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Limits of Directed Projective Systems of Probability Spaces

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Summary. The problem of limits of directed projective systems of probability spaces is treated from the categorical point of view, with certain equivalence classes of measurable measurepreserving mappings, or of regular conditional probabilities playing the role of morphisms. A.o. the existence of limits of such systems is established under the condition that the spaces carry compact generating pavings, without invoking conditions like Bochners' sequential maximality. There are some side results on liftings and general martingales.

1. Introduction

Projective systems of probability spaces were first introduced in the probabilistic literature by Bochner [1]. The importance of such projective systems in probability theory stems from the fact that the description of a stochastic process is in terms of a projective system (namely the projective systems of the finite-dimensional distributions of the process) on one hand and, on the other hand, from the problem of the structure of probability spaces (in particular, under what conditions is a probability space "completely determined" by the lattice of its finite partitions; this lattice being also a natural example of a projective system of probability spaces).

The problem treated and partially solved by Bochner and later by Raoult [9], Metivier [7], Choksy [3] and the present author [10] can be described as follows. $(\Omega, \mathfrak{A}, p; f; I)$ is a directed projective system of probability spaces, where the $f_{i,j}$ are measurable measurepreserving mappings (mmpm-'s) from Ω_j into Ω_i (cf. Section 2 for notations and precise definitions).

Let Λ be the set-theoretical projective limit of the projective system $(\Omega; f; I)$ of sets (cf. e.g. Bourbaki [2]) with canonical mappings $f_i\colon \Lambda \to \Omega_i$. Then the problem is to define a probability space structure on the set Λ in such a way that the f_i are measurable and measurepreserving. For a description of the various solutions of this problem we refer to the above mentioned references. Now, as is well known from the theory of separability of stochastic processes, there may be other and more useful "representations" of projective systems of probability spaces, a representation being any probability space $(X, \mathfrak{S}, \mu)$ with coherent mmpm-'s

$$g_i\colon X \to \Omega_i.$$

Thus, for example, the representation of Brownian motion on the space $X = C_0[0,1]$ of continuous functions on $[0,1]$, vanishing at 0, is much more informative than the representation on the space $\Lambda = \mathbb{R}^{[0,1]}$, offered in this case by the Bochner(-Kolmogorov) theorem.

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This then raises the question of a "best" representation. That is, is there a criterion which can select out of all possible representations a best one and if such a criterion exists, under what conditions does a projective system of probability spaces admit such a "best" representation.

The answer to the first question is given by the general categorical concept of projective limit, which is a "best" representation in the sense that every other representation factors through it in a unique way.

Partial answers to the second question were given in [10] and will be the object of the present paper. However the above discussion raises also the question whether mmpm-'s are the most natural "morphisms" to consider in probability theory. In fact this is not so, as one is in general not interested in mmpm-'s as such, but in equivalence classes of mmpm-'s under one of the equivalence relations "equal almost everywhere" and "originals of sets differ at most on a set of measure 0", denoted by $\sim$, resp. $\approx$.

The second of these relations is well known in the theory of stochastic processes and we know that $g\sim g' \Rightarrow g\approx g'$, but not conversely.

As we are dealing with projective systems with a completely arbitrary directed index set, it seems natural to consider, instead of mmpm-'s, equivalence classes of mmpm-'s under the equivalence relation $\approx$.

Thus we prove (Corollary 1 to Theorem 4.2) that any projective system of probability spaces which admit compact, generating pavings, has a projective limit in the sense described above, with mmpm-'s replaced by $\approx$-equivalence classes of mmpm-'s. The content of this theorem is in fact wider, as we prove, under the same conditions the existence of a projective limit in the larger category **RP** of regular conditional probabilities.

In Section 3 we give some results on projective limits without any conditions. It turns out that (Corollary 1 to Theorem 3.2) corresponding to any projective system of probability spaces there is a "representation space" (the existence of which is proved in Theorem 4.2), the L^∞-space of which is the projective limit of the L^∞-spaces of the objects of the projective system with the conditional expectation operators as connecting morphisms. As might be expected results like this are closely connected with the theory of martingales along a partially ordered index set (Theorem 3.2).

No attempt has been made to explore the relationship with the work of Mallory [6] who proves existence of representations of projective systems (not of projective limits) under slightly more general conditions (roughly speaking our hypothesis of the existence of a compact paving is in his work replaced by the existence of an $\aleph_0$-compact paving).

We intend to apply our results to the problem of the existence of regular conditional probabilities and to the representation of a stochastic process with an arbitrary time set in a later paper.

2. Notation and Terminology; Prerequisites

(i) *Notation for Projective and Inductive Systems*

If I is a directed set, we write

$$D_I=\{(i,j)\in I\times I \mid i\leqq j\}.$$

If $\mathbf{C}$ is a category, X and Y are objects in $\mathbf{C}$, $[X, Y]_{\mathbf{C}}$ will denote the set of morphisms from X into Y.

If $\mathbf{C}$ is a category, I a directed set, then $(X; f; I)$ will denote either a *projective system*, i.e.

X is a function $i \mapsto X_i$ from I into the class of objects of $\mathbf{C}$;

f is a function $(i,j) \mapsto f_{i,j}$ from D_I into $\bigcup_{(i,j)\in D_I} [X_j, X_i]_{\mathbf{C}}$, such that

$$\forall (i,j) \in D_I, \quad f_{i,j} \in [X_j, X_i]_{\mathbf{C}}; \qquad \forall (i,j) \in D_I, \ \forall (j,k) \in D_I, \quad f_{i,k} = f_{i,j} f_{j,k},$$

or an *inductive system*, that is

X is a function $i \mapsto X_i$ from I into the class of objects of $\mathbf{C}$;

f is a function $(i,j) \mapsto f_{i,j}$ from D_I into $\bigcup_{(i,j)\in D_I} [X_i, X_j]_{\mathbf{C}}$, such that

$$\forall (i,j) \in D_I, \quad f_{i,j} \in [X_i, X_j]_{\mathbf{C}}; \qquad \forall (i,j) \in D_I, \ \forall (j,k) \in D_I, \quad f_{i,k} = f_{j,k} f_{i,j}.$$

(ii) *Regular Conditional Probabilities and Desintegration of Regular Probabilities*

Suppose that $X = (\Omega_X, \mathfrak{A}_X, p_X)$ and $Y = (\Omega_Y, \mathfrak{A}_Y, p_Y)$ are probability spaces. A regular conditional probability (rcp) from X into Y will be a function

$$P\colon \ \Omega \times \mathfrak{B}_P \to [0, 1]$$

where $\mathfrak{B}_P$ is a sub σ-algebra of $\mathfrak{A}_Y$ which is p_Y-equivalent to $\mathfrak{A}_Y$ (that is, $\forall B \in \mathfrak{A}_Y, \exists B_0 \in \mathfrak{B}_P [p_Y(B \mathbin{\triangle} B_0) = 0]$), such that

a) $P(.\,, B)$ is $\mathfrak{A}_X$-measurable for every $B \in \mathfrak{A}_Y$.

b) $P(x, .)$ is a probability on $\mathfrak{B}_P$ for every $x \in \Omega_X$.

c) $\forall B \in \mathfrak{B}_P, \ \int P(.\,, B)\, dp_X = p_Y(B)$.

We shall say that two rcp-'s P and P' from X into Y are equivalent if

$$\forall B \in \mathfrak{B}_P, \quad \forall B' \in \mathfrak{B}_{P'} [p_Y(B \mathbin{\triangle} B') = 0 \Rightarrow P(.\,, B) = P'(.\,, B') p_X\text{-a.e.}].$$

Rcp-'s can be composed in the usual way and one shows easily that this composition respects equivalence. Hence there is a unique composition of equivalence classes. We shall call an equivalence class of rcp-'s from X into Y an **RP**-*morphism* from X into Y and the set of all **RP**-morphisms from X into Y will be denoted by $[X, Y]_{\mathbf{RP}}$. The **RP**-morphism from X into X containing the rcp 1_X defined by $1_X(x, A) = \chi_A(x)$, will be denoted by 1_X as well. It obviously plays the role of unit, that is

$$1_X \varphi = \varphi \quad \text{and} \quad \psi 1_X = \psi$$

whenever the compositions are defined.

If Ω_Y is a locally compact Hausdorff space, $\mathfrak{A}_Y$ the set of Borel subsets of Ω_Y and if p_Y is a regular probability on Ω_Y, we shall reserve a special name for those rcp-'s P from X into Y with the additional properly that $P(x, .)$ is, for every $x \in \Omega_X$, a regular probability on Ω_Y. These rcp-'s will be called *desintegrations* of p_Y with respect to X. If $f\colon \Omega_X \to \Omega_Y$ is a mmpm, then f induces a rcp,

$$f_*\colon X \to Y$$

as follows

$$f_*(x, A) = \chi_A(f(x)).$$

Denote by f^* the **RP**-morphism containing f_*. Then two mmpm-'s f and g from Ω_X into Ω_Y induce the same **RP**-morphism iff

$$\forall B \in \mathfrak{A}_Y, \quad p_X(f^{-1}B \triangle g^{-1}B) = 0,$$

that is, in the notation of Section 1, iff $f \approx g$.

Thus the set $[X, Y]_{\mathbf{F}}$ of all **RP**-morphisms induced by mmpm-'s is canonically isomorphic with the set of equivalence classes of mmpm-'s under the equivalence relation $\approx$.

(iii) *Probability Algebras*

If $X = (\Omega_X, \mathfrak{A}_X, p_X)$ is a complete probability space, we denote by $\mathfrak{N}_X$ the σ-ideal of null sets and by $\tilde{\mathfrak{A}}_X$ the (complete) Boolean algebra $\mathfrak{A}_X/\mathfrak{N}_X$. $A \mapsto \tilde{A}$ will denote the canonical mapping from $\mathfrak{A}_X$ onto $\tilde{\mathfrak{A}}_X$; $\tilde{p}_X$ is the measure on $\tilde{\mathfrak{A}}_X$ induced by p_X. Then $(\tilde{\mathfrak{A}}_X, \tilde{p}_X)$ is what we call a *probability algebra*, i.e. a Boolean algebra with a strictly positive normalized measure (i.e. finitely additive real-valued function). If we make the class of probability algebras into a category, **B** say, by admitting as morphisms measurepreserving homomorphisms, we have the following result on inductive limits

Lemma 2.1. *Every inductive system* $(X; \varphi; I)$ *in* **B**, *has an inductive limit* $\varphi_i\colon X_i \to \varinjlim_{\mathbf{B}} X_j$ *in* **B**.

Moreover, if the $\varphi_{i,j}$ *are* σ*-homomorphisms, then the* φ_i *are* σ*-homomorphisms as well.*

Proof. Straightforward verification shows that the set-theoretic inductive limit, endowed with the obvious probability algebra structure has the desired properties.

(iv) *Lifting and Desintegration of Regular Probabilities*

We shall use the lifting theorem in the following form. If $X = (\Omega_X, \mathfrak{A}_X, p_X)$ is a complete probability space, there is a lifting $a \mapsto \langle a \rangle$ of $\tilde{\mathfrak{A}}_X$ into $\mathfrak{A}_X$. That is, $a \mapsto \langle a \rangle$ is a Boolean algebra homomorphism which is a right inverse to $A \mapsto \tilde{A}$, i.e.

$$\forall a \in \tilde{\mathfrak{A}}_X, \quad \langle a \rangle^{\sim} = a.$$

Let $\mathfrak{M}_X$ be the image of $\tilde{\mathfrak{A}}_X$ under this lifting. Then we can extend this lifting to a lifting of $L^\infty(X)$ as follows. Let $f \mapsto \tilde{f}$ be the canonical map from $\mathscr{L}^\infty(X)$ onto $L^\infty(X)$. Then define, for $f \in \mathscr{L}^\infty(X)$

$$(l\tilde{f})(\omega) = \sup\{x \in \mathbb{R} \mid \omega \notin \langle f^{-1}(-\infty, x)^{\sim} \rangle\}.$$

Then one checks easily that (cf. Fillmore [4] for details)

$$(l\tilde{f})^{-1}(-\infty, r) = \bigcup_{\substack{q<r \\ q \text{ rational}}} \langle f^{-1}(-\infty, q)^{\sim} \rangle.$$

Thus there is, for every $r \in \mathbb{R}$, $f \in L^\infty(X)$ a sequence $(A_n)_{n \in \mathbb{N}}$ in $\mathfrak{M}_X$ such that

$$A_n \uparrow (lf)^{-1}(-\infty, r).$$

We have

Lemma 2.2. *Let $(\Omega, \mathfrak{A}, p)$ be a complete probability space. (X, μ) is a compact Hausdorff space with a regular probability measure μ; $\mathscr{B}_X$ is the class of Borel subsets of X. Furthermore suppose that*

$$E:\ L^\infty(X, \mu) \to L^\infty(\Omega, \mathfrak{A}, p)$$

is a positive linear map with $E1 = 1$, which preserves measure, i.e. $\int (Ef)\, dp = \int f\, d\mu$. Then there is a desintegration

$$\omega \mapsto \pi_\omega$$

of μ into regular probability measures π_ω, with respect to E, that is

$$\forall f \in L^\infty(X, \mu), \quad Ef = (\omega \mapsto \textstyle\int f\, d\pi_\omega) \quad p\text{-a.e.}$$

$(\pi_\omega)_{\omega \in \Omega}$ *is unique up to equivalence of* rcp-'s (*cf.* (ii)).

Proof. Consider, for compact $K \subset X$, the function $\Theta(.,K) \in \mathscr{L}^\infty(\Omega, \mathfrak{A})$ defined by

$$\Theta(.,K) = l E \chi_K$$

where l is the lifting described above. Then the properties of l and E imply that Θ satisfies

(a) $0 \leqq \Theta(\omega, K) \leqq 1$; $\Theta(\omega, \emptyset) = 0$; $\Theta(\omega, X) = 1$.

(b) $K \cap K' = \emptyset \Rightarrow \Theta(\omega, K \cup K') = \Theta(\omega, K) + \Theta(\omega, K')$.

(c) $K \subset K' \Rightarrow \Theta(\omega, K) \leqq \Theta(\omega, K')$.

It follows, by standard arguments of the theory of regular measures that then, for every $\omega \in \Omega$,

$$B \mapsto \pi_\omega(B) = \inf_{U \in \mathscr{U}(B)} \sup_{K \in \mathscr{C}(U)} \Theta(\omega, K)$$

is a regular probability measure on X

$$\big(\text{here } \mathscr{U}(B) = \{U \subset X \mid U \text{ open};\ B \subset U\},$$
$$\mathscr{C}(B) = \{K \subset X \mid K \text{ compact};\ K \subset B\}\big).$$

We shall prove now that, for compact $K \subset X$

$$\pi_.(K) = \Theta(.,K) \quad p\text{-a.e.}$$

Put, for compact $D \supset K$, $n \in \mathbb{N}$

$$A_D^n = \left\{\omega \in \Omega \,|\, \Theta(\omega, D) > \Theta(\omega, K) + \frac{1}{n}\right\}.$$

Then, clearly

$$A_K = \{\omega \in \Omega \mid \Theta(\omega, K) \neq \pi_\omega(K)\} = \bigcup_{n=1}^{\infty} \bigcap_{U \in \mathscr{U}(K)} \bigcup_{\substack{D \in \mathscr{C}(U) \\ K \subset D}} A_D^n.$$

Now, as $\Theta(.,K) = E\chi_K$ p-a.e., we have, for $U \in \mathscr{U}(K)$, $D \in \mathscr{C}(U)$,

$$\mu(U) \geqq \mu(D) = \int \chi_D \, d\mu = \int \Theta(.,D)\, dp = \int_{A_D^n} + \int_{(A_D^n)^c} \geqq \mu(K) + \frac{1}{n} p(A_D^n).$$

Hence

$$p(A_D^n) \leqq n(\mu(U) - \mu(K)).$$

Now the sets A_D^n are of the form $(l\tilde{f})^{-1}(-\infty, r)$ $\left(\text{indeed take } f = \Theta(.,K) - \Theta(.,D) \text{ and } r = -\frac{1}{n}\right)$, hence there is an increasing sequence $(b^k)_{k \in \mathbb{N}}$ in $\tilde{\mathfrak{A}}$ such that

$$A_D^n = \lim_{k \to \infty} \uparrow \langle b^k \rangle.$$

Let a_D^n be the supremum in $\tilde{\mathfrak{A}}$ of the sequence $(b^k)_{k \in \mathbb{N}}$. Then we have, for every $k \in \mathbb{N}$, $b^k \leqq a_D^n$, hence by the positivity of the lifting $\langle b^k \rangle \subset \langle a_D^n \rangle$ for all $k \in \mathbb{N}$. Therefore

$$A_D^n \subset \langle a_D^n \rangle.$$

Also clearly

$$p(\langle a_D^n \rangle) = \tilde{p}(a_D^n) = \sup_{k \in \mathbb{N}} \tilde{p}(b^k) = \lim_{k \to \infty} \uparrow p(\langle b^k \rangle) = p(A_D^n),$$

hence

$$\tilde{A}_D^n = a_D^n.$$

Next keep $U \in \mathscr{U}(K)$ and $n \in \mathbb{N}$ fixed. Then the set

$$\{A_D^n \mid D \in \mathscr{C}(U),\ K \subset D\}$$

is an increasing filter, therefore $\{a_D^n \mid D \in \mathscr{C}(U),\ K \subset D\}$ is an increasing filter in $\tilde{\mathfrak{A}}$. Let b_U^n be the supremum of this filter in $\tilde{\mathfrak{A}}$. Then we have, for all $D \subset \mathscr{C}(U)$, $K \subset D$, $a_D^n \leqq b_U^n$, therefore $\langle a_D^n \rangle \subset \langle b_U^n \rangle$, and then by the previous result $A_D^n \subset \langle b_U^n \rangle$, hence

$$B_U^n = \bigcup_{\substack{D \in \mathscr{C}(U) \\ K \subset D}} A_D^n \subset \langle b_U^n \rangle.$$

On the other hand

$$\begin{aligned} p(\langle b_U^n \rangle) = \tilde{p}(b_U^n) &= \sup\{\tilde{p}(a_D^n) \mid D \in \mathscr{C}(U),\ K \subset D\} \\ &= \sup\{p(A_D^n) \mid D \in \mathscr{C}(U),\ K \subset D\} \leqq n(\mu(U) - \mu(K)). \end{aligned}$$

Hence

$$p^*(B_U^n) \leqq p(\langle b_U^n \rangle) \leqq n(\mu(U) - \mu(K)).$$

Now, use the regularity of μ to choose, given an $\varepsilon > 0$, a sequence $(U_n)_{n \in \mathbb{N}} \subset \mathscr{U}(K)$ such that

$$\mu(U_n) - \mu(K) \leqq \frac{\varepsilon}{n\, 2^{n-1}}.$$

Then we have, clearly

$$A_K \subset \bigcup_{n=1}^{\infty} B^n_{U_n}.$$

Thus

$$p^*(A_K) \leqq \sum_{n=1}^{\infty} p^*(B^n_{U_n}) \leqq \sum_{n=1}^{\infty} n\big(\mu(U_n) - \mu(K)\big) \leqq \varepsilon.$$

As ε is arbitrary this proves that

$$p^*(A_K) = 0$$

and as $(\Omega, \mathfrak{A}, p)$ is complete it follows that

$$A_K \in \mathfrak{A} \quad \text{and} \quad p(A_K) = 0.$$

Therefore

$$\pi_{\bullet}(K) = \Theta(., K) \quad p\text{-a.e.}$$

Thus the class $\mathscr{F}$ of Borelsets $F \subset X$ for which $\pi_{\bullet}(F)$ is $\mathfrak{A}$-measurable and for which $\pi_{\bullet}(F) = E\chi_F$ p-a.e., contains all compact sets. As $\mathscr{F}$ is moreover clearly a σ-field, it follows that $\mathscr{F} = \mathscr{B}_X$. Hence

$$\pi_{\bullet}(B) = E\chi_B \quad p\text{-a.e.}$$

for every Borelset $B \subset X$. Hence

$$Ef = \int \pi_{\bullet}(dx) f(x) \quad p\text{-a.e.}$$

for every step function $f \in L^{\infty}(X, \mu)$. As the set of all f for which this equation holds, is obviously a closed subspace of $L^{\infty}(X, \mu)$ it follows that this set must be all of $L^{\infty}(X, \mu)$. This concludes the proof of the lemma.

As we have, for $B \in \mathscr{B}_X$, $E\chi_B \leqq 1$, it follows that

$$\psi(A, B) = \int_A (E\chi_B)\, dp \leqq p(A).$$

Now define, for $A \in \mathfrak{A}$

$$\varphi A = \bigcap_{\substack{K \text{ compact} \\ \psi(A,K) = p(A)}} K.$$

Then we have

Corollary. $\forall \omega \in \Omega$ $\operatorname{supp} \pi_{\omega} \subset \bigcap_{\substack{A \in \mathfrak{A} \\ \omega \in \langle \tilde{A} \rangle}} \varphi A.$

Proof. Let $A \in \mathfrak{A}$ and consider $\pi_{\omega}\big((\varphi A)^c\big)$. As $(\varphi A)^c$ is open, we have

$$\pi_{\omega}\big((\varphi A)^c\big) = \sup_{K \in \mathscr{C}((\varphi A)^c)} \Theta(\omega, K).$$

Now $K \in \mathscr{C}\big((\varphi A)^c\big)$ implies $K \cap \varphi A = \emptyset$, hence there is a K' compact with $K \cap K' = \emptyset$ and $\psi(A, K') = p(A)$. Therefore $\psi(A, K) = 0$. This implies $E\chi_K = 0$ p-a.e. on A. Thus, for every $r > 0$

$$\tilde{A} \leqq (E\chi_K)^{-1}(-\infty, r)^{\sim}$$

hence

$$\forall r > 0, \quad \langle \tilde{A} \rangle \subset \langle (E\chi_K)^{-1}(-\infty, r)^{\sim} \rangle.$$

Therefore, as $\Theta(.,K)\geqq 0$,

$$\omega\in\langle\tilde{A}\rangle \Rightarrow \Theta(\omega,K)=0.$$

Hence

$$\omega\in\langle\tilde{A}\rangle \Rightarrow \pi_\omega((\varphi A)^c)=0.$$

Therefore, finally

$$\omega\in\langle\tilde{A}\rangle \Rightarrow \pi_\omega(\varphi A)=1$$

or

$$\omega\in\langle\tilde{A}\rangle \Rightarrow \operatorname{supp}\pi_\omega\subset\varphi A.$$

(v) *Representation of Boolean Homomorphisms as* **F**-*Morphisms*

We consider the following particular case of the conditions for Lemma 2.2. Let again $(\Omega,\mathfrak{A},p)$ be a probability space; X is a compact Hausdorff space with a regular probability measure μ; ε: $\tilde{\mathfrak{A}}\to\mathscr{B}_X$ is a measurepreserving Boolean homomorphism. Then ε induces an E: $L^\infty(X,\mu)\to L^\infty(\Omega,\mathfrak{A},p)$ satisfying the conditions of Lemma 2.2, as follows. For $f\in L^\infty(X,\mu)$ the function $A\mapsto\int_{\varepsilon\tilde{A}} f\,d\mu$ from $\mathfrak{A}$ into $\mathbb{R}$ is clearly a bounded countably additive measure on $\mathfrak{A}$ (countably additive as $A_n\downarrow\emptyset$ in $\mathfrak{A}$ implies $\mu(\varepsilon\tilde{A}_n)\downarrow 0$ by the fact that ε is measurepreserving) which is absolutely continuous with respect to p. It follows, by the Radon-Nikodym theorem, that there is a unique $Ef\in L^\infty(\Omega,\mathfrak{A},p)$ such that

$$\forall A\in\mathfrak{A}\ \int_{\varepsilon\tilde{A}} f\,d\mu=\int_A (Ef)\,dp.$$

One verifies easily that this E satisfies the conditions of Lemma 2.2. Thus there is a family $(\pi_\omega)_{\omega\in\Omega}$ of regular probability measures on X such that

$$\forall A\in\mathfrak{A},\quad \forall B\in\mathscr{B}_X\ \mu(\varepsilon\tilde{A}\cap B)=\int_A p(d\omega)\,\pi_\omega(B).$$

The function φ in the corollary now takes the following form

$$\varphi A=\inf\{K\subset X|K \text{ compact},\ \mu(K\cap\varepsilon\tilde{A})=\mu(\varepsilon\tilde{A})\}.$$

This implies easily $(\varepsilon\tilde{A})^0\subset\varphi A\subset\overline{\varepsilon\tilde{A}}$.

We are interested in conditions ensuring that ε is induced by a mmpm g: $X\to\Omega$ (that is $\varepsilon\tilde{A}=g^{-1}A$ μ-a.e. for every $A\in\mathfrak{A}$).

We introduce the following terminology (cf. e.g. Meyer [8]). A *paving* $\mathfrak{C}$ of a probability space $(\Omega,\mathfrak{A},p)$ is a collection of measurable subsets of Ω with $\emptyset\in\mathfrak{C}$. A paving $\mathfrak{C}$ is *compact* if every subset $\mathfrak{S}$ of $\mathfrak{C}$ with the finite intersection property (i.e. for which $\mathfrak{S}_0\subset\mathfrak{S}$, $\mathfrak{S}_0$ finite, implies $\bigcap_{S\in\mathfrak{S}_0} S\neq\emptyset$), has non-empty intersection; it is *generating* if the σ-field generated by $\mathfrak{C}$ is p-equivalent to $\mathfrak{A}$; it will have "*approximate complements*" if we have

$$\forall C\in\mathfrak{C},\qquad p(C)+\sup\{p(D)|D\in\mathfrak{C},\ C\cap D=\emptyset\}=1.$$

Observe that, if a paving $\mathfrak{C}$ has one or more of these properties, then the collection $\mathfrak{C}'$ obtained by closing $\mathfrak{C}$ with respect to finite unions and intersections is a paving which retains all the properties of $\mathfrak{C}$. We shall say that $\mathfrak{C}'$ is a *sublattice* of $\mathfrak{A}$ with the appropriate adverbs.

Observe that the lattice of compact subsets of a locally compact Hausdorff space with a regular probability measure, has all the above mentioned properties.

With these conventions we have

Lemma 2.3. *Suppose that $(\Omega, \mathfrak{A}, p)$ is a complete probability space which admits a generating compact sublattice $\mathfrak{C} \subset \mathfrak{A}$; (X, μ) is a compact Hausdorff space with a regular probability measure μ; ε is a measure preserving lattice homomorphism from $\tilde{\mathfrak{C}} = \{\tilde{C} \in \tilde{\mathfrak{A}} \mid C \in \mathfrak{C}\}$ into $\mathscr{B}_X$, with $\varepsilon \emptyset = \emptyset$; $\overline{\mathscr{B}}_X$ is the completion of $\mathscr{B}_X$ with respect to μ. Then, if $\mathfrak{C}$ has approximate complements, there is a $\overline{\mathscr{B}}_X$-mmpm $g: X \to \Omega$, such that*

$$\forall C \in \mathfrak{C}, \quad g^{-1} C = \varepsilon \tilde{C} \quad \mu\text{-a.e.}$$

and

$$\forall C \in \mathfrak{C}, \quad \varepsilon \tilde{C} \subset g^{-1} C.$$

The F*-morphism $g^*: (X, \overline{\mathscr{B}}_X, \mu) \to (\Omega, \mathfrak{A}, p)$ induced by g is unique.*

Proof. Consider, for $x \in X$, the subset $d_x = \{C \in \mathfrak{C} \mid x \in \varepsilon \tilde{C}\}$ of $\mathfrak{C}$. As $\mathfrak{C}$ is a lattice, as ε is measurepreserving it follows from our assumption that d_x has the finite intersection property. Hence, by the compactness of $\mathfrak{C}$, $D_x = \bigcap_{C \in d_x} C \neq \emptyset$. Choose, for every $x \in X$, a $g x \in D_x$.

Then the function

$$g: X \to \Omega$$

satisfies,

$$\forall C \in \mathfrak{C}, \quad g^{-1} C = \varepsilon \tilde{C} \quad \mu\text{-a.e.}$$

Indeed, $x \in \varepsilon \tilde{C} \Rightarrow g x \in C$ by definition. Thus $\varepsilon \tilde{C} \subset g^{-1} C$. Moreover, as $\mathfrak{C}$ has approximate complements, we have

$$\mu(\varepsilon \tilde{C}) + \sup \{\mu(\varepsilon \tilde{D}) \mid D \in \mathfrak{C},\ C \cap D = \emptyset\} = 1.$$

Hence there is a nullset $N \subset X$ such that

$$x \notin \varepsilon \tilde{C} \cup N \Rightarrow \exists D \in \mathfrak{C} [C \cap D = \emptyset,\ x \in \varepsilon \tilde{D}] \Rightarrow g x \notin C.$$

Thus $\varepsilon \tilde{C} \subset g^{-1} C \subset \varepsilon \tilde{C} \cup N$, which proves the assertion. The facts that g is measurable and measurepreserving and that g^* is unique are now proved in a straightforward way and we omit further details.

Observe that, if $\mathfrak{C}$ is a generating sublattice, a measurepreserving lattice homomorphism $\varepsilon: \tilde{\mathfrak{C}} \to \mathscr{B}_X$ induces a measurepreserving homomorphism $\check{\varepsilon}: \tilde{\mathfrak{A}} \to \overline{\mathscr{B}}_X$ with $\forall C \in \mathfrak{C}$, $\varepsilon \tilde{C} = \check{\varepsilon} \tilde{C}$ μ-a.e., as follows. ε can be algebraically extended to the field $\mathfrak{A}_0$ generated by $\mathfrak{C}$. As $\mathfrak{C}$ generates $\mathfrak{A}$ mod p, it follows that $\tilde{\mathfrak{A}}_0$ is dense in $\tilde{\mathfrak{A}}$ (in the metric $\rho(\tilde{A}, \tilde{B}) = p(A \vartriangle B)$). $\tilde{\varepsilon}: \tilde{\mathfrak{A}}_0 \to \tilde{\mathscr{B}}_X$, defined by $\tilde{\varepsilon} \tilde{C} = (\varepsilon \tilde{C})^{\sim}$, being measurepreserving, is uniformly continuous with respect to this metric and to the metric $\rho'(\tilde{A}, \tilde{B}) = \mu(A \vartriangle B)$ in $\tilde{\mathscr{B}}_X$. Hence $\tilde{\varepsilon}$ has a unique extension (also denoted by $\tilde{\varepsilon}$) to $\tilde{\mathfrak{A}}$, and this extension still preserves measure. Now let $b \mapsto \langle b \rangle$ be a lifting of $\tilde{\mathscr{B}}_X$ and define

$$\check{\varepsilon}: \tilde{\mathfrak{A}} \to \overline{\mathscr{B}}_X$$

by

$$\check{\varepsilon} \tilde{A} = \langle \tilde{\varepsilon} \tilde{A} \rangle.$$

This $\check{\varepsilon}$ then clearly has the desired properties.

With this definition we have

Corollary 1. *The correspondence* $\varepsilon \mapsto g^*$ *of Lemma* 2.3 *is functorial in the following sense. If* $(\Omega, \mathfrak{A}, p)$, $\mathfrak{C}$, ε *and* $(\Omega', \mathfrak{A}', p')$, $\mathfrak{C}'$, ε' *both satisfy the requirements of the lemma, whilst there is, moreover, a* mmpm $h\colon \Omega \to \Omega'$ *such that*

$$\forall C \in \mathfrak{C}', \quad \varepsilon' \tilde{C} = \check{\varepsilon}(h^{-1} C)^{\sim} \quad \mu\text{-a.e.},$$

then the mmpm-'s g *and* g' *associated with* ε *and* ε' *respectively, satisfy*

$$g'^* = (h \circ g)^* = h^* g^*.$$

Proof. We know that

$$\forall C \in \mathfrak{C}, \quad g^{-1} C = \varepsilon \tilde{C} \quad \mu\text{-a.e.}$$

This clearly implies $\forall A \in \mathfrak{A}$, $g^{-1} A = \check{\varepsilon} \tilde{A}$ μ-a.e. Hence we have, for every $C \in \mathfrak{C}'$, μ-a.e.,

$$(h \circ g)^{-1} C = g^{-1}(h^{-1} C) = \check{\varepsilon}(h^{-1} C)^{\sim} = \varepsilon' \tilde{C} = (g')^{-1} C.$$

Thus, by the uniqueness assertion of the lemma

$$g'^* = (h \circ g)^* = h^* g^*.$$

Corollary 2. *If* $(\Omega, \mathfrak{A}, p)$, $\mathfrak{C}$, ε *satisfy the conditions of Lemma* 2.3, *then there is a desintegration* $\omega \mapsto \pi_\omega$ *of* μ *with respect to the induced map* g, *for which*

$$\operatorname{supp}(\pi_\omega) \subset g^{-1} \omega$$

if the following conditions are satisfied.

a) $\varepsilon \tilde{C}$ *is, for every* $C \in \mathfrak{C}$, *closed in* X,

b) *there is a lifting* $a \mapsto \langle a \rangle$ *of* $\tilde{\mathfrak{A}}$ *such that* $\mathfrak{C}_0 = \{C \in \mathfrak{C} \mid C \subset \langle \tilde{C} \rangle\}$ *separates* Ω.

Proof. According to the remarks made in the beginning of this paragraph ε induces a desintegration (π_ω) for which

$$\forall A \in \mathfrak{A}, \quad (\varepsilon A)^0 \subset \varphi A \subset \overline{\varepsilon \tilde{A}}.$$

Thus we have, using the lifting of the statement of the corollary, according to the corollary to Lemma 2.2 and to Lemma 2.3 for $\omega \in \Omega$

$$\begin{aligned} \operatorname{supp}(\pi_\omega) &\subset \bigcap_{\substack{A \in \mathfrak{A} \\ \omega \in \langle \tilde{A} \rangle}} \varphi A \subset \bigcap_{\substack{C \in \mathfrak{C} \\ \omega \in \langle \tilde{C} \rangle}} \varphi C \subset \bigcap_{\substack{C \in \mathfrak{C} \\ \omega \in \langle \tilde{C} \rangle}} \overline{\varepsilon \tilde{C}} = \bigcap_{\substack{C \in \mathfrak{C} \\ \omega \in \langle \tilde{C} \rangle}} \varepsilon \tilde{C} \\ &\subset \bigcap_{\substack{C \in \mathfrak{C} \\ \omega \in \langle \tilde{C} \rangle}} g^{-1} C \subset \bigcap_{\substack{C \in \mathfrak{C}_0 \\ \omega \in C}} g^{-1} C = g^{-1} \omega. \end{aligned}$$

3. Representation Spaces for Arbitrary F-Projective Systems

Let $(\Omega, \mathfrak{A}, p; f^*; I)$ be a directed **F**-projective system (we shall consistently use this notation, which implies that we are actually given mmpm-'s $f_{i,j}$ such that the **F**-morphisms $f_{i,j}^*$ constitute the projective system). Denote, for $(i,j) \in D_I$, by $\varphi_{i,j}\colon \tilde{\mathfrak{A}}_i \to \tilde{\mathfrak{A}}_j$ the Boolean homomorphism induced by $f_{i,j}$. Then $(\tilde{\mathfrak{A}}, \tilde{p}; \varphi; I)$ is clearly an inductive system in **B**. Denote its inductive limit (cf. Section 2) by $\varphi_i\colon (\tilde{\mathfrak{A}}_i, \tilde{p}_i) \to (\mathfrak{A}, \pi)$. Then, by Lemma 2.1, $\mathfrak{A}$ is the set-theoretic inductive limit of the $\tilde{\mathfrak{A}}_i$ and the φ_i are measurepreserving σ-homomorphisms. We shall say that a compact Hausdorff space X with a regular probability measure p is a *representation space* for the projective system if the following conditions are satisfied ($\mathscr{B}_X$ is the σ-field of Borel subsets of X).

1. There is a subalgebra $\mathfrak{A}_0$ of $\mathfrak{A}$, which is dense in $\mathfrak{A}$ (in the metric $d(A,B)=\pi(A \triangle B)$ and an injective homomorphism $A \mapsto \hat{A}$ from $\mathfrak{A}_0$ into $\mathscr{B}_X$ which is measurepreserving (i.e. $p(\hat{A})=\pi(A)$)). We denote the image of this homomorphism by $\hat{\mathfrak{A}}_0$.

2. For every pair (K, U) of subsets of X, where K is compact and U is open, such that $K \subset U$, there is an $A \in \mathfrak{A}_0$ such that

$$K \subset \hat{A} \subset U.$$

As π is strictly positive, we have as an immediate consequence of 2 that every open set has positive measure and that, hence $\operatorname{supp}(p)=X$. Moreover, 2 implies clearly that $\hat{\mathfrak{A}}_0$ generates $\mathscr{B}_X \bmod p$.

For the moment we leave out the question of the existence of such representation spaces, merely stating here that one verifies easily that the well known Stone space representation of $(\mathfrak{A}, \pi)$ satisfies 1 and 2. However we shall establish the existence of representation spaces in Section 4 (Theorem 4.2).

Meanwhile we have

Lemma 3.1. *Suppose that $(v_i)_{i \in I}$ is a collection of measures, i.e. v_i is a measure on $(\Omega_i, \mathfrak{A}_i)$ such that*

(i) *the v_i are non-negative and uniformly bounded with respect to the p_i, i.e.*

$$\exists M>0, \quad \forall i \in I, \quad \forall A \in \mathfrak{A}_i, \quad v_i(A) \leqq M p_i(A).$$

(ii) $\forall (i,j) \in D_I$, $\forall A \in \mathfrak{A}_i$, $v_j(f_{i,j}^{-1} A) \leqq v_i(A)$.

Then there is a unique regular measure v on X such that we have for every $i \in I$ and every $A \in \mathfrak{A}_i$, such that $\varphi_i \tilde{A} \in \mathfrak{A}_0$

$$v\big((\varphi_i \tilde{A})^\wedge\big) = \lim_{j \geqq i} \downarrow v_j(f_{i,j}^{-1} A)$$

and

$$\forall B \in \mathscr{B}_X, \quad 0 \leqq v(B) \leqq M p(B).$$

Proof. Condition (i) shows that $v_i \ll p_i$, hence $\tilde{v}_i$ can be defined unambiguously on $\tilde{\mathfrak{A}}_i$ by

$$\tilde{v}_i(\tilde{A}) = v_i(A).$$

Now define $v\colon \hat{A}_0 \to \mathbb{R}_+$ as follows. For $A=\varphi_i A_i \in \mathfrak{A}_0$, $A_i \in \tilde{\mathfrak{A}}_i$, put

$$v(\hat{A}) = \lim_{j \geqq i} \tilde{v}_j(\varphi_{i,j} A_i).$$

This limit exists, as the generalized sequence on the right hand side is decreasing and non-negative by hypothesis; it is moreover independent of the choice of the representative A_i of A, by the definition of $\mathfrak{A}$. Indeed, suppose that $A=\varphi_i A_i = \varphi_j A_j$. Then there is an index $k \in I$, such that $k \geqq i$, $k \geqq j$ and

$$\varphi_{i,k} A_i = \varphi_{j,k} A_j.$$

Consequently

$$\lim_{l\geqq i} \tilde{v}_l(\varphi_{i,l} A_i) = \lim_{l\geqq k} = \lim_{l\geqq k} \tilde{v}_l(\varphi_{k,l}\,\varphi_{i,k} A_i) = \lim_{l\geqq k} \tilde{v}_l(\varphi_{k,l}\,\varphi_{j,k} A_j)$$
$$= \lim_{l\geqq k} \tilde{v}_l(\varphi_{j,l} A_j) = \lim_{l\geqq j} \tilde{v}_l(\varphi_{j,l} A_j).$$

Moreover v so defined is obviously monotonic, finitely additive and subadditive on $\hat{\mathfrak{A}}_0$. Furthermore, by definition and our assumption

$$\forall A\in\hat{\mathfrak{A}}_0, \quad v(A)\leqq M\,p(A).$$

Finally, let $A\in\hat{\mathfrak{A}}_0$ be arbitrary. Choose, using the regularity of p, $K\subset A$ compact and $U\supset A$ open in such a way that

$$p(U)-p(K)<\varepsilon/M.$$

Then we have for every $B\in\hat{\mathfrak{A}}_0$ with $K\subset B\subset U$, $p(A\,\triangle\,B)<\varepsilon/M$, hence

$$|v(A)-v(B)|\leqq v(A\,\triangle\,B)<\varepsilon.$$

Therefore v has, by Th. 4.4.5 of Bourbaki, Intégration, a unique extension to a regular measure on X, which then has clearly the required properties.

Next we introduce the *(mixed) conditional expectation operators*

$$E^{i,j}\colon\ L^\infty(\Omega_j,\mathfrak{A}_j,p_j)\to L^\infty(\Omega_i,\mathfrak{A}_i,p_i)$$

defined by

$$\forall g\in L^1(\Omega_i,\mathfrak{A}_i,p_i), \quad \forall h\in L^\infty(\Omega_j,\mathfrak{A}_j,p_j)\left[\int g\,E^{i,j}h\,dp_i=\int (g\circ f_{i,j})\,h\,dp_j\right]$$

or equivalently

$$\forall A\in\mathfrak{A}_i, \quad \forall h\in L^\infty(\Omega_j,\mathfrak{A}_j,p_j)\left[\int_A E^{i,j}h\;dp_i=\int_{f_{i,j}^{-1}A} h\,dp_j\right].$$

(Observe that the last equation implies that $E^{i,j}$ depends only on the equivalence class of $f_{i,j}$.) Then, obviously

$$i\leqq j\leqq k\Rightarrow E^{i,j}=E^{i,j}\circ E^{j,k}.$$

Moreover each $E^{i,j}$ is the adjoint of the composition operator $F^{i,j}\colon L^1(\Omega_i,\mathfrak{A}_i,p_i)\to L^1(\Omega_j,\mathfrak{A}_j,p_j)$ (defined by $F^{i,j}g=g\circ f_{i,j}$) and left inverse to the restriction of $F^{i,j}$ to L^∞, i.e.

$$E^{i,j}\circ F^{i,j}|_{L^\infty(\Omega_i,\mathfrak{A}_i,p_i)}=1_{L^\infty(\Omega_i,\mathfrak{A}_i,p_i)}.$$

Furthermore, as is well known, the $E^{i,j}$ are linear, positive, normcontracting operators of norm 1, and can be extended uniquely to $L^1(\Omega_j,\mathfrak{A}_j,p_j)$.

Theorem 3.1. *Let (X,p) be a representation space for the* **F**-*projective system $(\Omega,\mathfrak{A},p;f^*;I)$. Then there is, for every $i\in I$ a positive linear contraction*

$$E^i\colon\ L^\infty(X,p)\to L^\infty(\Omega_i,\mathfrak{A}_i,p_i)$$

with $E^i 1=1$ and such that

$$\forall i\in I, \quad \forall A\in\mathfrak{A}_i, \quad \forall h\in L^\infty(X,p)\left[\varphi_i\tilde{A}\in\mathfrak{A}_0\Rightarrow\int_A (E^i h)\,dp_i=\int_{(\varphi_i\tilde{A})^\wedge} h\,dp\right].$$

E^i satisfies moreover

i) $i \leqq j \Rightarrow E^i = E^{i,j} \circ E^j$.

ii) *E^i induces a desintegration of p, i.e. there is a family $(\pi_i(\omega, .))_{\omega \in \Omega_i}$ of regular probability measures on X, such that*

a) $\forall h \in L^\infty(X, p),\ E^i h = \int h(x)\, \pi_i(., dx)$.

b) $\forall B \in \mathscr{B}_X,\ E^{i,j} \pi_j(., B) = \pi_i(., B)$.

Proof. Observe that, as $\mathfrak{A}_0$ is dense in $\mathfrak{A}$, the homomorphism $A \mapsto \hat{A}$ from $\mathfrak{A}_0$ into $\mathscr{B}_X$ can clearly be extended to a measurepreserving "*almost homomorphism*", again denoted $A \mapsto \hat{A}$ from $\mathfrak{A}$ into $\bar{\mathscr{B}}_X$ (i.e. $p(\hat{A}) = \pi(A)$; $(A \wedge B)^\wedge = \hat{A} \cap \hat{B}$ μ-a.e.; $(A \vee B)^\wedge = \hat{A} \cup \hat{B}$ μ-a.e.; and $A'^\wedge = A^{\wedge c}$ μ-a.e.). This almost homomorphism in turn induces measurepreserving almost homomorphisms $A \mapsto \check{A}$ from $\mathfrak{A}_i$ into $\bar{\mathscr{B}}_X$, for every $i \in I$. Consider then, for fixed $i \in I$, $h \in L^\infty(X, p)$ the function

$$A \mapsto \int_{\check{A}} h\, dp.$$

This is obviously a bounded signed measure on $\mathfrak{A}_i$ which is absolutely continuous with respect to p_i, hence there is a unique $E^i h \in L^\infty(\Omega_i, \mathfrak{A}_i, p_i)$ such that

$$\int_A (E^i h)\, dp_i = \int_{\check{A}} h\, dp.$$

The fact that E^i so defined is linear, positive, normcontracting, has norm 1 and that $E^i 1 = 1$, is proved in the usual straightforward way. We omit the details.

If $i \leqq j$ and $h \in L^\infty(X, p)$, consider $(E^{i,j} \circ E^j)(h)$. We have, for $A \in \mathfrak{A}_i$

$$\begin{aligned}\int_A E^{i,j}(E^j h)\, dp_i &= \int_{f_{i,j}^{-1} A} (E^j h)\, dp_j = \int_{(f_{i,j}^{-1} A)^\vee} h\, dp \\ &= \int_{\check{A}} h\, dp = \int_A (E^i h)\, dp_i.\end{aligned}$$

Hence

$$E^{i,j} \circ E^j = E^i.$$

The existence of the desintegration $(\pi_i(\omega, .))_{\omega \in \Omega_i}$ of p follows now directly from our Lemma 2.2, whilst property b) of this desintegration is an immediate consequence of the fact that, according to this lemma, $\pi_i(., B) = E^i \chi_B$ p_i-a.e. This concludes the proof of the theorem.

Now we want to prove a minimality property of representation spaces. For this we shall need the concept of martingale with respect to an **F**-projective system. We shall say that a family $(g_i)_{i \in I}$ is an $(\Omega, \mathfrak{A}, p; f^*; I)$-*martingale* if

(i) $\forall i \in I,\ g_i \in L^1(\Omega_i, \mathfrak{A}_i, p_i)$.

(ii) $i \leqq j \Rightarrow E^{i,j} g_j = g_i$.

We shall denote the collection of all $(\Omega, \mathfrak{A}, p; f^*; I)$-martingales by $M(\Omega, \mathfrak{A}, p; f^*; I)$. It is in an obvious way a linear space. The coordinate functions from $M(\Omega, \mathfrak{A}, p; f^*; I)$ into $L^1(\Omega_i, \mathfrak{A}_i, p_i)$ will be denoted by $g \mapsto g_i$. The subspace of $M(\Omega, \mathfrak{A}, p; f^*; I)$ consisting of all $(\Omega, \mathfrak{A}, p; f^*; I)$-martingales g which are terminally uniformly integrable, i.e. for which

$$\forall \varepsilon > 0,\ \exists i_0 \in I,\ \exists R > 0,\ \forall i \geqq i_0 \Big[\int_{\{|g_i| > R\}} |g_i|\, dp_i < \varepsilon\Big],$$

will be denoted by $L^1(\Omega, \mathfrak{A}, p; f^*; I)$. The subspace of $L^1(\Omega, \mathfrak{A}, p; f^*; I)$ consisting of all uniformly bounded $(\Omega, \mathfrak{A}, p; f^*; I)$-martingales will be denoted by $L^\infty(\Omega, \mathfrak{A}, p; f^*; I)$. That is $g \in L^1(\Omega, \mathfrak{A}, p; f^*; I)$ is in $L^\infty(\Omega, \mathfrak{A}, p; f^*; I)$ iff the sequence $(\|g_i\|_\infty)_{i\in I}$ of norms is bounded.

We have

Lemma 3.2. a) *An* $(\Omega, \mathfrak{A}, p; f^*; I)$*-martingale* g *is terminally uniformly integrable iff*

(i) $\sup_{i\in I} \|g_i\|_1 < \infty$ *(uniform boundedness).*

(ii) $\forall \varepsilon > 0, \exists \delta > 0, \exists i_0 \in I, \forall i \geqq i_0, \forall A \in \mathfrak{A}_i [p_i(A) < \delta \Rightarrow \int_A |g_i| \, dp_i < \varepsilon]$ *(terminal uniform absolute continuity).*

b)
$$g \mapsto \sup_{i\in I} \|g_i\|_1 = |||g|||_1$$
defines a norm on $L^1(\Omega, \mathfrak{A}, p; f^*; I)$; $L^1(\Omega, \mathfrak{A}, p; f^*; I)$ *is a Banach space under this norm.*

c)
$$g \mapsto \sup_{i\in I} \|g_i\|_\infty = |||g|||_\infty$$
defines a norm on $L^\infty(\Omega, \mathfrak{A}, p; f^*; I)$. *This space is a Banach space under this norm.*

d) *Under coordinatewise ordering (i.e.* $g \leqq h \Leftrightarrow \forall i \in I, g_i \leqq h_i$*)* $L^1(\Omega, \mathfrak{A}, p; f^*; I)$ *and* $L^\infty(\Omega, \mathfrak{A}, p; f^*; I)$ *are complete Banach lattices;* L^1 *is even an L-space (in the sense of Kakutani).*

e) $L^\infty(\Omega, \mathfrak{A}, p; f^*; I)$ *is dense in* $L^1(\Omega, \mathfrak{A}, p; f^*; I)$.

Proof. All these statements are either included in or are direct consequences of results in Krickeberg and Pauc [5], in particular, Cor. to Prop. 1.4.3 (p. 466), Prop. 2.5.1 (p. 494), Th. 5 (p. 499) and Th. 6 (p. 500).

Observe now that there is a canonical map
$$E\colon L^1(X, p) \to L^1(\Omega, \mathfrak{A}, p; f^*; I)$$
defined by
$$(E g)_i = E^i g \qquad (i \in I).$$
Indeed, by Theorem 3.1, $E g$ is an $(\Omega, \mathfrak{A}, p; f^*; I)$-martingale and it is terminally uniformly integrable as
$$\sup_{i\in I} \|E^i g\|_1 \leqq \|g\|_1$$
and
$$\int_A |E^i g| \, dp_i \leqq \int_A E^i |g| \, dp_i = \int_A |g| \, dp.$$
Thus the absolute continuity of the integral $\int_B |g| \, dp$ proves the terminal uniform absolute continuity of $E g$.

Furthermore E is clearly positive and linear. Also the restriction of E to $L^\infty(X, p)$ maps into $L^\infty(\Omega, \mathfrak{A}, p; f^*; I)$.

Moreover

Theorem 3.2. *E is an (L-space) isomorphism from $L^1(X, p)$ onto $L^1(\Omega, \mathfrak{A}, p; f^*; I)$. The restriction E_∞ of E to $L^\infty(X, p)$ is a Banach lattice isomorphism from $L^\infty(X, p)$ onto $L^\infty(\Omega, \mathfrak{A}, p; f^*; I)$.*

Proof. Observe that it is, by Lemma 3.2e, sufficient to prove that E and E_∞ are isometries and that E_∞ is bijective. Moreover in order to prove that E and E_∞ preserve L^1- and L^∞-norm respectively it is sufficient to prove that E preserves both norms for $\hat{\mathfrak{A}}_0$-measurable stepfunctions. Thus, let $g \in L^1(X, p)$ be an $\hat{\mathfrak{A}}_0$-measurable stepfunction,

$$g = \sum_{l=1}^{n} \alpha_l \chi_{A_l}$$

where $\alpha_l \in \mathbb{R}$ and $A_l \in \hat{\mathfrak{A}}_0$, disjoint. Then there is an $i \in I$ and $B_1, \ldots, B_n \in \mathfrak{A}_i$ disjoint, such that

$$A_l = (\varphi_i \tilde{B}_l)^\wedge \qquad (l = 1, \ldots, n).$$

Then we have, if $j \geqq i$

$$E^j g = \sum_{l=1}^{n} \alpha_l \chi_{f_{i,j}^{-1} B_l}$$

hence

$$\|E^j g\|_1 = \sum_{l=1}^{n} |\alpha_l| \, p_j(f_{i,j}^{-1} B_l) = \sum_{l=1}^{n} |\alpha_l| \, p(A_l) = \|g\|_1$$

and

$$\|E^j g\|_\infty = \max_{l=1,\ldots,n} |\alpha_l| = \|g\|_\infty .$$

Thus

$$|||E g|||_1 = \|g\|_1 \quad \text{and} \quad |||E g|||_\infty = \|g\|_\infty .$$

Therefore it remains to prove that E_∞ is bijective. E_∞ is clearly injective, for $E_\infty g = E_\infty h$ implies, by definition of E and E^i, that we have for every $A \in \hat{\mathfrak{A}}_0$

$$\int_A g \, dp = \int_A h \, dp .$$

Hence $h = g$ p-a.e. In order to prove that E_∞ is surjective remark that it is sufficient to find a pre-image for a positive uniformly bounded martingale, because of Lemma 3.2d. Thus let g be a positive element of $L^\infty(\Omega, \mathfrak{A}, p; f^*; I)$. Consider then, for $i \in I$, the measure

$$A \mapsto \nu_i A = \int_A g_i \, dp_i .$$

These measures are coherent, i.e.

$$\nu_i A = \int_A g_i \, dp_i = \int_A (E^{i,j} g_j) \, dp_i = \int_{f_{i,j}^{-1} A} g_j \, dp_j = \nu_j(f_{i,j}^{-1} A)$$

and uniformly bounded with respect to the p_i, indeed

$$0 \leqq \nu_i A \leqq \|g_i\|_\infty \, p_i(A) \leqq |||g|||_\infty \, p_i(A).$$

Therefore there is, by Lemma 3.1, a unique regular measure ν on X such that

$$\forall B \in \mathscr{B}_X, \qquad 0 \leqq \nu B \leqq |||g|||_\infty \, p(B)$$

and

$$\forall i \in I, \quad \forall A \in \mathfrak{A}_i [\varphi_i A \in \mathfrak{A}_0 \Rightarrow \nu(\varphi_i \tilde{A})^\wedge = \nu_i A] .$$

This clearly implies that $\nu \ll p$ and that $\forall i \in I$, $\forall A \in \mathfrak{A}_i$ $\nu \check{A} = \nu_i A$ (cf. the notation on p. 72). Let h be the Radon-Nikodym derivative of ν with respect to p and consider the functions $h_i = E^i h$. We have, for $A \in \mathfrak{A}_i$

$$\int_A h_i \, dp_i = \int_A (E^i h) \, dp_i = \int_{\check{A}} h \, dp = \nu \check{A} = \nu_i A = \int_A g_i \, dp_i .$$

Hence $h_i = g_i$ p_i-a.e. and thus $g = E h$. Therefore E_∞ is surjective. This concludes the proof of the theorem.

Corollary 1. 1. *If Y is a Banach space and if there is, for every $i \in I$ a continuous linear map G_i: $Y \to L^\infty(\Omega_i, \mathfrak{A}_i, p_i)$ such that*

$$i \leqq j \Rightarrow G_i = E^{i,j} \circ G_j,$$

and

$$\sup_{i \in I} \|G_i\| = M < \infty$$

then there is a unique continuous linear

$$G\colon \; Y \to L^\infty(X, p)$$

such that

$$G_i = E^i \circ G \quad \text{and} \quad \|G\| \leqq M.$$

2. *If Y is moreover a Banach lattice and the G_i are positive then G is positive.*

3. *Thus E^i: $L^\infty(X, p) \to L^\infty(\Omega_i, \mathfrak{A}_i, p_i)$ is the projective limit of $(L^\infty(\Omega, \mathfrak{A}, p); E; I)$ in the category of Banach spaces with contractions and also in the category of Banach lattices with positive contractions.*

Proof. 1. Consider, for $y \in Y$, the sequence $(G_i y)_{i \in I}$. By our assumptions this is a uniformly bounded $(\Omega, \mathfrak{A}, p; f^*; I)$-martingale, hence there is, by the theorem, a unique $G y \in L^\infty(X, p)$ such that

$$E^i(G y) = G_i y$$

G is then clearly linear and moreover, again by the theorem

$$\|G y\|_\infty = |||(G_i y)_{i \in I}|||_\infty = \sup_{i \in I} \|G_i y\|_\infty \leqq \sup_{i \in I} \|G_i\| \cdot \|y\| = M \cdot \|y\| .$$

2. If $y \in Y$ is positive then, by assumption $G_i y \geqq 0$ for every $i \in I$ and hence, by Theorem 3.2 $G y \geqq 0$. Hence G is positive.

3. This is merely a restatement of 1. and 2.

Corollary 2. *Suppose that Y is a compact Hausdorff space with a regular probability measure q, such that there is, for every $i \in I$ a desintegration $\omega \mapsto \varphi_i(\omega, .)$ of q in regular probability measures $\varphi_i(\omega, .)$ on Y (i.e $q(.) = \int p_i(d\omega) \, \varphi_i(\omega, .)$), which is coherent, that is*

$$i \leqq j \Rightarrow \forall B \in \mathscr{B}_y, \quad \varphi_i(., B) = E^{i,j} \varphi_j(., B).$$

Then there is a unique (up to equivalence) desintegration $x \mapsto \varphi(x, .)$ of q over X, such that

$$\forall i \in I, \quad \forall B \in \mathscr{B}_y, \quad \varphi_i(., B) = \int \pi_i(., dx) \, \varphi(x, B) \quad p_i\text{-a.e.}$$

Proof. This is a direct consequence of Lemma 2.2 and the previous corollary.

4. Representation by Mappings

We show first that a representation space (X, p) for an **F**-projective system $(\Omega, \mathfrak{A}, p; f^*; I)$ which admits coherent mmpm-'s

$$f_i\colon X \to \Omega_i$$

such that the E^i (or alternatively the π_i) are induced by the f_i (i.e. E^i is the conditional expectation operator induced by f_i) must be an **RP**-projective limit of $(\Omega, \mathfrak{A}, p; f^*; I)$.

Theorem 4.1. *Suppose that the* **F**-*projective system* $(\Omega, \mathfrak{A}, p; f^*; I)$ *is such that there is a representation space* (X, p) *and* mmpm-'s

$$f_i\colon X \to \Omega_i$$

such that the positive contractions E^i are induced by the f^i, i.e.

$$\forall h \in L^1(\Omega_i, \mathfrak{A}_i, p_i),\ \forall g \in L^\infty(X, p), \quad \int h(E^i g)\, dp_i = \int (h \circ f_i)\, g\, dp.$$

Then $\{f_i^*\colon (X, \bar{\mathscr{B}}_X, p) \to (\Omega_i, \mathfrak{A}_i, p_i)\}$ *is both an* **RP**- *and an* **F**-*projective limit of* $(\Omega, \mathfrak{A}, p; f^*; I)$. *In particular the f_i^* are then coherent.*

Proof. We have to show that, for any complete probability space $(\Lambda, \mathfrak{B}, q)$ with **RP**-morphisms

$$Q_i\colon (\Lambda, \mathfrak{B}, q) \to (\Omega_i, \mathfrak{A}_i, p_i)$$

which are coherent, i.e.

$$\forall (i,j) \in D_I, \quad Q_i = f_{i,j}^* Q_j$$

there is a unique **RP**-morphism $Q\colon (\Lambda, \mathfrak{B}, q) \to (X, \mathscr{B}_X, p)$ such that

$$\forall i \in I, \quad Q_i = f_i^* Q$$

and that, moreover, if the Q_i are actually **F**-morphisms, then Q is an **F**-morphism as well.

Observe first that we have for every $i \in I$ and $A \in \mathfrak{A}_i$

$$f_i^{-1} A = \check{A} \quad p\text{-a.e.}$$

Indeed, by definition of E^i and because the f_i induce E^i, we have for every $g \in L^\infty(X, p)$

$$\int_{\check{A}} g\, dp = \int_A (E^i g)\, dp_i = \int (\chi_A \circ f_i)\, g\, dp = \int_{f_i^{-1} A} g\, dp$$

and this implies clearly $p(\check{A} \,\triangle\, f_i^{-1} A) = 0$. From this it follows easily that the f_i^* are coherent.

Let us assume then that the Q_i are given. Select for every $i \in I$ a rcp $\sigma_i \in Q_i$. The coherence condition states, in terms of the σ_i

$$\forall (i,j) \in D_I, \forall A \in \mathfrak{A}_i, \quad \sigma_i(., A) = \sigma_j(., f_{i,j}^{-1} A) \quad q\text{-a.e.}$$

Consider, for $B\in\mathfrak{B}$ the measures

$$\psi_i(B,.)=\int_B q(d\lambda)\,\sigma_i(\lambda,.)$$

on $\mathfrak{A}_i$. As $\psi_i(\Lambda,A)=p_i(A)$, because σ_i is a rcp, we have clearly

$$\forall B\in\mathfrak{B},\qquad \psi_i(B,.)\leqq p_i$$

and, by the coherence condition

$$\forall (i,j)\in D_I,\qquad \psi_i(.,A)=\psi_j(.,f_{i,j}^{-1}A).$$

Hence there is, by Lemma 3.1, for every $B\in\mathfrak{B}$, a unique regular measure $\psi(B,.)$ on X such that

$$\forall i\in I,\ \forall A\in\mathfrak{A}_i,\qquad \psi(.,\check{A})=\psi_i(.,A)$$

$$\forall B\in\mathscr{B}_X,\qquad \psi(.,B)\leqq p(B).$$

It follows that $\psi(C,.)\ll p$, thus, by what we proved already

$$\forall i\in I,\ \forall A\in\mathfrak{A}_i,\qquad \psi(.,f_i^{-1}A)=\psi_i(.,A).$$

Also $\psi(.,B)$ is then a measure on $\mathfrak{B}$ for every $B\in\hat{\mathfrak{A}}_0$. As this is, moreover, trivially true for all nullsets $B\in\mathscr{B}_X$ and as $\hat{\mathfrak{A}}_0$ generates $\mathscr{B}_X$ mod p, a well known argument then shows that $\psi(.,B)$ is a measure on $\mathfrak{B}$ for every Borelset $B\subset X$.

Now we can construct a positive linear contraction

$$G\colon\ L^\infty(X,p)\to L^\infty(\Lambda,\mathfrak{B},q)$$

as follows. Consider, for $g\in L^\infty(X,p)$, the bounded signed measure

$$B\mapsto\int g(x)\,\psi(B,dx).$$

This measure is clearly absolutely continuous with respect to q (as $\psi_i(B,A)\leqq q(B)$, hence $\psi(B,.)\leqq q(B)$). Hence there is a unique $G\,g\in L^\infty(\Lambda,\mathfrak{B},q)$ such that

$$\forall B\in\mathfrak{B},\qquad \int g(x)\,\psi(B,dx)=\int_B (G\,g)\,dq$$

G is clearly a positive linear contraction. Also, obviously, $G1=1$, and moreover G is "integralpreserving", as

$$\int (G\,g)\,dq=\int g(x)\,\psi(\Lambda,dx)=\int g\,dp.$$

Thus Lemma 2.2 can be applied and we find a desintegration $(\sigma(\lambda,.))_{\lambda\in\Lambda}$ of p such that

$$\forall g\in L^\infty(X,p),\qquad G\,g=\int\sigma(.,dx)\,g(x).$$

Now, consider the composition $f_{i*}\sigma\colon(\Lambda,\mathfrak{B},q)\to(\Omega_i,\mathfrak{A}_i,p_i)$. We have, for $\lambda\in\Lambda$ and $A\in\mathfrak{A}_i$

$$(f_{i*}\sigma)(\lambda,A)=\int\sigma(\lambda,dx)\,f_{i*}(x,A)=\int\sigma(\lambda,dx)(\chi_A\circ f_i)(x)=G(\chi_A\circ f_i)(\lambda).$$

Hence $(f_{i*}\sigma)(.,A)$ satisfies

$$\forall B\in\mathfrak{B}, \quad \int_B (f_{i*}\sigma)(.,A)\,dq = \int (\chi_A\circ f_i)(x)\,\psi(B,dx) = \psi(B, f_i^{-1}A) = \psi_i(B,A) = \int_B \sigma_i(.,A)\,dq.$$

Hence

$$(f_{i*}\sigma)(.,A) = \sigma_i(.,A) \quad q\text{-a.e.}$$

or

$$f_i^* Q = Q_i,$$

if Q is the **RP**-morphism containing the rcp σ.

Next we prove uniqueness. Thus suppose that Q' also satisfies

$$\forall i\in I, \quad f_i^* Q' = Q_i$$

and let σ' be a rcp in Q'. Then we have

$$\forall i\in I,\ \forall A\in\mathfrak{A}_i, \quad \sigma(.,f_i^{-1}A) = \sigma'(.,f_i^{-1}A) \quad q\text{-a.e.}$$

This implies, as σ and σ' are both measurepreserving and as

$$f_i^{-1}A = \check{A} \quad p\text{-a.e.},$$

that

$$\forall A\in\hat{\mathfrak{A}}_0, \quad \sigma(.,A) = \sigma'(.,A) \quad q\text{-a.e.}$$

As $\hat{\mathfrak{A}}_0$ generates $\mathscr{B}_X$ mod p, this implies

$$\forall A\in\mathscr{B}_X, \quad \sigma(.,A) = \sigma'(.,A) \quad q\text{-a.e.}$$

or

$$Q = Q'.$$

It remains to prove that Q is an **F**-morphism if Q_i is an **F**-morphism for every $i\in I$. Choose mmpm-'s $u_i\colon \Lambda\to\Omega_i$ such that $u_i^* = Q_i$ and a rcp σ in Q. Then we have

$$f_i^* Q = Q_i = u_i^*,$$

hence

$$\forall i\in I,\ \forall A\in\mathfrak{A}_i, \quad \sigma(.,f_i^{-1}A) = \chi_A\circ u_i \quad q\text{-a.e.}$$

This implies easily (again using that $f_i^{-1}A = \check{A}$ p-a.e.) that

$$\forall B\in\mathscr{B}_X \qquad \text{ess range } \sigma(.,B)\subset\{0,1\}.$$

Let $b\mapsto\langle b\rangle$ be a lifting of $\mathfrak{B}$ and consider, for $B\in\mathscr{B}_X$

$$\alpha B = \langle\{\lambda\in\Lambda \mid \sigma(\lambda,B)=1\}^{\sim}\rangle.$$

As $\sigma(\lambda,.)$ is a regular probability measure on X and because ess range $\sigma(.,B)\subset\{0,1\}$, it follows that $\alpha\colon \mathscr{B}_X\to\mathfrak{B}$ is a Boolean homomorphism, with the σ-ideal of Borel nullsets as kernel. Therefore $\sigma(.,B) = \chi_{\alpha B}$ q-a.e., hence

$$q(\alpha B) = \int_{\alpha B} dq = \int_{\alpha B} \sigma(.,B)\,dq = \int \sigma(.,B)\,dq = p(B),$$

thus α is measurepreserving as well. Define now, for compact $K \subset X$

$$\bar{\alpha} K = \bigcap_{U \in \mathscr{U}(K)} \alpha U.$$

Then, obviously $\alpha K \subset \bar{\alpha} K$, and the regularity of p implies easily

$$q(\alpha K) = q(\bar{\alpha} K).$$

Consider then, for $\lambda \in \Lambda$, the set

$$d_\lambda = \{K \subset X \mid K \text{ compact, } \lambda \in \bar{\alpha} K\}.$$

d_λ is clearly an ultrafilter in the collection of compact subsets of X (if $K \subset X$ is compact and $\lambda \notin \bar{\alpha} K$, there is an open $U \subset X$, such that $K \subset U$ and $\lambda \notin \alpha U$. Hence $\lambda \in \alpha U^c \subset \bar{\alpha} U^c$). Hence the intersection of all elements of d_λ consists of exactly one point, $h\lambda$ say. The function

$$h \colon \Lambda \to X$$

so defined satisfies clearly, for compact $K \subset X$

$$h^{-1} K = \bar{\alpha} K.$$

Thus, for compact K

$$h^{-1} K = \alpha K \quad q\text{-a.e.}$$

Then it follows along well known lines that

$$\forall B \in \mathscr{B}_X, \quad h^{-1} B = \alpha B \quad q\text{-a.e.}$$

(using the fact that α, although not a σ-homomorphism, has the property that $\bigcup_{n=1}^{\infty} \alpha B_n = \alpha \bigcup_{n=1}^{\infty} B_n$ q-a.e., because it is measurepreserving). Hence h is measurable and measurepreserving. Finally we have for every $B \in \mathscr{B}_X$, outside some nullset $N \subset \Lambda$,

$$h\lambda \in B \Leftrightarrow \lambda \in h^{-1} B \Leftrightarrow \lambda \in \alpha B \Leftrightarrow \sigma(\lambda, B) = 1.$$

Hence

$$\forall B \in \mathscr{B}_X, \quad \sigma(.\,, B) = h_*(.\,, B) \quad q\text{-a.e.}$$

or

$$Q = h^*.$$

This concludes the proof of the theorem.

We proceed now by establishing the existence of representation spaces.

Theorem 4.2. *Let $(\Omega, \mathfrak{A}, p; f^*; I)$ be an* **F**-*projective system. $\mathfrak{A}$ is the inductive limit Boolean algebra with measure π, described in the beginning of Section* 3. *Then, corresponding to any dense subalgebra $\mathfrak{A}_0$ of $\mathfrak{A}$ there is a representation space (X, p) for $(\Omega, \mathfrak{A}, p; f^*; I)$.*

Proof. Let X be the Stone space of the Boolean algebra $\mathfrak{A}_0$. $A \mapsto \hat{A}$ will be the canonical map from $\mathfrak{A}_0$ into the power set of X, i.e.

$$\hat{A} = \{u \in X \mid A \in u\}.$$

Define, for $A \in \mathfrak{A}_0$

$$p(\hat{A}) = \pi(A).$$

It is well known that p can then be extended to a regular probability measure on the compact Hausdorff space X, for which $\hat{\mathfrak{A}}_0$ is a basis of clopen sets. Then (X, p) is clearly a representation space.

Corollary 1. *If the* F*-projective system admits a system* $(\mathfrak{C}_i)_{i \in I}$ *of generating compact sublattices, which have approximate complements, then it has an* **RP** *and* F*-projective limit.*

Proof. This is a direct consequence of Lemma 2.3, Theorems 4.1 and 4.2. Indeed, let $\mathfrak{A}_0$ be any dense subalgebra of $\mathfrak{A}$ such that $\varphi_i \tilde{C} \in \mathfrak{A}_0$ for every $i \in I$ and every $C \in \mathfrak{C}_i$, and let (X, p) be the representation space, the existence of which follows from Theorem 4.2. Then, for each $i \in I$, $(\Omega_i, \mathfrak{A}_i, p_i)$, $\mathfrak{C}_i$, φ_i satisfy the requirements of Lemma 2.3. Hence there is a mmpm f_i: $X \to \Omega_i$ such that $\forall C \in \mathfrak{C}_i$, $f_i^{-1} C = \varphi_i \tilde{C}$ a.e. This easily implies that the f_i induce the E^i and hence that Theorem 4.1 is applicable.

Corollary 2. *Let* $(\Omega, \mathfrak{A}, p; f; I)$ *be an* F*-projective system. Suppose that the* Ω_i *are locally compact Hausdorff spaces, and that the* p_i *are regular measures. Then, there is an* **RP**- *and* F*-projective limit*

$$\{f_i^*: (X, \bar{\mathscr{B}}_X, p) \to (\Omega_i, \mathfrak{A}_i, p_i)\}.$$

Proof. One checks without difficulty that $(\mathfrak{C}_i)_{i \in I}$, where $\mathfrak{C}_i$ is the lattice of compact subsets of Ω_i, satisfies the requirements of Corollary 1.

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