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Ratio Comparisons of Supremum and Stop Rule Expectations

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Summary. Suppose $X_1, X_2, \ldots, X_n$ are independent non-negative random variables with finite positive expectations. Let T_n denote the stop rules for $X_1, \ldots, X_n$. The main result of this paper is that $E(\max\{X_1, \ldots, X_n\}) < 2\sup\{EX_t: t \in T_n\}$. The proof given is constructive, and sharpens the corresponding weak inequalities of Krengel and Sucheston and of Garling.

§ 1. Introduction

Let $X_1, X_2, \ldots, X_n$ be independent non-negative random variables on a probability space $(\Omega, \mathfrak{A}, P)$, and let T_n denote the set of stop rules for $X_1, \ldots, X_n$. The "prophet" inequality $E(\max\{X_1, \ldots, X_n\}) \leqq k \sup\{EX_t: t \in T_n\}$ has been studied in the theory of semiamarts (e.g., [2-5]). Krengel and Sucheston [3] discovered that $2 \leqq k \leqq 4$ for all n and all $X_1, \ldots, X_n$, and Garling's proof ([3], p. 237) shows that $k=2$, and that 2 is the best possible bound.

The purpose of this note is to offer a constructive proof that $k=2$, using extremal random variables called "long shots", and to show that in fact strict inequality holds in all non-trivial situations. The main result is

Theorem 1. *Let $n > 1$, and $X_1, X_2, \ldots, X_n$ be independent non-negative random variables with positive finite expectations. Then $E(\max\{X_1, \ldots, X_n\}) < 2\sup\{EX_t: t \in T_n\}$.*

§ 2. Proof of Theorem 1

Throughout this section, all random variables are assumed to be non-negative with positive finite expectations. EX will denote the expectation of X, $X \vee Y$ the maximum of X and Y, $(X-Y)^+$ the positive part $((X-Y) \vee 0)$ of $X-Y$, $V(X_1, \ldots, X_n) = \sup\{EX_t: t \in T_n\}$, and

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$$R(X_1, \ldots, X_n) = E(X_1 \vee \ldots \vee X_n)/V(X_1, \ldots, X_n).$$

With this notation, the conclusion of Theorem 1 is that $R(X_1, \ldots, X_n) < 2$ for all $n > 1$ and all $X_1, \ldots, X_n$.

Essential in the construction to follow is the notion of a "long shot", a two-valued random variable which is nearly always zero, but is very large on a set of small probability.

Definition. A *long shot* is a random variable L defined by $L=0$ with probability $1-p$ and $=\mu$ with probability p, where $\mu > 10^6$ and $0 < p < 10^{-6}$. (Any "large" and "small" constants will do.)

Lemma 1. *Given $n > 2$ and independent random variables $X_1, \ldots, X_n$ there exists a long shot L satisfying $R(\lambda, X_2, \ldots, X_{n-2}, L) > R(X_1, \ldots, X_n)$, where $\lambda = V(X_2, \ldots, X_n)$.*

Since Lemma 1 reduces the number of random variables by one and since

$$R(X_1, X_2) = E(X_1 \vee X_2)/V(X_1, X_2) < (EX_1 + EX_2)/\max\{EX_1, EX_2\} \leqq 2,$$

the proof of Theorem 1 will be complete once Lemma 1 is established.

Proof of Lemma 1. First it is shown that X_1 may be replaced by the constant $\lambda = V(X_2, \ldots, X_n)$, that is,

$$\begin{aligned} &R(X_1, \ldots, X_n) \\ &\quad \leqq [E(\lambda \vee X_2 \vee \ldots \vee X_n) + E(X_1 - \lambda)^+]/V(X_1, \ldots, X_n) \\ &\quad = [E(\lambda \vee X_2 \vee \ldots \vee X_n) + E(X_1 - \lambda)^+]/[V(\lambda, X_2, \ldots, X_n) + E(X_1 - \lambda)^+] \\ &\quad \leqq R(\lambda, X_2, \ldots, X_n). \end{aligned} \tag{1}$$

The first inequality in (1) follows since

$$\begin{aligned} E(X_1 \vee \ldots \vee X_n) &\leqq E(X_1 \vee \lambda \vee X_2 \vee \ldots \vee X_n) \\ &= E(\lambda \vee X_2 \vee \ldots \vee X_n) + E(X_1 - \lambda \vee X_2 \vee \ldots \vee X_n)^+ \\ &\leqq E(\lambda \vee X_2 \vee \ldots \vee X_n) + E(X_1 - \lambda)^+; \end{aligned}$$

the equality in (1) since (as an easy consequence of [1], p. 50) $V(X_1, \ldots, X_n) = V(X_2, \ldots, X_n) + E(X_1 - \lambda)^+$ and $V(\mu, X_2, \ldots, X_n) = V(X_2, \ldots, X_n)$; and the last inequality since $0 < V(\lambda, X_2, \ldots, X_n) \leqq E(\lambda \vee X_2 \vee \ldots \vee X_n)$ and since $(a+\delta)/(b+\delta) \leqq a/b$ for $a \geqq b > 0$ and $\delta \geqq 0$.

Next, it will be shown that the last two random variables X_{n-1} and X_n may be replaced by some long shot L. Let L_p be a long shot independent of $X_2, \ldots, X_{n-2}$ with $P(L_p = V(X_{n-1}, X_n)/p) = p > 0$. Clearly $V(\lambda, X_2, \ldots, X_{n-2}, L_p) = V(\lambda, X_2, \ldots, X_n)$. As $p \searrow 0$,

$$\begin{aligned} E(\lambda \vee X_2 \vee \ldots \vee X_{n-2} \vee L_p) &\nearrow E(\lambda \vee X_2 \vee \ldots \vee X_{n-2}) + EL_p \\ &= E(\lambda \vee X_2 \vee \ldots \vee X_{n-2}) + EX_n + E(X_{n-1} - EX_n)^+ \\ &\geqq E(\lambda \vee X_2 \vee \ldots \vee X_{n-2}) + EX_n + E(X_{n-1} - \lambda \vee X_2 \vee \ldots \vee X_{n-2})^+ \\ &= E(\lambda \vee X_2 \vee \ldots \vee X_{n-1}) + EX_n > E(\lambda \vee X_2 \vee \ldots \vee X_n). \end{aligned}$$

Thus for p' sufficiently small, the long shot $L = L_{p'}$ satisfies

$$R(\lambda, X_2, \ldots, X_{n-2}, L) > R(\lambda, X_2, \ldots, X_n), \tag{2}$$

which, with (1), completes the proof of the lemma. □

§ 3. Remarks

An easy consequence of Theorem 1 is the result of Garling for infinite sequences:

Corollary 1. *Let $X_1, X_2, \ldots$ be independent non-negative random variables. Then $E(X_1 \vee X_2 \vee \ldots) \leqq 2V(X_1, X_2, \ldots)$.*

If the independence assumption is dropped, the proportionate advantage a prophet enjoys over a gambler in an n-step game is at most n.

Proposition 1. *If $X_1, X_2, \ldots, X_n$ are non-negative, then $E(X_1 \vee \ldots \vee X_n) \leqq n V(X_1, \ldots, X_n)$, and the bound n is sharp.*

Proof. Since $E(X_1 \vee \ldots \vee X_n) \leqq EX_1 + \ldots + EX_n$ and $V(X_1, \ldots, X_n) \geqq \max\{EX_1, \ldots, EX_n\}$ it follows that $E(X_1 \vee \ldots \vee X_n) \leqq n V(X_1, \ldots, X_n)$. For $n = 1$, $EX_1 = V(X_1)$. To show that the bound n is sharp for $n > 1$, let $p \in (0, 1)$ be given and define random variables $X_1, \ldots, X_n$ jointly by $P[(X_1, \ldots, X_n) = (p^0, p^{-1}, \ldots, p^{-j}, 0, \ldots, 0)] = p^j - p^{j+1}$ if $0 \leqq j \leqq n-2$, and $= p^{n-1}$ for $j = n-1$. Then $X_1, \ldots, X_n$ is a martingale and $V(X_1, \ldots, X_n) = EX_1 = 1$. Observe that $E(X_1 \vee \ldots \vee X_n) = (n-1)(1-p) + 1$, and let $p \searrow 0$. □

If one drops the non-negativity assumption, on the other hand, the prophet's proportionate advantage may be arbitrarily high in even a 2-step game of independent random variables.

Example 1. Fix $M \gg 0$. Let $X_1 \equiv 1$, and define X_2 by $P(X_2 = 2M) = P(X_2 = -2M) = 1/2$. Then $V(X_1, X_2) = 1$, and $E(X_1 \vee X_2) = M + 1/2$.

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