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On the Critical Percolation Probabilities*

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Summary. We prove that the critical probabilities of site percolation on the square lattice satisfy the relation $p_c + p_c^* = 1$. Furthermore we prove the continuity of the function "percolation probability".

1. Introduction

It was conjectured in [1] that in any pair of dual graphs the critical probabilities of percolation, p_c and p_c^*, satisfy the relation

$$p_c + p_c^* = 1. \tag{1.1}$$

If, in particular, as in the bond percolation in the square lattice, the graph is self-dual, so that $p_c = p_c^*$, (1.1) becomes

$$p_c = 1/2. \tag{1.2}$$

In the case of bond percolation in the square lattice (1.2) has been recently proved by H. Kesten [2]. Here we extend his result by proving (1.1). The present paper deals only with site percolation in the square lattice, but it seems possible to extend his results to other regular planar graphs.

We call μ_x the Bernoulli probability measure according to which each element of the graph is equal to $+1$ ("open" in the bond terminology) with probability x. In his paper Kesten determines the μ_x-probability of suitable events, whose $\mu_{\frac{1}{2}}$-probability is known, by a sequence of modifications of the measure $\mu_{\frac{1}{2}}$. In particular he uses the fact that, by self-duality, the $\mu_{\frac{1}{2}}$-probability of the crucial events $A_{L,1}^{+}$ (defined in Sect. 2) for any L equals 1/2. The main new tool in our proof of (1.1) is an uniform bound on the functions $\mu_x(A_{L,K}^{+})$, for $x \in [1 - p_c^*, p_c]$. This bound, proved in Sect. 3, allows us to prove also the continuity of the function "percolation probability" (we remark that in the self-dual case, conversely, this last statement is a simple consequence of (1.2)). Section 4 contains a remark which allows us to simplify the main proof. The main result is proved in Sect. 5.

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2. Definitions and Some Preliminary Results

We shall employ the following terminology and notations. Two elements i, j of Z^2 are adjacent if $|i_1-j_1|+|i_2-j_2|=1$, they are *adjacent if $\mathrm{Max}(|i_1-j_1|, |i_2-j_2|)=1$. A finite sequence $(i_1, \dots, i_n)$ of distinct elements of Z^2 is a (self-avoiding) chain [*chain] if i_r and i_s are adjacent [*adjacent] if and only if $|r-s|=1$ (throughout this paper chains and *chains will always be understood to be self-avoiding, in the above specified sense). $(i_1, \dots, i_n)$ is a circuit [*circuit] if for any $r \in (1, \dots, n)$ $(i_r, i_{r+1}, \dots, i_n, i_1, \dots, i_{r-2})$ is a chain [*chain]. A set $X \subset Z^2$ is connected [*connected] if for any pair i, j of points in X there is a chain [*chain] made up of points in X having i, j as terminal points.

We consider the configuration space $\Omega=\{-1,1\}^{Z^2}$. We define in Ω the partial order $\leqq$ by putting $\omega_1 \leqq \omega_2$ if and only if $\forall i \in Z^2$ $\omega_1(i) \leqq \omega_2(i)$ and we call positive [negative] an event A if this characteristic function is non-decreasing [non-increasing]. We put $E_i^+ [E_i^-]=\{\omega \in \Omega | \omega(i)=1[-1]\}$. For every $K \subset Z^2$, we call $\mathscr{B}_K$ the σ-algebra generated by the events E_i^+, $i \in K$.

If $\omega \in \Omega$ the $(+)$clusters $[(+*)$clusters$]$ in ω are the maximal connected [*connected] components of $\omega^{-1}(1)$; $(-)$clusters and $(-*)$clusters are defined in the same way. We call $(+)$chain in $\omega \in \Omega$ any chain included in $\omega^{-1}(1)$; $(-)$chains, $(-*)$chains, $(+)$circuits, and so on, are defined in an analogous way.

For any $x \in [0,1]$ v_x is the measure on $\{-1,1\}$ which assigns weights x and $1-x$ respectively to 1 and -1. We put $\mu_x=\prod_{i \in Z^2} v_x$. $P_\infty^+(x)$ $[P_\infty^{+*}(x)]$ is the μ_x-probability that a given element of Z^2 belongs to an infinite $(+)$cluster $[(+*)$cluster$]$. $P_\infty^-(x)$ and $P_\infty^{-*}(x)$ are defined in an analogous way.

The critical points are defined by:

$$p_c=\mathrm{Sup}\{x \in [0,1] | P_\infty^+(x)=0\}, \quad p_c^*=\mathrm{Sup}\{x \in [0,1] | P_\infty^{+*}(x)=0\}.$$

Note that

$$1-p_c^*=\mathrm{Inf}\{x \in [0,1] | P_\infty^{-*}(x)=0\}.$$

Hence Harris' theorem [3] implies

$$p_c+p_c^* \geqq 1. \tag{2.1}$$

We put, for any pair of positive integers L, K:

$$\Lambda_{L,K}=\{i \in Z^2 | |i_1| \leqq KL, \ |i_2| \leqq L\}.$$

We shall consider in particular the square $\Lambda_{L,1}$ and the rectangles $\Lambda_{L,2}$, $\Lambda_{L,3}$. For any rectangle Λ we call $S(\Lambda)$ $[S^*(\Lambda)]$ the set of chains [*chains] contained in Λ starting on the "left side" of Λ and ending on its "right side". If $c \in S(\Lambda)$ $[c \in S^*(\Lambda)]$ we call $A(c)$ the set of elements of Λ which are "above c" and we consider in $S(\Lambda)$ $[S^*(\Lambda)]$ the partial order defined by putting $c_1 \geqq c_2$ if $A(c_1) \subseteq A(c_2)$. We call $S_\Lambda^+(\omega)$ $[S_\Lambda^{-*}(\omega)]$ the set of elements of $S(\Lambda)$ $[S^*(\Lambda)]$ which are included in $\omega^{-1}(1)$ $[\omega^{-1}(-1)]$, and we put

$$A_{L,K}^+=\{\omega \in \Omega | S_{\Lambda_{L,K}}^+(\omega) \neq \emptyset\}, \quad A_{L,K}^{-*}=\{\omega \in \Omega | S_{\Lambda_{L,K}}^{-*}(\omega) \neq \emptyset\};$$

$$R_{L,K}^+(x)=\mu_x(A_{L,K}^+), \quad R_{L,K}^{-*}(x)=\mu_x(A_{L,K}^{-*}).$$

It is not too difficult to prove that if $S_\Lambda^+(\omega) \neq \emptyset$ $[S_\Lambda^{-*}(\omega) \neq \emptyset]$ there is in $S_\Lambda^+(\omega)$ $[S_\Lambda^{-*}(\omega)]$ one and only one minimal element (a formal proof of this statement is in [2]): we call it the lowest $(+)$chain $[(-*)$chain$]$ in $S(\Lambda)$ $[S^*(\Lambda)]$ in the configuration ω.

The functions $R_{L,K}^+(x)$, $R_{L,K}^{-*}(x)$, introduced independently in [4] and in [5], play a considerable role in percolation theory. The following lemma contains some inequalities relating $R_{L,2}^+(x)$, $R_{L,3}^+(x)$ $[R_{L,2}^{-*}(x)$, $R_{L,3}^{-*}(x)]$ with $R_{L,1}^+(x)$ $[R_{L,1}^{-*}(x)]$. These inequalities are very similar (but slightly stronger) to the analogous inequalities proved in [5] for bond percolation. We give an independent proof of them because our proof seems simpler than the one given in [5].

Lemma 1. *For any positive integer L and for any* $x \in [0,1]$:

$$R_{L,2}^+(x) \geqq R_{L,1}^+(x)\,[1-(1-R_{L,1}^+(x))^{\frac{1}{2}}]^6, \tag{2.2}$$

$$R_{L,2}^{-*}(x) \geqq R_{L,1}^{-*}(x)\,[1-(1-R_{L,1}^{-*}(x))^{\frac{1}{2}}]^6, \tag{2.2a}$$

$$R_{L,3}^+(x) \geqq [R_{L,1}^+(x)]^3\,[1-(1-R_{L,1}^+(x))^{\frac{1}{2}}]^{12}, \tag{2.3}$$

$$R_{L,3}^{-*}(x) \geqq [R_{L,1}^{-*}(x)]^3\,[1-(1-R_{L,1}^{-*}(x))^{\frac{1}{2}}]^{12}. \tag{2.3a}$$

Proof. We consider, besides $\Lambda_{L,1}$, the other square in Z^2:

$$\Lambda'_{L,1} = \{i \in Z^2 \mid 0 \leqq i_1 \leqq 2L,\ |i_2| \leqq L\}.$$

Furthermore we put

$$a_l = \{i \in Z^2 \mid i_1 = 0,\ -L \leqq i_2 \leqq 0\}, \qquad a_u = \{i \in Z^2 \mid i_1 = 0,\ 0 \leqq i_2 \leqq L\},$$
$$\Lambda_{\Lambda,3/2} = \Lambda_{L,1} \cup \Lambda'_{L,1}.$$

In other words a_l and a_u are the lower and the upper halves of the left side $a = a_l \cup a_u$ of $\Lambda'_{L,1}$. We call $A_{L,3/2}^+$ the event "there exists at least one $(+)$chain in $\Lambda_{L,3/2}$ connecting its left side with its right side" and we put $R_{L,3/2}^+(x) = \mu_x(A_{L,3/2}^+)$. If $s = (s_1, \ldots, s_n)$ is a chain in $S(\Lambda_{L,1})$ (ordered from the left side of $\Lambda_{L,1}$ to the right one) we call s^a the last intersection of s with a and we put $s^r = s^a, \ldots, s_n)$. Furthermore we call S_l the set of chains $s \in S(\Lambda_{L,1})$ such that $s^a \in a_l$ and $s^{r'}$ the chain obtained by reflecting s^r with respect to the line $i_1 = L$.

Now we consider the following events:

$E_s = \{s$ is the lowest $(+)$chain in $S(\Lambda_{L,1})\}$,

$F_s = \{$there is in $\Lambda'_{L,1} \cap A(s^r \cup s^{r'})$ a $(+)$chain starting on the upper side of $\Lambda'_{L,1}$ and ending in $s^r\}$,

$G = \bigcup_{s \in S_l} (E_s \cap F_s)$,

$H_u[H_l] = \{$there is in $\Lambda'_{L,1}$ a $(+)$chain starting in $a_u[a_l]$ and ending on the right side of $\Lambda'_{L,1}\}$.

By using Harris' inequality [3] and the symmetry properties of μ_x we get

$$[1-\mu_x(H_u)]^2 = [\mu_x(\Omega \setminus H_u)]^2 \leqq \mu_x((\Omega \setminus H_u) \cap (\Omega \setminus H_l)) = 1 - R_{L,1}^+(x).$$

Hence

$$\mu_x(H_u) \geqq 1-(1-R^+_{L,1}(x))^{\frac{1}{2}}. \tag{2.4}$$

In an analogous way, by considering the event $\bigcup_{s \in S_l} E_s$ and the event obtained by reflecting it with respect to the line $i_2=0$ we get

$$\sum_{s \in S_l} \mu_x(E_s) \geqq 1-(1-R^+_{L,1}(x))^{\frac{1}{2}}. \tag{2.5}$$

Furthermore it is easy to prove that $\mu_x(F_s|E_s) \geqq \mu_x(F_s)$. Hence, by using once more the same argument, we get, for any $s \in S(\Lambda_{L,1})$:

$$\mu_x(F_s|E_s) \geqq 1-(1-R^+_{L,1}(x))^{\frac{1}{2}}. \tag{2.6}$$

Since G and H_u are positive events Harris' inequality implies:

$$\mu_x(G \cap H_u) \geqq \mu_x(H_u)\, \mu_x(G) = \mu_x(H_u) \sum_{s \in S_l} \mu_x(F_s|E_s)\, \mu_x(E_s)$$
$$\geqq [1-(1-R_{L,1}(x))^{\frac{1}{2}}]^3.$$

By observing that $G \cap H_u \subset A^+_{L,3/2}$ we get

$$R^+_{L,3/2}(x) \geqq [1-(1-R^+_{L,1}(x))^{\frac{1}{2}}]^3. \tag{2.7}$$

If we consider the rectangle $\Lambda'_{L,3/2} = \{i \in Z^2 \mid -2L \leqq i_1 \leqq L,\ |i_2| \leqq L\}$, it is easy to convince oneself that if there is in $\Lambda_{L,3/2}$ a $(+)$chain connecting left and right sides of $\Lambda_{L,3/2}$, there is in $\Lambda'_{L,3/2}$ a $(+)$chain connecting left and right sides of $\Lambda'_{L,3/2}$, and there is a $(+)$chain in $\Lambda_{L,1}$ connecting its upper side with its lower side, then the event $A^+_{L,2}$ occurs. By using Harris' inequality we get

$$R^+_{L,2}(x) \geqq R^+_{L,1}(x)\, [R^+_{L,3/2}(x)]^2. \tag{2.8}$$

(2.7) and (2.8) imply (2.2); in the same way one can get (2.3) from (2.2). The proof of (2.2a) and (2.3a) is analogous.

3. Continuity of the Percolation Probability

In this section we prove the following proposition.

Proposition 1. *$P^+_\infty(x)$ and $P^{-*}_\infty(x)$ are continuous functions.*

The proof of Proposition 1 is based on the following lemma.

Lemma 2. *If for some L $R^+_{L,3}(x) > 1-5^{-4}$ $[R^{-*}_{L,3}(x) > 1-5^{-4}]$, then $P^+_\infty(x) > 0$ $[P^{-*}_\infty(x) > 0]$.*

Proof. We suppose $R^+_{L,3}(x) > 1-5^{-4}$. The proof works in the same way under the hypothesis $R^{-*}_{L,3}(x) > 1-5^{-4}$.

Besides $\Lambda_{L,3}$ we consider the other rectangle

$$\Lambda'_{L,3} = \{i \in Z^2 \mid |i_1| \leqq L,\ |i_2| \leqq 3L\},$$

and, for any $i \in Z^2$, we define $\Lambda_{L,3}^{(i)}$ by putting

$$\text{if } i_1 - i_2 \text{ is even} \quad \Lambda_{L,3}^{(i)} = T_2^{2L(i_1+i_2)} T_1^{2L(i_1-i_2)} \Lambda_{L,3}$$
$$\text{if } i_1 - i_2 \text{ is odd} \quad \Lambda_{L,3}^{(i)} = T_2^{2L(i_1+i_2)} T_1^{2L(i_1-i_2)} \Lambda'_{L,3}$$

where T_1, T_2 are the one-step translations along the two axes of Z^2. Furthermore we call z_i the characteristic function of the event

$$A_{L,3}^{(i)+} = \{\text{there exists in } \Lambda_{L,3}^{(i)} \text{ a } (+)\text{chain connecting its opposite smaller sides}\}.$$

The hypothesis of the lemma and the symmetry properties of the measure μ_x imply:

$$\forall i \in Z^2 \qquad \delta \equiv 1 - \mu_x(A_{L,3}^{(i)+}) < 5^{-4}. \tag{3.1}$$

Remark 1. If $i, j \in Z^2$ are adjacent and $z_i(\omega) = z_j(\omega) = 1$, the $(+)$chains connecting the opposite smaller sides of $\Lambda_{L,3}^{(i)}$ and $\Lambda_{L,3}^{(j)}$ belong to the same $(+)$cluster. Hence if $s = (i_1, \ldots, i_n)$ is a chain in Z^2 and for any $i \in s$ $z_i(\omega) = 1$, there is in ω a $(+)$chain starting in $\Lambda_{L,3}^{(i_1)}$ and ending in $\Lambda_{L,3}^{(i_n)}$.

Remark 2. If $i, j \in Z^2$ are not *adjacent, since $\Lambda_{L,3}^{(i)} \cap \Lambda_{L,3}^{(j)} = \emptyset$, z_i and z_j are independent. Hence we can divide Z^2 in four distinct sublattices such that the z's associated to each sublattice form a set of mutually independent random variables.

Now we apply the Peierls' argument to the variables z's. If l is a *circuit in Z^2 surrounding the origin, we put

$$B_l = \{\omega \in \Omega \mid \forall i \in l, \; z_i(\omega) = 0\}.$$

Since at least $|l|/4$ elements of l belong to the same sublattice, (3.1) and Remark 2 imply

$$\mu_x(B_l) < \delta^{|l|/4} \tag{3.2}$$

where $|l|$ is the number of elements of l. Furthermore it is easy to check that the number of self-avoiding *circuits in Z^2 surroinding the origin and of length k is less than $k^2 5^k$ (note that, since each point has eight *neighbors, each given open *chain can be prolonged in seven different ways, but at most five of them give rise to a still self-avoiding *chain). Hence for any integer k $\mu_x(\bigcup_{l:\,|l|=k} B_l) \leq k^2 5^k \delta^{k/4}$. Using (3.1) and Borel-Cantelli lemma we get that μ_x-a.s. only a finite number of B_l occur. Hence μ_x-a.s. there is in Z^2 an infinite connected subset C such that $\forall i \in C$ $z_i = 1$. Then Remark 1 implies that μ_x-a.s. there is in Z^2 an infinite $(+)$cluster.

Proof of Proposition 1. Lemma 2 and the definition of p_c imply that for any L and for any $x < p_c$ $R_{L,3}^+(x) \leq 1 - 5^{-4}$. For any L, since $R_{L,3}^+(x)$ is a continuous function of x (namely a polynomial), in the limit $x \to p_c - 0$ we get $R_{L,3}^+(p_c) \leq 1 - 5^{-4}$. On the other hand $P_\infty^+(p_c) > 0$ implies $\lim_{L\to\infty} R_{L,3}^+(p_c) = 1$ (see [4], Lemma 4). Hence we get $P_\infty^+(p_c) = 0$. The last equality means that $P_\infty^+(x)$ is left-continuous in the point $x=p_c$. Since it is known [4] that $P_\infty^+(x)$ is a right-continuous function and that it is continuous in $[0,1]\setminus\{p_c\}$, $P_\infty^+(x)$ is a continuous function. The continuity of $P_\infty^{-*}(x)$ can be proved in the same way.

We remark that Proposition 1 and (2.1) imply that the set of x such that $P_\infty^+(x)=P_\infty^{-*}(x)=0$ is the non-empty closed interval $[1-p_c^*, p_c]$.

4. A Remark on the Probability of Positive Events

In this section we prove a simple equality which will be useful in the next section.

Let Λ be a finite set and let $\Omega_\Lambda=\{-1,1\}^\Lambda$. For any $i\in\Lambda$ we define S_i: $\Omega_\Lambda\to\Omega_\Lambda$ by putting

$$\forall k\neq i \quad (S_i\omega)(k)=\omega(k); \quad (S_i\omega)(i)=-\omega(i).$$

If $i\in\Lambda$, $A\subset\Omega_\Lambda$, the event $\delta_i A$ is defined by:

$$\delta_i A=\{\omega\in A\,|\,S_i\omega\notin A\}\cup\{\omega\notin A\,|\,S_i\omega\in A\}.$$

If $\omega\in\delta_i A$ we call i a critical point of the configuration ω for the event A. The number of critical points of the configuration ω for the event A is, of course:

$$n(A)(\omega)=\sum_{i\in\Lambda}\chi_{\delta_i A}(\omega), \tag{4.1}$$

where we have used the symbol χ_E for the characteristic function of the event E. $\langle n(A)\rangle_{\mu_x}$ denotes the expectation value, for the measure μ_x, of $n(A)$; if A is a positive event it has the simple meaning given by the following lemma.

Lemma 3. *If A is a positive event*

$$\frac{d}{dx}\mu_x(A)=\langle n(A)\rangle_{\mu_x}. \tag{4.2}$$

Proof. If $\mathbf{x}=(x_1,\ldots,x_{|\Lambda|})$ we put $\mu_{\mathbf{x}}=\prod_{i\in\Lambda}\nu_{x_i}$. For any event $A\subset\Omega_\Lambda$ we have

$$\mu_{\mathbf{x}}(A)=\mu_{\mathbf{x}}(A\cap\delta_i A)+\mu_{\mathbf{x}}(A\setminus\delta_i A);$$

since $A\setminus\delta_i A\in\mathscr{B}_{\Lambda\setminus\{i\}}$, we have

$$\frac{\partial}{\partial x_i}\mu_{\mathbf{x}}(A)=\frac{\partial}{\partial x_i}\mu_{\mathbf{x}}(A\cap\delta_i A).$$

If A is positive, then

$$A\cap\delta_i A=E_i^+\cap\delta_i A; \quad \mu_{\mathbf{x}}(A\cap\delta_i A)=x_i\,\mu_{\mathbf{x}}(\delta_i A);$$

furthermore $\delta_i A\in\mathscr{B}_{\Lambda\setminus\{i\}}$. Hence we get

$$\frac{\partial}{\partial x_i}\mu_{\mathbf{x}}(A)=\mu_{\mathbf{x}}(\delta_i A).$$

If, in particular, we consider the vector $\mathbf{x}=(x,x,\ldots,x)$, and sum the last relation over i, we obtain (4.2).

5. Proof of the Relation $p_c + p_c^* = 1$

We consider the event

$D_k^L = \{$there are in $\Lambda_{2L,1}$ at least k disjoint $(-*)$circuits surrounding the origin$\}$.

The following lemma is a transcription, in the language introduced in the preceding section, of the main idea of the proof of Ref. 2. For convenience of the reader we insert here a short proof of it. The reader interested in further details is referred to [2].

Lemma 4. $\forall x \in [0, 1]$, $\forall L$

$$\mu_x(\omega \in \Omega \,|\, n(A_{2L,1}^+)(\omega) \geqq k) \geqq R_{L,2}^+(x)\, R_{L,2}^{-*}(x)\, \mu_x(D_k^L).$$

Proof. We consider the following events:

$E_s = \{s$ is the lowest $(+)$chain in $S(\Lambda_{2L,1})\}$;

$N_{s,t} = \{t$ is the left-most $(-*)$chain contained in $\Lambda_{2L,1} \cap A(s)$ starting in the upper side of $\Lambda_{2L,1}$ and ending in a point *adjacent to some point in $s\}$;

$L_{s,t} = E_s \cap N_{s,t}$;

$M_{s,t}^k = \{$at least k different points of s are *adjacent to the ending point of a $(-*)$chain contained in $R(s,t)$ and starting in a point *adjacent to some point in $t\}$;

$Q_{s,t}^k = L_{s,t} \cap M_{s,t}^k$;

where $R(s,t)$ is the set of points in $\Lambda_{2L,1}$ which are "above s" and "on the right of t".

Furthermore we put

$$\Lambda_1 = \Lambda_{2L,1} \cap \{i \in Z^2 \,|\, i_2 \leqq 0\}, \qquad \Lambda_2 = \Lambda_{2L,1} \cap \{i \in Z^2 \,|\, i_1 \leqq 0\}, \qquad Q^k = \bigcup_{\substack{s \subset \Lambda_1 \\ t \subset \Lambda_2}} Q_{s,t}^k.$$

It is easy to check that if $\omega \in Q^k$, then $n(A_{2L,1}^+)(\omega) \geqq k$; on the other hand we have

$$\begin{aligned}
\mu_x(Q^k) &= \sum_{\substack{s \subset \Lambda_1 \\ t \subset \Lambda_2}} \mu_x(Q_{s,t}^k) = \sum_{\substack{s \subset \Lambda_1 \\ t \subset \Lambda_2}} \mu_x(M_{s,t}^k \,|\, L_{s,t})\, \mu_x(L_{s,t}) \\
&= \sum_{\substack{s \subset \Lambda_1 \\ t \subset \Lambda_2}} \mu_x(M_{s,t}^k)\, \mu_x(L_{s,t}) \geqq \sum_{\substack{s \subset \Lambda_1 \\ t \subset \Lambda_2}} \mu_x(D_k^L)\, \mu_x(L_{s,t}) \\
&= \mu_x(D_k^L) \sum_{s \subset \Lambda_1} \sum_{t \subset \Lambda_2} \mu_x(N_{s,t} \,|\, E_s)\, \mu_x(E_s) \geqq \mu_x(D_k^L) \sum_{s \subset \Lambda_1} \sum_{t \subset \Lambda_2} \mu_x(N_{s,t})\, \mu_x(E_s) \\
&\geqq \mu_x(D_k^L) \sum_{s \subset \Lambda_1} \mu_x(E_s)\, R_{L,2}^{-*}(x) \geqq R_{L,2}^+(x)\, R_{L,2}^{-*}(x)\, \mu_x(D_k^L).
\end{aligned}$$

Lemma 5. *There exists $\alpha > 0$ such that, $\forall L$, $\forall x \in [1 - p_c^*, p_c]$*

$$R_{L,2}^+(x) \geqq \alpha, \qquad R_{L,2}^{-*}(x) \geqq \alpha.$$

Proof. If $x \in [1-p_c^*, p_c]$, by proposition 1, $P_\infty^+(x) = P_\infty^{-*}(x) = 0$. Hence, by lemma 2, $R_{L,3}^+(x) \leqq 1 - 5^{-4}$, $R_{L,3}^{-*}(x) \leqq 1 - 5^{-4}$; by using lemma 1 we get $R_{L,1}^+(x) \leqq \beta$, $R_{L,1}^{-*}(x) \leqq \beta$, where β is the root in $[0,1]$ of the equation $x^3[1-(1-x)^{\frac{1}{2}}]^{12} = 1 - 5^{-4}$. Since $R_{L,1}^+ + R_{L,1}^{-*}(x) = 1$, we have $R_{L,1}^+(x) \geqq 1 - \beta$, $R_{L,1}^{-*}(x) \geqq 1 - \beta$. If we choose $\alpha = (1-\beta)(1-\beta^{\frac{1}{2}})^6$, the statement of the lemma easily follows.

Lemma 6. *For any positive integer k, there exists $L_0(k)$ such that if $x \in [1-p_c^*, p_c]$, $L \geqq L_0(k)$, then $\mu_x(D_k^L) \geqq 1/2$.*

Proof. Since D_k^L is a negative event, for any $x \in [1-p_c^*, p_c]$, we have $\mu_x(D_k^L) \geqq \mu_{p_c}(D_k^L)$; on the other hand since, by proposition 1, $P_\infty^+(p_c) = 0$, μ_{p_c}-a.s. there are infinitely many disjoint $(-*)$circuits surroinding the origin; hence, for any k, $\lim_{L\to\infty} \mu_{p_c}(D_k^L) = 1$.

Now we can easily prove our main result.

Theorem 1. $p_c + p_c^* = 1$.

Proof. Suppose $p_c > 1 - p_c^*$; then we can choose an integer $\bar{k}$ such that

$$\bar{k} > 2[(p_c + p_c^* - 1)\alpha^2]^{-1} \tag{5.1}$$

(where α is the number defined in the proof of Lemma 5). Furthermore we choose an integer $L > L_0(\bar{k})$ (where $L_0(k)$ is the function defined in Lemma 6). Then Lemmas 4, 5, 6 imply:

$$\forall x \in [1-p_c^*, p_c] \qquad \mu_x(\omega \in \Omega \,|\, n(A_{2L,1}^+)(\omega) \geqq \bar{k}) > \alpha^2/2.$$

Hence (5.1) implies:

$$\forall x \in [1-p_c^*, p_c] \qquad \langle n(A_{2L,1}^+)\rangle_{\mu_x} > \bar{k}\alpha^2/2 > (p_c + p_c^* - 1)^{-1}.$$

By using Lemma 3 we get

$$\mu_{p_c}(A_{2L,1}^+) - \mu_{1-p_c^*}(A_{2L,1}^+) > (p_c + p_c^* - 1) \min_{x \in [1-p_c^*, p_c]} \frac{d}{dx} \mu_x(A_{2L,1}^+) > 1.$$

The last inequality gives a contradiction, since $\forall x \in [0,1]$ $\mu_x(A_{2L,1}^+) \in [0,1]$. This proves Theorem 1.

We call $S^+(x)$ $[S^{-*}(x)]$ the mean size, with respect to the measure μ_x, of the finite $(+)$clusters $[(-*)$clusters$]$. Theorem 1, together with results of Refs. 4 and 5 implies the following theorem

Theorem 2. *There exists $p_c \in (0,1)$ such that:*

a) *If* $x < p_c$ $\quad P_\infty^+(x) = 0$, $P_\infty^{-*}(x) > 0$, $S^+(x) < \infty$, $S^{-*}(x) < \infty$,
$\lim_{L\to\infty} R_{L,1}^+(x) = 0$, $\quad \lim_{L\to\infty} R_{L,1}^{-*}(x) = 1$.

b) *If* $x = p_c$ $\quad P_\infty^+(x) = P_\infty^{-*}(x) = 0$; $\quad S^+(x) = S^{-*}(x) = \infty$.

c) *If* $x > p_c$ $\quad P_\infty^+(x) > 0$, $P_\infty^{-*}(x) = 0$, $S^+(x) < \infty$, $S^{-*}(x) < \infty$,
$\lim_{L\to\infty} R_{L,1}^+(x) = 1$, $\quad \lim_{L\to\infty} R_{L,1}^{-*}(x) = 0$.

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