Z. Wahrscheinlichkeitstheorie verw. Geb. 19, 319–322 (1971)


# On the Convergence of Martingale Transforms

LUIS BÁEZ-DUARTE

## 1. Introduction

Let $(\Omega, \mathscr{F}, P)$ be a probability space and $\{\mathscr{F}_n\}_{n\geqq 0}$ an increasing sequence of $\sigma$-algebras contained in $\mathscr{F}$. For a martingale $\{x_n\}_{n\geqq 1}$ and a "multiplier sequence" $\{v_n\}_{n\geqq 1}$, where each $v_n \in L_\infty(\mathscr{F}_{n-1})$, Burkholder has defined the *transform* of $\{x_n\}$ as the martingale $\{y_n\}$ given by

$$y_n = \sum_{k=1}^{n} v_k(x_k - x_{k-1}), \qquad n \geqq 1, \qquad x_0 = 0.$$

and then has established [2, Theorem 1] the following

**Theorem 1.** *If* $\{x_n\}$ *is* $L_1$*-bounded, then* $\{y_n\}$ *converges almost everywhere in the set where* $\sup_n |v_n|$ *is finite.*

Building from this result using a difficult technique developed in an earlier paper [3], Burkholder arrived at some very interesting weak-type inequalities, among them the following *maximal theorem for martingale transforms* [2, Theorem 6]: If $\sup |v_n| \leqq 1$, then there is a universal constant $M$ such that

$$P(\sup |y_n| > \lambda) \leqq \frac{M}{\lambda} \sup \|x_n\|_1 \tag{1}$$

for all $\lambda > 0$.

Weak-type inequalities such as this are often the crucial fact needed in order to prove almost everywhere convergence of a wide variety of processes, as is the case in ergodic theory, and in the theory of orthogonal expansions. In [1] the author gave a proof of Doob's martingale convergence theorem based on the relevant maximal inequality (for yet another approach see Isaac [6]). We have been able to extend our method to cover the case of martingale transforms. So in this paper we show how Theorem 1 can be deduced from the inequality (1). Thus once more a maximal inequality turns out to be equivalent to pointwise convergence. In view of this result one should like to see a direct proof of the maximal inequality for martingale transforms completely independent of their convergence a.e. In this respect we would like to remark that Gundy [5] has identified a large class of inequalities including those of Burkholder, and proved them with "elementary" martingale techniques (e.g. stopping times). His proof however is based on a somewhat intricate decomposition theorem for $L_1$-bounded martingales which by itself implies Theorem 1.

## 2. The Convergence Theorem

As remarked by Burkholder himself it is easy to see that it suffices to establish Theorem 1 in the case where $\sup \|v_n\|_\infty \leqq 1$. For convenience we introduce the following linear operators

$$T_{m,n} = \sum_{k=m}^{n} v_k (E_k - E_{k-1}), \qquad 1 \leqq m \leqq n,$$

where $E_k = E(\cdot|\mathscr{F}_k)$, $k \geqq 1$, and $E_0 = 0$. Setting $T_{1,n} = T_n$ we have $y_n = T_n x_n$, $n \geqq 1$, is the Burkholder transform of $\{x_n\}$. It is important to note that $m \leqq k \leqq n$ implies $E_k T_{m,n} = T_{m,n} E_k = T_{m,k}$. From Theorems 3.1 and 2.1 of [1] one may assume without loss of generality that the $L_1$-bounded martingale $\{x_n\}$ is *measure dominated*, that is, there is an integrable random variable $x_\infty$, and a finite signed measure $v$, *singular with respect to P*, such that its restrictions to each $\mathscr{F}_n$ are absolutely continuous with density $z_n$, and

$$x_n = E_n x_\infty + z_n, \qquad n \geqq 1,$$

$$\int_{E_n} z_n \, dP = v(E_n), \qquad \text{for all } E_n \in \mathscr{F}_n. \tag{2}$$

Clearly $T_n x_n = T_n x_\infty + T_n z_n$ so the following two propositions accomplish the proof of the theorem.

**Proposition 1.** *If* $x_\infty \in L_1$, $\sup |v_n| \leqq 1$, *then* $T_n x_\infty$ *converges almost everywhere.*

*Proof.* Here one could apply Banach's principle ([4], Theorem 2, p. 332) to the operators $\{T_n\}$, however we prefer a direct argument. Set $y_n = E_n x_\infty$, and note that $\|x_\infty - y_n\|_1 \to 0$ assuming without loss of generality that $\mathscr{F}$ is generated by the $\{\mathscr{F}_n\}$. Now select a sequence of integers $m_k \uparrow \infty$, such that $\sum k \|x_\infty - y_{m_k}\|_1 < \infty$. Next observe that $\{y_n - y_{m_k}\}_{n \geqq m_k}$ is a martingale indexed by $n$, and therefore (1) gives

$$P\left(\sup_{n \geqq m_k} |T_n y_n - T_{m_k} y_{m_k}| > \frac{1}{k}\right) \leqq M k \|x_\infty - y_{m_k}\|_1.$$

Thus adding for $k = 1, 2, \ldots$ we get

$$\sum_{k=1}^{\infty} P\left(\sup_{n \geqq m_k} |T_n y_n - T_{m_k} y_{m_k}| > \frac{1}{k}\right) < \infty.$$

Now using the Borel-Cantelli lemma we conclude that for almost every $\omega$ the events $\left(\sup_{n \geqq m_k} |T_n y_n - T_{m_k} y_{m_k}| \geqq \frac{1}{k}\right)$ occur only for a finite number of $k$'s. That is, $T_n y_n(\omega)$ is a Cauchy sequence $P$-a.e. But $T_n y_n(\omega) = T_n x_\infty(\omega)$. q.e.d.

**Proposition 2.** *Let $v$ be a finite signed singular measure on $\mathscr{F}$ which dominates a martingale $\{z_n\}$, i.e., (2) is satisfied. If* $\sup_n |v_n| \leqq 1$ *then* $T_n z_n$ *converges P-almost everywhere.*

Without loss of generality we assume that the algebra of sets $\mathscr{E}=\bigcup_{n=1}^{\infty}\mathscr{F}_n$ generates $\mathscr{F}$. The next lemma was used without proof in [1].

**Lemma.** *For every* $\delta_1>0$, $\delta_2>0$ *one can find a set* $E\in\mathscr{E}$ *such that* $P(E)<\delta_1$, *and* $|v|(E^c)<\delta_2$.

*Proof of the Lemma.* Let $N$ be a support of $v$, then from the theory of outer measures we know there is a set $F\in\mathscr{E}_\sigma$ such that $N\subseteq F$ and $P(F)<\delta_1$. On the other hand we have $|v|(F)=|v|(N)=|v|(\Omega)$. Since $F=\bigcup_{m=1}^{\infty}E_m$, with $E_m\in\mathscr{E}$, find $p$ large enough so that $|v|\left(\bigcup_{m=1}^{p}E_m\right)\geqq|v|(\Omega)-\delta_2$. Now set $E=\bigcup_{m=1}^{p}E_m$. q.e.d.

*Proof of Proposition* 2. Given a preassigned $\varepsilon>0$, by repeated applications of the lemma we arrive at a sequence of sets $\{E_k\}$ such that $E_k\in\mathscr{F}_{m_k}$, $m_k\uparrow\infty$, and

$$\sum_{k=1}^{\infty}P(E_k)<\varepsilon, \tag{i}$$

$$\sum_{k=1}^{\infty}k\,|v|(E_k^c)<\infty. \tag{ii}$$

For $n\geqq m_k$ one easily sees that the commutation relations of the $T_{m,n}$ with conditional expectations yield

$$(T_n x_n - T_{m_k}x_{m_k})\,\chi_{E_k^c}=T_{m_k,n}(x_n\,\chi_{E_k^c}).$$

But, for each $k$, the sequence $\{x_n\,\chi_{E_k^c}\}_{n\geqq m_k}$ is a martingale indexed by $n$, so (1) gives

$$P\left(\sup_{n\geqq m_k}|T_n x_n - T_{m_k}x_{m_k}|>\frac{1}{k}\right)$$
$$\leqq M k \sup_{n\geqq m_k}\int_{E_k^c}|x_n|\,dP\leqq M k\,|v|(E_k^c),$$

since $E_k^c\in\mathscr{F}_n$ for all $n\geqq m_k$, and therefore $\int_{E_k^c}|x_n|\,dP$ is the total variation of $v$ restricted to $\mathscr{F}_n$ over $E_k^c$. Now adding for $k=1,2,\ldots$ we get taking account of (ii)

$$\sum_{k=1}^{\infty}P\left(\sup_{n\geqq m_k}|T_n x_n - T_{m_k}x_{m_k}|\,\chi_{E_k^c}>\frac{1}{k}\right)<\infty.$$

As in Proposition 1 resort is made to the Borel-Cantelli lemma to conclude that for almost every $\omega$ there is only a finite number of $k$'s (depending on $\omega$) for which $\sup_{n\geqq m_k}|T_n x_n - T_{m_k}x_{m_k}|\,\chi_{E_k^c}>\frac{1}{k}$. Thus $T_n x_n(\omega)$ is a Cauchy sequence for almost every $\omega$ in $\bigcap_{k\geqq 1}E_k^c$, i.e., outside $D(\varepsilon)=\bigcup_{k\geqq 1}E_k$, with $P(D(\varepsilon))<\varepsilon$ according to (i). So $T_n x_n(\omega)$ converges a.e. outside the null set $\bigcap_{r\geqq 1}D\left(\frac{1}{r}\right)$. q.e.d.

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Luis Báez-Duarte
Instituto Venezolano
de Investigaciones Científicas
Departamento de Matemáticas
Apartado 1827
Caracas, Venezuela


*(Received July 20, 1970)*