9) is

$$E^\mu \int W^\alpha f(X_{s+\lambda})\, Y_{s+\lambda} dn(\lambda) = E^\mu \int W^\alpha f(X_{s+\lambda})\, Y_{s+\lambda} 1_{[R,\infty)}(\lambda)\, dn(\lambda),$$

where R is defined in (4.4). Since $s>0$, $\lambda \to W^\alpha f(X_{s+\lambda})$ is almost surely right continuous on $[R,\infty)$ by the comment below (4.7). Consequently this last expression may be written

$$\begin{aligned} &E^\mu\{W^\alpha f(X_s)\, Y_s n^+(0);\, R=0\} \\ &\quad + \lim_{k\to\infty} \sum_{j\geq 1} E^\mu\{W^\alpha f(X_{s+(j,k)})\, Y_{s+(j,k)} 1_{[R,\infty)}[(j,k)]\, \Delta n(j,k)\}. \end{aligned}$$

But $n^+(0)=0$ if $R>0$ and $\Delta n(j,k)=0$ if $(j,k)<R$. Therefore this last expression which is the Laplace transform of the right hand side of (4.9) becomes

$$\begin{aligned} &E^\mu\{W^\alpha f(X_s)\, Y_s n^+(0)\} \\ &\quad + \lim_{k\to\infty} \sum_{j\geq 1} E^\mu\{W^\alpha f(X_{s+(j,k)})\, Y_{s+(j,k)} \Delta n(j,k)\}. \end{aligned} \tag{4.11}$$

Comparing (4.10) and (4.11) establishes (4.8).

(4.12) *Remark.* Exactly the same argument shows that (4.9) holds whenever Y is bounded and well measurable relative to the system $(\Omega, \mathscr{F}^\mu, \mathscr{F}_t^\mu, P^\mu)$ which satisfies the "usual hypotheses" of the general theory of processes. See [3].

We now are ready to show how to use n to birth a Markov process. Define $\hat{\Omega} = [0,\infty] \times \Omega$ and write $\hat{\omega} = (\lambda, \omega)$, $\lambda \in [0,\infty]$, $\omega \in \Omega$ for the generic point in $\hat{\Omega}$. Let $\mathscr{G} = \mathscr{B} \otimes \mathscr{F}^*$ where $\mathscr{B}$ now denotes the σ-algebra of Borel subsets of $\bar{\mathbb{R}}^+ = [0,\infty]$. Define $\hat{X}_t(\hat{\omega}) = \hat{X}_t((\lambda,\omega)) = X_{t+\lambda}(\omega)$ for $t \geq 0$. Then each $\hat{X}_t\colon \hat{\Omega} \to E_\Delta$ and $t \to \hat{X}_t(\hat{\omega})$ is right continuous on $\mathbb{R}^+$, has Δ as a cemetery, and $\hat{X}_t(\infty,\omega) = \Delta$ for all t. If $\hat{\theta}_t \hat{\omega} = \hat{\theta}_t(\lambda,\omega) = (\lambda, \theta_t \omega)$ and $\hat{k}_t \hat{\omega} = (\lambda, k_{\lambda+t}\omega)$, then it is easy to check that $\hat{X}_t \circ \hat{\theta}_s = \hat{X}_{t+s}$ and $\hat{X}_t \circ \hat{k}_s = \hat{X}_t$ for $t<s$ while $\hat{X}_t \circ \hat{k}_s = [\Delta]$ for $t \geq s$. For each $t \in \mathbb{R}^+$ define a σ-algebra $\hat{\mathscr{G}}_t$ on $\hat{\Omega}$ as follows: $\hat{Y}$ is $\hat{\mathscr{G}}_t$ measurable if and only if it is $\hat{\mathscr{G}}$ measurable and there exists an $(\mathscr{F}_{t+}^*)$ well measurable process Y such that $\hat{Y}(\hat{\omega}) = \hat{Y}(\lambda,\omega) = Y_{t+\lambda}(\omega)$ for $\lambda \in \mathbb{R}^+$. It is immediate that $(\hat{\mathscr{G}}_t)_{t\geq 0}$ is an increasing family of sub-σ-algebras of $\hat{\mathscr{G}}$, and since $\hat{X}_t(\omega) = X_{t+\lambda}(\omega)$ it is clear that each $\hat{X}_t$ is $\hat{\mathscr{G}}_t$ measurable. The family $(\hat{\mathscr{G}}_t)$ need not be right continuous but this causes no problems.

Finally for each μ on E_Δ we define a probability $\hat{P}^\mu$ on $(\hat{\Omega}, \hat{\mathscr{G}})$ as follows

$$\hat{E}^\mu(\hat{Y}) = E^\mu \int_{\mathbb{R}^+} \hat{Y}(\lambda, \cdot)\, dn(\lambda) + E^\mu\{\hat{Y}(\infty, \cdot)\,[1 - n(\infty)]\}, \tag{4.13}$$

for $\hat{Y} \in b\hat{\mathscr{G}}$.

(4.14) **Theorem.** *For each μ on E_Δ, the process $(\hat{\Omega}, \hat{\mathscr{G}}, \hat{\mathscr{G}}_{t+}, \hat{X}_t, \hat{P}^\mu)_{t>0}$ is a strong Markov process with transition semigroup (K_t). The restriction $t>0$ is essential.*

Proof. Fix μ on E_Δ. We begin by showing that $(\hat{\Omega}, \hat{\mathscr{G}}_t, \hat{X}_t, \hat{P}^\mu)_{t>0}$ is a simple Markov process. To this end fix $s>0$ and $t \geqq 0$, and let $\hat{Y} \in \mathrm{b}\hat{\mathscr{G}}_s$. By definition there exists $Y = (Y_u)$ well measurable with respect to $(\mathscr{F}^*_{u+})$ such that $\hat{Y}(\lambda, \omega) = Y_{s+\lambda}(\omega)$. Let f be a bounded continuous function on E. Then $f(\Delta) = 0$. Consequently from (4.13) and (4.8) we have

$$\begin{aligned}\hat{E}^\mu[f(\hat{X}_{t+s})\,\hat{Y}] &= E^\mu \int f(X_{t+s+\lambda})\, Y_{s+\lambda}\, dn(\lambda)\\ &= E^\mu \int K_t f(X_{s+\lambda})\, Y_{s+\lambda}\, dn(\lambda) = \hat{E}^\mu[K_t f(\hat{X}_s)\,\hat{Y}],\end{aligned}$$

and so $(\hat{\Omega}, \hat{\mathscr{G}}_t, \hat{X}_t, \hat{P}^\mu)_{t>0}$ is a Markov process with transition semigroup (K_t).

It is a standard fact in the theory of Markov processes that to complete the proof of (4.14), it suffices to show that $t \to W^\alpha f(\hat{X}_t)$ is $\hat{P}^\mu$ almost surely right continuous on $(0, \infty)$ for each bounded continuous f on E. See I-8.11 of [2]. To this end let $f \in b\mathscr{E}^*$ and $h = W^\alpha f$. Then as remarked below (4.7), $t \to h(X_t)$ is almost surely right continuous on (R, ∞). Let $\hat{\Gamma} \subset [0, \infty] \times \Omega$ be the set of $\hat{\omega}$ such that $t \to h(\hat{X}_t(\hat{\omega}))$ is not right continuous on $(0, \infty)$. Since $\hat{X}_t[(\infty, \omega)] = \Delta$, $\hat{\Gamma} \subset \mathbb{R}^+ \times \Omega$. By definition $h(\hat{X}_t(\omega)) = h(X_{t+\lambda}(\omega))$ and so $(\lambda, \omega) \in \hat{\Gamma}$ if and only if $t \to h(X_t(\omega))$ is not right continuous on (λ, ∞). Let $\hat{Y}$ be the indicator of $\hat{\Gamma}$. It is a standard fact that for each λ, $\omega \to \hat{Y}(\lambda, \omega)$ is $\mathscr{F}^*$ measurable. (See, for example, the argument in [6].) Moreover, it is evident that $\lambda \to \hat{Y}(\lambda, \omega)$ is decreasing for each ω. Also if (λ_n) is a sequence that decreases to λ with $\lambda_n > \lambda$ for each n and if $\hat{Y}(\lambda_n, \omega) = 0$ for all n, then $\hat{Y}(\lambda, \omega) = 0$. Consequently $\lambda \to \hat{Y}(\lambda, \omega)$ is right continuous for each ω, and so $\hat{Y}$ is $\hat{\mathscr{G}}$ measurable. Hence

$$\hat{P}^\mu(\hat{\Gamma}) = \mathrm{E}^\mu \int \hat{Y}(\lambda, \cdot)\, dn(\lambda) = E^\mu \int_{[R, \infty)} \hat{Y}(\lambda, \cdot)\, dn(\lambda).$$

But if Λ is the set of ω's such that $t \to h(X_t(\omega))$ is not right continuous on $(R(\omega), \infty)$, and if $\lambda \geqq R(\omega)$, then $\hat{Y}(\lambda, \omega) \leqq 1_{[R(\omega), \infty)}(\lambda)\, 1_\Lambda(\omega)$ and so $\hat{P}^\mu(\hat{\Gamma}) \leqq P^\mu(\Lambda) = 0$. This completes the proof of Theorem (4.14).

(4.15) *Remark.* Of course, the argument in the last paragraph of the proof of (4.14) shows that if h is α-excessive for the semigroup (K_t), then $t \to h(\hat{X}_t)$ is almost surely right continuous on $(0, \infty)$.

We close this section by sketching very briefly another method for constructing the birthed process corresponding to n. This method is analogous to to that used by Meyer in [5] to kill a process. Let $q(d\lambda) = e^{-\lambda} d\lambda$ on $\mathbb{R}^+$. Let $\check{\Omega} = \mathbb{R}^+ \times \Omega$ and $\check{\mathscr{G}} = \mathscr{B}^+ \otimes \mathscr{F}^*$. Define $\check{P}^\mu$ on $\check{\mathscr{G}}$ by $\check{P}^\mu = q \times P^\mu$. Let $\check{\omega} = (\lambda, \omega)$ and put $X_t(\check{\omega}) = X_t(\omega)$. Next define

$$L(\check{\omega}) = L(\lambda, \omega) = \sup\{t \colon -\log n_t(\omega) > \lambda\},$$

and $\check{X}_t = X_{t+L}$. Then one easily checks that

$$\check{E}^\mu[f(\check{X}_t)] = E^\mu \int f(X_{t+\lambda}) dn(\lambda).$$

Now making use of (4.8) one can show that $(\check{X}_t, \check{P}^\mu)_{t>0}$ is a Markov process with transition semigroup (K_t) provided one excercises a bit of care in defining the appropriate σ-algebras. Moreover if one defines $\check{\theta}_t(\lambda, \omega) = (\lambda, \theta_t \omega)$ and

$$\check{k}_t(\lambda, \omega) = ([\lambda + \log n_t(\omega)]^+, k_t \omega),$$

then L becomes a coterminal time in the sense of [8], and one can use the results of [8] to conclude that $(\check{X}_t, \check{P}^\mu)_{t>0}$ is a strong Markov process with transition semigroup (K_t). However, to carry this program through entails some difficulties and we prefer the approach given here.

5. Duality

In order to complete the circle of ideas begun in Section 3 we are going to show in this section how to construct a comf, n, from a *mf*, m, in such a way that applying the construction in Section 3 to n gives the original m, at least if m is exact. However, in view of Theorem (3.11) we can only hope to accomplish this if the m we start with satisfies (3.11)(i) and (3.11)(ii). Before coming to the construction we list some elementary consequences of Definition (3.4). If m is a *mf*, then $m_\infty = \lim_{t\to\infty} m_t$ and $m_\infty = m_t m_\infty \circ \theta_t$ for each $t \geqq 0$. Also recall that $m_t = m_\zeta$ if $t \geqq \zeta$.

(5.1) **Proposition.** *Let m be a mf. Then*

(i) $m_\infty \circ k_s = m_t \circ k_s = m_s \circ k_s$ *for* $0 \leqq s < t$;

(ii) $t \to m_\infty \circ \theta_t$ *is increasing on* $\mathbb{R}^+$.

Proof. Since $\theta_t k_t \omega = [\Delta]$, we have

$$m_\infty \circ k_t = m_t \circ k_t m_\infty \circ \theta_t \circ k_t = m_t \circ k_t,$$

and composing with k_s, $s<t$, gives $m_t \circ k_s = m_\infty \circ k_s = m_s \circ k_s$ proving (i). Also $m_\infty \circ \theta_s = m_t \circ \theta_s m_\infty \circ \theta_{t+s} \leqq m_\infty \circ \theta_{t+s}$, proving (ii).

It will be convenient to introduce the notation

$$m(s,t] = m_{t-s} \circ \theta_s, \qquad 0 \leqq s < t. \tag{5.2}$$

Then $t \to m(s,t]$ is right continuous and decreasing on (s, ∞), while $s \to m(s,t]$ is increasing on $[0, t)$ and even right continuous if m is exact. It is immediate from (3.4)(ii) that for $0 \leqq r < s < t$

$$m(r,t] = m(r,s]\, m(s,t]. \tag{5.3}$$

Define $m(t-,t] = \lim_{s \uparrow t} m(s,t]$ when $t>0$. Then property (3.11)(ii) may be written

$$m(t-,t] \circ k_t = 0 \quad \text{or} \quad 1, \tag{5.4}$$

(5.5) **Proposition.** *Let m be a mf. Then m satisfies* (5.4) *if and only if for each* $t>0$ *either* $m_t \circ k_t = m_{t-}$ *or* $m_t \circ k_t = 0$. *If* $m_{\zeta-} = m_\zeta$, *then m satisfies* (5.4).

Proof. Letting $s \uparrow t$ in $m_t = m_s m(s,t]$ gives $m_t = m_{t-} m(t-, t]$ when $t > 0$. But $m_s \circ k_t = m_s$ if $s < t$ and so $m_{t-} \circ k_t = m_{t-}$. Therefore if $t > 0$

$$m_t \circ k_t = m_{t-} \, m(t-, t] \circ k_t, \tag{5.6}$$

and hence (5.4) implies that either $m_t \circ k_t = m_{t-}$ or $m_t \circ k_t = 0$. Conversely if we define $m^*(s,t) = \lim_{u \uparrow t} m(s,u] = m(s, t-]$ for $0 \leqq s < t$, then letting $s \uparrow t$ in (5.3) gives $m(r,t] = m^*(r,t) \, m(t-, t]$. It is evident that $m^*(r,t) \circ k_t = m^*(r,t)$, and so

$$m(r,t] \circ k_t = m^*(r,t) \, m(t-, t] \circ k_t. \tag{5.7}$$

Now $r \to m(r,t] \circ k_t$ is increasing on $(0,t)$ and if it vanishes on this interval, then certainly $m(t-, t] \circ k_t = 0$. If on the other hand for some $r_0 < t$, $m(r,t] \circ k_t > 0$ for $r_0 < r < t$, then

$$0 < m(r,t] \circ k_t = m_{t-r} \circ \theta_r \circ k_t = m_{t-r} \circ k_{t-r} \circ \theta_r.$$

But by hypothesis for each $u > 0$, either $m_u \circ k_u = m_{u-}$ or zero, and so for $r_0 < r < t$

$$m(r,t] \circ k_t = m_{(t-r)-} \circ \theta_r = m^*(r,t).$$

Substituting this expression for $m^*(r,t)$ into (5.7) and letting $r \uparrow t$ shows that $m(t-, t] \circ k_t$ is either zero or one. This establishes the first assertion in (5.5).

For the second note that $\zeta \circ k_t = \zeta \wedge t$. Using this one easily checks that $m_{\zeta-} = m_\zeta$ implies that $m_t \circ k_t = m_{t-}$ for each $t > 0$, completing the proof of (5.5).

A positive $\mathscr{F}^*$ measurable random variable T is called a *terminal time* if it satisfies:

(i) $\{T \leqq t\} \in \mathscr{F}^*_{t+}$ for each $t \geqq 0$

(ii) $t + T \circ \theta_t = t$ on $\{T > t\}$,

(iii) $T([\Delta]) = \infty$.

It is not difficult to see that (i) is equivalent to

(i′) $T < t$ implies $T \circ k_t = T$.

If T is a terminal time, then one checks readily that $m_t = 1_{[0,T)}(t)$ is a *mf*. Note that condition (5.4) is automatically satisfied in this case since m only takes the two values zero or one.

In view of (5.1)(i) property (3.11)(i) may be written

$$m_\infty \circ k_t = m_t \circ k_t \geqq m_t; \qquad t \geqq 0. \tag{5.8}$$

If $m_t = 1_{[0,T)}(t)$ with T a terminal time, then (5.8) is equivalent to

(iv) $t < T$ implies $T \circ k_t = \infty$.

But (iv) is a condition imposed on a terminal time in [8]. Here we prefer not to make (iv) part of the definition of a terminal time; rather we shall impose (iv), or equivalently (5.8) as needed.

We now fix a mf, m, that satisfies both (5.4) *and* (5.8). We are going to construct a comf, n, from m by a procedure that is dual to that used in Section 3 but is slightly more complicated. Define

$$\begin{aligned}&\text{(a)}\quad n_t^{\#}=\lim_{s\to\infty} m(t,s]=m_\infty\circ\theta_t; \qquad t\geqq 0\\&\text{(b)}\quad n_t^{*}=\lim_{s\uparrow t} n_s^{\#}=n_{t-}^{\#}; \qquad t>0.\end{aligned}\tag{5.9}$$

In view of (5.1)(ii) both $n^{\#}$ and n^* are increasing, and n^* is left continuous on $(0,\infty)$. Also it is clear that $n_t^{\#}\circ\theta_s=n_{t+s}^{\#}$ for $t,s\geqq 0$, and consequently $n_t^*\circ\theta_s=n_{t+s}^*$ for $t>0$, $s\geqq 0$. Since $\theta_t k_s\omega=[\Delta]$ if $s\leqq t$, we have $n_t^{\#}\circ k_s=1$ for $s\leqq t$ and $n_t^*\circ k_s=1$ for $s<t$. Using (5.3) we see that

$$n_t^{\#}=m(t,s]\,n_s^{\#}, \qquad 0\leqq t<s. \tag{5.10}$$

Composing with k_s and using $n_s^{\#}\circ k_s=1$ gives $n_t^{\#}\circ k_s=m(t,s]\circ k_s$, $0\leqq t<s$. Taking left limits on s in (5.10) yields $n_t^{\#}=m^*(t,s)n_s^*$, $0\leqq t<s$ where $m^*(t,s)=m(t,s-]$ was defined in the proof of (5.5). Composing this last equality with k_s and comparing with the previous expression for $n_t^{\#}\circ k_s$ we find $m(t,s]\circ k_s=m^*(t,s)\,n_s^*\circ k_s$ when $0\leqq t<s$. Now taking left limits on t we obtain

$$m(t-,s]\circ k_s=m^*(t-,s)n_s^*\circ k_s, \qquad 0<t\leqq s, \tag{5.11}$$

where, of course, $m^*(t-,s)=\lim_{u\uparrow t} m^*(u,s)$. Next taking left limits first on s and then on t in (5.10) we obtain

$$n_t^*=m^*(t-,s)\,n_s^*, \qquad 0<t\leqq s. \tag{5.12}$$

Composing this with k_s and using (5.11) we see that $n_t^*\circ k_s=m(t-,s]\circ k_s$. Let $t\uparrow s$. Since $t\to m(t-,s]$ and $t\to m(t,s]$ have the same limit as $t\uparrow s$ we find

$$n_s^*\circ k_s=m(s-,s]\circ k_s, \qquad s>0. \tag{5.13}$$

From (5.12), $n_t^*\circ k_s=m^*(t-,s)\,n_s^*\circ k_s$. By (5.13) and (5.4), $n_s^*\circ k_s$ is either 0 or 1. If $n_s^*\circ k_s=1$, then $n_t^*\circ k_s=m^*(t-,s)$, $0<t\leqq s$, and substituting this into (5.12) yields

$$n_t^*=n_t^*\circ k_s n_s^*, \qquad 0<t\leqq s. \tag{5.14}$$

On the other hand if $n_s^*\circ k_s=0$, then $n_t^*\circ k_s=0$ for $t\leqq s$, and so in order to establish (5.14) in this case it suffices to show that $n_t^*=0$. But $n_t^*\circ k_s=0$ for $0<t\leqq s$ implies $n_t^{\#}\circ k_s=0$ for $0\leqq t<s$. From (5.9)(a) and (5.8) for $t<s$,

$$0=n_t^{\#}\circ k_s=m_\infty\circ\theta_t\circ k_s=m_\infty\circ k_{s-t}\circ\theta_t\geqq m_{s-t}\circ\theta_t.$$

In other words $m(t,s]=0$, and so by (5.10), $n_t^{\#}=0$ for $0<t<s$. Consequently $n_t^*=0$ for $0<t\leqq s$, which establishes (5.14). Therefore, in view of the remarks below (5.9), if we define $n_0^*=0$, then $n^*=(n_t^*)_{t\geqq 0}$ is a comf.

Finally we define n to be the exact regularization of n^* as in (3.3), that is,

$$n_t=\inf_{s>0} n_t^*\circ k_s=\lim_{s\to\infty} n_t^*\circ k_s, \qquad t\geqq 0. \tag{5.15}$$

By an argument dual to the proof of (3.10) one obtains for $t>0$

$$n_t=\lim_{s\to\infty}\lim_{u\uparrow t} m_{s-u}\circ\theta_u\circ k_s=\lim_{s\to\infty} m(t-,s]\circ k_s. \tag{5.16}$$

Now (5.8) implies that $t \to m_t \circ k_t$ is decreasing and so $m_{s-u} \circ \theta_u \circ k_s = m_{s-u} \circ k_{s-u} \circ \theta_u$ decreases as s increases. Therefore

$$n_{0+} = \lim_{s\to\infty} \lim_{t\downarrow 0} m(t-, s] \circ k_s = \lim_{s\to\infty} \lim_{t\downarrow 0} m(t, s] \circ k_s. \tag{5.17}$$

If m is exact the limit on t in (5.17) is just $m_s \circ k_s$ and so $n_{0+} = \lim_{s\to\infty} m_s \circ k_s$. By (5.8), $m_s \circ k_s \geqq m_s$ and therefore $n_{0+} \geqq m_\infty$. But (5.4) and (5.5) imply that $m_s \circ k_s \leqq m_{s-}$ for all $s>0$, and so $n_{0+} \leqq m_\infty$. Therefore $n_{0+} = m_\infty$.

We now summarize what we have proved.

(5.18) **Theorem.** *Let m be a mf satisfying* (5.4) *and* (5.8). *Then n_t defined by* (5.16) *for $t>0$ and $n_0=0$ defines an exact comf. If m is exact, then $n_{0+}=m_\infty$.*

Finally suppose we begin with a comf, n, and apply (3.11) to obtain an exact *mf*, m, satisfying (5.4) and (5.8). Next apply (5.18) to m to obtain an exact comf, $\bar{n}$. Using (5.16) and (3.12) one may write for $t>0$

$$\begin{aligned}\bar{n}_t &= \lim_{s\to\infty} \lim_{u\uparrow t} m_{s-u} \circ \theta_u \circ k_s \\ &= \lim_{s\to\infty} \lim_{u\uparrow t} n_{u+} \circ k_s = \lim_{s\to\infty} n_t \circ k_s,\end{aligned}$$

and so $\bar{n}$ is the exact regularization of n. In particular, if n is exact, then $\bar{n}=n$.

Conversely suppose m is a *mf* satisfying (5.4) and (5.8). Apply (5.18) to obtain an exact comf, n, and then apply (3.11) to n to obtain an exact *mf*, $\bar{m}$. We claim that $\bar{m}$ is the exact regularization of m. If $t>0$, (3.10) and (5.17) imply

$$\begin{aligned}\bar{m}_t &\geqq \lim_{u\downarrow t} n_{0+} \circ k_u = \lim_{u\downarrow t} \lim_{s\to\infty} \lim_{v\downarrow 0} m(v, s] \circ k_u \\ &\geqq \lim_{u\downarrow t} m_\infty \circ k_u \geqq \lim_{u\downarrow t} m_u = m_t\end{aligned}$$

where we have used (5.8). Thus $\bar{m}_t \geqq m_t$ for all t. If $\tilde{m}$ is the exact regularization of m, it follows from this that $\bar{m} \geqq \tilde{m}$. For the reverse inequality note that (5.4) and (5.5) imply $m_t \circ k_t \leqq m_{t-}$ for all $t>0$. Since n is the exact regularization of n^* defined in (5.9), $n_t \circ k_s = n_t^* \circ k_s$, and it was shown below (5.12) that $n_t^* \circ k_s = m(t-, s] \circ k_s$. Combining these remarks with (3.10) we obtain

$$\begin{aligned}\bar{m}_t &= \lim_{s\downarrow 0} \lim_{u\downarrow t} m(s-, u] \circ k_u \leqq \lim_{s\downarrow 0} \lim_{u\downarrow t} m(s, u] \circ k_u \\ &= \lim_{s\downarrow 0} \lim_{u\downarrow t} m_{u-s} \circ k_{u-s} \circ \theta_s \\ &\leqq \lim_{s\downarrow 0} \lim_{u\downarrow t} m_{(u-s)-} \circ \theta_s = \lim_{s\downarrow 0} m_{t-s} \circ \theta_s = \tilde{m}_t\end{aligned}$$

for $t>0$. Consequently $\bar{m} = \tilde{m}$ proving the claim at the beginning of this paragraph.

We now have stablished a complete duality between exact comf's and exact *mf*'s satisfying (5.4) and (5.8).

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