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A Unification of Strassen's Law and Lévy's Modulus of Continuity*

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1. Introduction

Let $B(t)$ be 1-dimensional Brownian motion. The law of the iterated logarithm asserts that

$$\limsup_{t\uparrow\infty} \frac{|B(t)|}{(2t\log\log\ )^{1/2}}=1.$$

This result and many similar theorems are consequences of Strassen's version of the law of the iterated logarithm [6]. He shows that if

$$\mathscr{C}(t)=\left\{f(x)=\frac{B(xt)}{(2t\log\log t)^{1/2}}\right\}$$

where $x\in[0,1]$, then in the uniform topology, the set of limit points of $\mathscr{C}(t)$ as $t\uparrow\infty$ is the set K of functions f absolutely continuous on $[0,1]$ which satisfy

$$f(0)=0$$
$$\int_0^1 (f'(x))^2\,dx\leqq 1.$$

More precisely, $f\in K$ iff we can find functions $g_{t(n)}(x)\in\mathscr{C}(t(n))$ with $t(n)\uparrow\infty$ such that $g_{t(n)}$ converges to f. This implies that if Φ is a continuous functional on $\mathscr{C}[0,1]$, then

$$\limsup_{t\uparrow\infty}\Phi\left(\frac{B(xt)}{(2t\log\log t)^{1/2}}\right)=\sup_{f\in K}\Phi(f).$$

Setting $\Phi(f)=|f(1)|$, we obtain the classical law of the iterated logarithm.

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Lévy's modulus of continuity for Brownian motion asserts that

$$\limsup_{t\downarrow 0} \sup_{0\leqq s\leqq 1-t} \frac{|B(s+t)-B(s)|}{(2t\log 1/t)^{1/2}}=1.$$

This theorem is closely related to the law of the iterated logarithm. While the law of the iterated logarithm involves the time intervals $[0,t]$, Lévy's modulus involves the intervals $[s,s+t]$ with $0\leqq s\leqq 1-t$. Motivated by this similarity, we prove the following Strassen-type law for Lévy's modulus. If

$$\mathscr{C}(t)=\left\{f(x)=\frac{B(s+xt)-B(s)}{(2t\log 1/t)^{1/2}}: 0\leqq s\leqq 1-t\right\}$$

then, in the uniform topology, the set of limit points of $\mathscr{C}(t)$ as $t\uparrow\infty$ is Strassen's limit class K. This result implies Lévy's modulus in the same way that Strassen's law implies the law of the iterated logarithm.

We prove a stronger theorem which includes both Strassen's law and the previous result as special cases. Here is an important corollary, in which we consider the time intervals $[s,s+t]$ with $0\leqq s\leqq R(t)$. Let

$$\mathscr{C}(t)=\left\{f(x)=\frac{B(s+xt)-B(s)}{(2th(t))^{1/2}}: 0\leqq s\leqq R(t)\right\}.$$

If $R(t)$ and $h(t)$ satisfy monotonicity conditions, and if $h(t)$ satisfies an integral test, then, in the uniform topology, the set of limit points of $\mathscr{C}(t)$ as $t\uparrow\infty$ is Strassen's limit class K.

Setting $R(t)=0$ yields a form of Strassen's law for decreasing intervals. The result for Lévy's modulus is obtained by setting $R(t)=1-t$.

Using the preceding ideas, we give a test for upper and lower functions which includes both Kolmogorov's test and the Chung-Erdös-Sirao test [3] as special cases. If $R(t)$ and $\psi(t)$ satisfy certain monotonicity and growth conditions, then

$$P\left\{\sup_{0\leqq s\leqq R(t)}\left|B\left(s+\frac{1}{t}\right)-B(s)\right|<\frac{\psi(t)}{t^{1/2}},\ t\uparrow\infty\right\}=1 \quad \text{or} \quad 0$$

according as

$$\int^{\infty}\left(R(t)\,\psi^3(t)+\frac{\psi(t)}{t}\right)\exp\left(-\tfrac{1}{2}\psi^2(t)\right)dt$$

converges or diverges.

Finally, using the Komlos-Major-Tusnady theorem [5], we generalize Strassen's invariance principle. Let $X_1, X_2, \ldots$ be i.i.d. random variables with mean 0 and variance σ^2. Suppose that $|EX_1|^{2\alpha}<\infty$, $\alpha\geqq 1$. Let $Z(n)=X_1+\ldots+X_n$, and define $Z(t)$ for nonintegral t by linear interpolation. Let

$$h(t)=\begin{cases}(\alpha-1)\log t & \text{if } \alpha>1\\ \log\log t & \text{if } \alpha=1\end{cases}$$

If

$$\mathscr{C}(t)=\left\{f(x)=\frac{Z(s+xt)-Z(s)}{\sigma(2th(t))^{1/2}}: s\leqq t^{\alpha}\right\}$$

then, in the uniform topology, the set of limit points of $\mathscr{C}(t)$ as $t\uparrow\infty$ is K.

In Sect. 2, we prove the generalization of Strassen's law. Section 3 gives several important examples, including the recent work of Chan, Csörgö, and Révész [1]. Section 4 contains the generalization of Kolmogorov's test and the Chung-Erdös-Sirao test, and Sect. 5 proves the invariance principle.

These ideas will be extended to Gaussian processes in a future paper.

2. A Generalization of Strassen's Law

In this section we generalize Strassen's law. The proof uses many ideas due to Strassen [6], which are clearly explained in Freedman's book [4]. Several important applications are given in Sect. 3.

Notation

We will represent intervals as follows. With the interval $[s, s+l]$ associate the point $p=(s, l)$. Let $\mathscr{I}$ and $\mathscr{I}(t)$ be arbitrary index sets.

With each set of points $\mathscr{P}=\{p_i\}_{i\in\mathscr{I}}$ associate the following area. First, surround each point $p_i=(s_i, l_i)$ with the rectangle

$$R_r(p_i)=\{(s, l):\ e^{-r}\leqq l/l_i\leqq e^r,\ |s-s_i|\leqq l_i r\}.$$

Next, let $A_r(\mathscr{P})$ be the area of the union of these rectangles under the measure $\frac{ds\,dl}{l^2}$. That is,

$$A_r(\mathscr{P})=\int_{\bigcup_{i\in\mathscr{I}} R_i}\frac{ds\,dl}{l^2}.$$

The law of the iterated logarithm states that

$$\limsup_{t\uparrow\infty}\frac{|B(t)-B(0)|}{(2t\log\log t)^{1/2}}=1.$$

For each t, Brownian motion over the time interval $[0, t]$ is considered. Suppose that instead, several intervals are considered for each t. Let $\mathscr{P}_t=\{(s_i, l_i)\}_{i\in\mathscr{I}(t)}$ be the set of points representing these intervals. Again, surround each point p_i with the rectangle $R_r(s_i, l_i)$. Let $A_r(t)$ be the measure of the union of the rectangles up to time t. That is,

$$A_r(t)=\int_{\bigcup_{u\leqq t}\bigcup_{p\in\mathscr{P}_u} R_r(p)}\frac{du\,dl}{l^2}.$$

We now generalize Strassen's law.

Theorem 1. *Suppose that*

(1) $h(t)$ *is increasing and* $\lim_{t\to\infty} h(t)=\infty$,

(2) $\int_0^\infty \exp(-a h(t))\, dA_1(t)$ *converges for* $a>1$, *and diverges for* $a<1$.

Let

$$\mathscr{C}(t)=\left\{f(x)=\frac{B(s+xl)-B(s)}{(2lh(t))^{1/2}}:\ (s,l)\in\mathscr{P}_t\right\}$$

(*where* $x\in[0,1]$). *Then, in the uniform topology, the set of limit points of* $\mathscr{C}(t)$ *as* $t\uparrow\infty$ *is the set of functions* $g(x)$ *absolutely continuous on* $[0,1]$ *satisfying* $g(0)=0$ *and*

$$\int_0^1 (g'(x))^2\, dx \leqq 1.$$

Proof. Let K be the above set of functions, and let $\|\ \|$ denote the sup norm. Let

$$f_{s,l}(x)=\frac{B(s+xl)-B(s)}{\sqrt{2l}}.$$

First, we will show that for almost all ω, $\mathscr{C}(t)$ approaches K. In other words, if K_ε is the set of functions which have distance $<\varepsilon$ from K,

$$P\{\mathscr{C}(t)\nsubseteq K_\varepsilon \text{ i.o.}\}=0.$$

This claim will be established through a series of lemmas.

Lemma 1. *If* $0<\varepsilon<1$, *then*

$$P\{\sup_{\substack{1\leqq a\leqq 2\\ -\varepsilon<\Delta<\varepsilon}} |B(a+\Delta)-B(a)|>L\}\leqq \frac{6}{\sqrt{\pi}}\,\varepsilon^{-3/2}\exp(-L^2/16\varepsilon).$$

Proof. Break up $[0,1]$ into intervals of length $\frac{1}{N}$, where $\frac{1}{N}\leqq\varepsilon\leqq\frac{1}{N-1}$. Then,

$$\begin{aligned}
&P\{\sup_{\substack{1\leqq a\leqq 2\\ -\varepsilon\leqq\Delta\leqq\varepsilon}} |B(a+\Delta)-B(a)|>L\}\\
&\leqq P\left\{\sup_{\substack{0\leqq n\leqq N\\ -\varepsilon\leqq\Delta\leqq\varepsilon}}\left|B\left(1+\frac{n}{N}+\Delta\right)-B\left(1+\frac{n}{N}\right)\right|>\frac{L}{2}\right\}\\
&\leqq (N+1)\,P\left\{\sup_{0\leqq\Delta\leqq 2\varepsilon}|B(\Delta)|>\frac{L}{2}\right\}\\
&\leqq (N+1)\,4\,\frac{1}{\sqrt{2\pi\, 2\varepsilon}}\exp\left(-\frac{1}{2}\,\frac{(L/2)^2}{2\varepsilon}\right)\\
&\leqq \frac{6}{\sqrt{\pi}}\,\varepsilon^{-3/2}\exp\left(-\frac{L^2}{16\varepsilon}\right).
\end{aligned}$$

Lemma 2. *Fix r. Let $\varepsilon=\max(r, e^{2r}-1)$. Then, for large M, the following estimate holds uniformly for all rectangles $R_r(p)$*

$$P\{\sup_{\substack{(s',l')\in R_r(p)\\(s'',l'')\in R_r(p)}}\|f_{s',l'}-f_{s'',l''}\|>M\}\leqq C\varepsilon^{-3/2}\exp(-M^2/2^{10}\varepsilon^2).$$

Proof. Let (s', l') and (s'', l'') be in $R_r(p)$. Let $p=(s, l)$. Then

$$\begin{aligned}\|f_{s',l'}-f_{s'',l''}\|\leqq&\sup_{\substack{(s',l')\\(s'',l'')}}\frac{|B(s')-B(s'')|}{\sqrt{2l'}}\\&+\sup_{\substack{(s',l')\\(s'',l'')}}\frac{\sup_{x\in[0,1]}|B(s'+xl')-B(s''+xl'')|}{\sqrt{2l'}}\\&+\sup_{\substack{(s',l')\\(s'',l'')}}\left|\frac{1}{\sqrt{2l'}}-\frac{1}{\sqrt{2l''}}\right|(\sup_{x\in[0,1]}|B(s''+xl'')-B(s'')|)\\=&I+II+III.\end{aligned}$$

Note that $I\leqq II$. Thus,

$$P\{\sup_{\substack{(s',l')\in R_r(p)\\(s'',l'')\in R_r(p)}}\|f_{s',l'}-f_{s'',l''}\|>M\}\leqq P\left\{2(II)>\frac{M}{2}\right\}+P\left\{III>\frac{M}{2}\right\}.$$

First,

$$P\left\{III>\frac{M}{2}\right\}=P\left\{\sup_{\substack{x\in[0,1]\\(s',l')\\(s'',l'')}}|B(x)|\cdot|\sqrt{l''/l'}-1|>\frac{M}{\sqrt{2}}\right\}.$$

By assumption $|\sqrt{l''/l'}-1|\leqq e^r-1\leqq\varepsilon$. So,

$$P\left\{III>\frac{M}{2}\right\}\leqq 4\frac{1}{\sqrt{2\pi}}\frac{\sqrt{2}\varepsilon}{M}\exp\left(-\frac{M^2}{4\varepsilon^2}\right).$$

Next, we evaluate $P\left\{2(II)>\frac{M}{2}\right\}$. Note that if $x\in[0,1]$,

$$\begin{aligned}\frac{|(s'+xl')-(s''-xl'')|}{l'}&\leqq 2r\frac{l}{l'}+\left|1-\frac{l''}{l'}\right|\\&\leqq 2\varepsilon(1+\varepsilon)+e^{2r}\\&\leqq 4\varepsilon.\end{aligned}$$

Thus, using Lemma 1,

$$P\left\{2(II)>\frac{M}{2}\right\}\leqq P\left\{\sup_{\substack{1\leqq a\leqq 2\\-4\varepsilon<\Delta<4\varepsilon}}|B(a+\Delta)-B(a)|>\frac{M}{4}\right\}\leqq C\varepsilon^{-3/2}\exp\left(-\frac{M^2}{2^{10}\varepsilon}\right).$$

Combining these two estimates, we get Lemma 2.

Now we will show that $\mathscr{C}(t)$ approaches K. In the quarterplane $\{(s, l)\colon s \geqq 0, l>0\}$ we construct a grid of rectangles $R_{n,m}$ $(n, m \in \mathbb{Z}, n \geqq 0)$. Let $R_{n,m}$ be the rectangle with the following boundaries:

$$e^{mr} \leqq l \leqq e^{(m+1)r}$$
$$n r e^{mr} \leqq s \leqq (n+1) e^{mr}.$$

We will write $f(x) \approx g(x)$ if there exist constants C_1 and C_2 such that for all x,

$$C_1 f(x) \leqq g(x) \leqq C_2 f(x).$$

It is easy to check that if

$$d\mu = \frac{ds\, dl}{l^2}$$

then, for $r<1$,

$$\mu(R_{m,n}) \approx \frac{r e^{mr}(e^{(m+1)r} - e^{mr})}{(e^{mr})^2}$$
$$= r(e^r - 1)$$
$$\approx r^2.$$

Intuitively, we wish to choose times $t_1 \leqq t_2 \leqq \dots$ such that the points $\mathscr{P}(t)$ first enter a new rectangle R_{m_k, n_k} $(k>0)$ of the grid at time t_k. Formally, we will choose $\{t_k\}$ by induction. Note that since the integral

$$\int_0^\infty \exp(-a h(t))\, dA_1(t)$$

converges, only finitely many rectangles have been entered up to any given time t. Let $t_0 = 0$. Let R_{m_0, n_0} be any rectangle. Suppose we have chosen $t_0 \leqq t_1 \leqq \dots \leqq t_k$ and $R_{m_1, n_1}, \dots, R_{m_k, n_k}$. Let

$$t_{k+1} = \inf\{t \geqq t_k\colon \mathscr{P}(t) \not\subset R_{m_1, n_1} \cup \dots \cup R_{m_k, n_k}\}.$$

Choose $R_{m_{k+1}, n_{k+1}}$ such that for any $\varepsilon>0$, we can find t for which $t_{k+1} < t < t_{k+1} + \varepsilon$ and

$$\mathscr{P}(t) \cap R_{m_{k+1}, n_{k+1}} \neq \phi.$$

This is possible because for any $T>0$, $\bigcup_{t \leqq T} \mathscr{P}(t)$ is contained in a finite union of rectangles in $\{R_{m,n}\}$.

Next, let $A'_r(t)$ be the measure of the rectangles in $\{R_{m,n}\}$ entered up to time t. For $r<\frac{1}{2}$, $(s, l) \in R_{m,n}$ implies that

$$R_{m,n} \subset R_1(s, l).$$

Therefore

$$A'_r(t) \leqq A_1(t).$$

If $a>1$, then since $e^{-ah(t)}$ is decreasing, condition (2) of Theorem 1 implies that

$$\sum_{k=1}^{\infty} \exp(-ah(t_k)) \approx \frac{1}{r^2} \int_0^{\infty} \exp(-ah(t))\, dA'_r(t)$$

converges.

Let (s_k, l_k) be the center of R_{m_k, n_k}. Fix m. For $n=0, \ldots, m$ let

$$f_{(s,l)}^{(m)}\left(\frac{n}{m}\right) = f_{(s,l)}\left(\frac{n}{m}\right).$$

Define $f_{(s,l)}^{(m)}$ by linear interpolation between these points.

To finish the first part of the proof, it suffices to show that for all $\varepsilon>0$,

I. $P\left\{\frac{f_{(s_k,l_k)}^{(m)}}{(h(t_k))^{1/2}} \notin K_\varepsilon \text{ for infinitely many } k\right\}=0$

II. For $\varepsilon^2 m>4$,

$$P\left\{\frac{\|f_{(s_k,l_k)}^{(m)} - f_{(s_k,l_k)}\|}{|(h(t_k))^{1/2}|} > \varepsilon \text{ for infinitely many } k\right\}=0$$

III. Let $\delta=\max(r, e^{2r}-1)$. If $\frac{\varepsilon^2}{\delta^2}>2^9$

$$P\left\{\sup_{(s,l)\in R_{m_k,n_k}} \frac{\|f_{(s_k,l_k)} - f_{(s,l)}\|}{(h(t_k))^{1/2}} > \varepsilon \text{ for infinitely many } k\right\}=0$$

These statements will be proved using the Borel-Cantelli lemmas.

Proof of I. If $A(k,\delta)$ is the event that

$$\sum_{n=1}^{m} m \left(\frac{f_{(s_k,l_k)}^{(m)}\left(\frac{n-1}{m}\right) - f_{(s_k,l_k)}^{(m)}\left(\frac{n}{m}\right)}{(h(t_k))^{1/2}}\right)^2 > 1+\delta$$

it suffices to show that for all $\delta>0$, $A(k,\delta)$ occurs only finitely often. Now by standard probability estimates,

$$\begin{aligned} P(A(k,\delta)) &= P(\chi_m^2 > 2h(t_k)(1+\delta)) \\ &\leqq \exp(-(1+2\delta)^2 h(t_k)) \end{aligned}$$

for $h(t_k)$ sufficiently large. This converges when summed over k, so by the Borel-Cantelli lemma,

$$P\{A(k,\delta) \text{ for infinitely many } k\}=0.$$

Proof of II. Let $A(k,m,\varepsilon)$ be the event that

$$\sup_{\substack{0\leqq n\leqq m \\ -\frac{1}{m}<\Delta<\frac{1}{m}}} \frac{\left|f_{(s_k,l_k)}\left(\frac{n}{m}\right) - f_{(s_k,l_k)}\left(\frac{n}{m}+\Delta\right)\right|}{(h(t_k))^{1/2}} > \varepsilon.$$

It suffices to show that $A(k,m,\varepsilon)$ occurs for only finitely many k. But, by Lemma 1,

$$P(A(k,m,\varepsilon)) \leqq \frac{6}{\sqrt{\pi}} m^{3/2} \exp\left(-\frac{\varepsilon^2 m}{4} h(t_k)\right).$$

This converges when summed over k, since $\frac{\varepsilon^2 m}{4} > 1$. Therefore, by the Borel-Cantelli lemma,

$$P\{A(k,m,\varepsilon) \text{ for infinitely many } k\} = 0.$$

Proof of III. Let $A(k,\varepsilon)$ be the event that

$$\sup_{(s,l)\in R_{m_k,n_k}} \frac{\|f_{(s_k,l_k)} - f_{(s,l)}\|}{(h(t_k))^{1/2}} > \varepsilon.$$

By Lemma 2,

$$P\{A(k,\varepsilon)\} \leqq C\,\delta^{-3/2} \exp\left(-\frac{\varepsilon^2}{2^9\,\delta^2} h(t_k)\right).$$

This converges when summed over k, since $\frac{\varepsilon^2}{2^9\,\delta^2} > 1$. By the Borel-Cantelli lemmas,

$$P\{A(k,\varepsilon) \text{ for infinitely many } k\} = 0.$$

To complete the proof of Theorem 1, we need to show that for almost every path and for all $g\in K$, we can find a sequence $t'_k \nearrow \infty$ and points $(s'_k, l'_k)\in\mathscr{P}(t'_k)$ such that

$$\lim_{n\to\infty} \left\| \frac{f_{(s'_k,l'_k)}}{(h(t'_k))^{1/2}} - g \right\| = 0. \tag{$*$}$$

For each k, choose t'_k and $((s'_k, l'_k)$ such that $t_k \leqq t'_k < t_k + 1$, and

$$(s'_k, l'_k) \in \mathscr{P}(t'_k) \cap R_{m_k,n_k}.$$

By the assumptions of Theorem 1,

$$\frac{h(t'_k)}{h(t_k)} \to 1 \quad \text{as} \quad k \nearrow \infty.$$

This will be used in the reasoning which follows.

Fix m. Let $K(\varepsilon,m)$ be the set of functions for which $\int_0^1 (g'(x))^2\,dx \leqq 1-\varepsilon$ and which are linear in the intervals $\frac{n-1}{m} \leqq x \leqq \frac{n}{m}$.

Now $\bigcup_{m,\varepsilon} K(\varepsilon,m)$ is dense in K, so it suffices to show that $(*)$ holds for all $g\in K(\varepsilon,m)$. Fix $g\in K(\varepsilon,m)$. Let $A(K)$ be the event that

$$\left\| \frac{f^{(m)}_{(s'_k,l'_k)}}{(h(t_k))^{1/2}} - g \right\| < \frac{\varepsilon}{2}.$$

Strassen derived the following procedure. If $g\left(\frac{i}{m}\right)-g\left(\frac{i-1}{m}\right)=a_i$, choose $\delta_1, \dots, \delta_m$ such that

$$\delta_1+\dots+\delta_m<\frac{\varepsilon}{2}$$

and

$$m\sum_{i=1}^{m}(a_i\pm\delta_i)^2=\theta<1$$

where $(a\pm\delta)^2$ denotes $\max[(a+\delta)^2,(a-\delta)^2]$. Let F_i be the event that

$$a_i-\delta_i<\frac{f^{(m)}_{(s'_k,t'_k)}\left(\frac{i}{m}\right)-f^{(m)}_{(s'_k,t'_k)}\left(\frac{i-1}{m}\right)}{(h(t_k))^{1/2}}<a_i+\delta_i.$$

Let $E=\bigcap_{i=1}^{m}F_i$. Clearly $E\subseteq A(k)$. Also, by the independence of Brownian increments,

$$P\{E\}=\prod_{i=1}^{m}P\{F_i\}.$$

But, by standard estimates,

$$P\{F_i\}\geqq\exp(-m(a_i\pm\delta)^2h(t_k)).$$

Therefore,

$$P\{A(k)\}\geqq P\{E\}\geqq\exp(-\theta h(t_k)).$$

This diverges when summed over n. We wish to use the second Borel-Cantelli lemma, but the events $A(k)$ are not independent. Instead, we use the following theorem of Chung and Erdös [2]:

Theorem 2. *Let $\{A(k)\}$ be a sequence of events satisfying the following conditions.*

(i) $\sum_{k=1}^{\infty}P\{A(k)\}=\infty$

(ii) *For every pair of positive integers h, n with $n>h$, there exist $c(h)$ and $H(n,h)>n$ such that for every $m>H(n,h)$ we have*

$$P\{A(m)|A(h)^c,\dots,A(n)^c\}>c(h)P\{A(m)\}$$

where A^c denotes the complement of A.

(iii) *There exist two absolute constants c_1 and c_2 with the following property: to each $A(j)$ there corresponds a set of events $A(j_1),\dots,A(j_s)$ belonging to $\{A(k)\}$ such that*

(a) $\sum_{i=1}^{s}P\{A(j)A(j_i)\}\leqq c_1P\{A(j)\}$ *and if $k>j$ but $A(k)$ is not among the $A(j_i)$ $(1\leqq i\leqq s)$ then*

(b) $P\{A(j)A(k)\}\leqq c_2P\{A(j)\}P\{A(k)\}$.

Then $P\{A(k)$ i.o.$\}=1$.

In our case, (ii) is satisfied if r is large enough, and for $H(n,h)$ sufficiently large, since then the interval involved in $A(m)$ will either not intersect that of $A(k)$, $h \leqq k \leqq n$, or will be much smaller or much larger.

If r is large enough and if $\{A(j_1), \ldots, A(j_s)\}$ is the empty set then condition (iii) is also satisfied.

Therefore $P\{A(n) \text{ i.o.}\}=0$. It remains to show that if $A(n,m)$ is the event that

$$\left\| \frac{f^{(m)}_{(s_n,l_n)} - f_{(s_n,l_n)}}{(h(t_h))^{1/2}} \right\| < \frac{\varepsilon}{2}$$

then $P\{A(n,m)$ for infinitely many $n\}=0$ provided $\frac{m\varepsilon^2}{4}<1$. But this follows from assertion II of the earlier part of the proof.

3. Examples

This section gives several examples of Theorem 1. Only the first two examples will be proved, since all of the proofs are similar.

In the following, we use the uniform topology. Recall that K is the set of functions $g(x)$ absolutely continuous on $[0, 1]$ satisfying $g(0)=0$ and

$$\int_0^1 g'(t)^2\, dt \leqq 1.$$

Example 1. Strassen's law for Lévy's modulus. Let

$$\mathscr{C}(t)=\left\{ f(x)=\frac{B(s+xt)-B(s)}{(2t \log 1/t)^{1/2}} \colon 0 \leqq s \leqq 1-t \right\}.$$

Then the set of limit points of $\mathscr{C}(t)$ as $t \downarrow 0$ is the set K.

Proof. Set $u=1/t$. Then

$$\mathscr{P}(u)=\{(s,l) \colon l=1/u,\, 0 \leqq s \leqq 1-l\}.$$

Now

$$\bigcup_{1 \leqq v \leqq u} \mathscr{P}(v)=\{(s,l) \colon 1 \leqq l \leqq 1/u,\, 0 \leqq s \leqq 1-l\}.$$

It is easily checked that $dA_1(u)$ is comparable to

$$-dl \int_0^1 \frac{ds}{l^2}\bigg|_{l=1/u} = du.$$

If $h(u)=\log u$, then $h(u)$ satisfies the conditions of Theorem 1.

Example 2. Strassen's Law. Let

$$\mathscr{C}(t)=\left\{ f(x)=\frac{B(xt)}{(2t \log\log t)^{1/2}} \right\}.$$

Then the set of limit points of $\mathscr{C}(t)$ as $t \uparrow \infty$ is K.

Proof. $\mathscr{P}(t)=\{(0,t)\}$.

Let $S(t)$ be the region

$$\{(s,l)\colon 0\leqq l\leqq t,\ -l\leqq s\leqq l\}.$$

It is easily seen that $S(t)$ is close enough to the region $\bigcup_{s\leqq t} R_1((0,t))$ so that $dA_1(t)$ is comparable to

$$dl\int_{-l}^{l}\frac{ds}{l^2}\bigg|_{l=t}=2\frac{dt}{t}.$$

If $h(t)=\log\log t$, then $h(t)$ satisfies the conditions of Theorem 1.

Example 3. Intervals of Increasing Size. Suppose that $R(t)\uparrow t$ is increasing, $h(t)$ is increasing, and that

$$\int_0^\infty \frac{R(t)}{t^2}\exp(-ah(t))\,dt$$

diverges when $a<1$ and converges when $a>1$. If

$$\mathscr{C}(t)=\left\{f(x)=\frac{B(s+xt)-B(s)}{(2th(t))^{1/2}}\colon 0\leqq s\leqq R(t)-t\right\}$$

then the set of limit points of $\mathscr{C}(t)$ as $t\uparrow\infty$ is K.

Example 4. Intervals of Decreasing Size. Suppose that $R(t)$ is increasing or decreasing $h(t)$ is increasing, and that

$$\int_0^\infty \frac{R(t)}{t^2}\exp(-ah(t))\,dt$$

diverges when $a<1$ and converges when $a>1$. If

$$\mathscr{C}(t)=\left\{f(x)=\frac{B(s+xt)-B(s)}{(2th(1/t))^{1/2}}\colon 0\leqq s\leqq R(t)-\frac{1}{t}\right\}$$

then the set of limit points of $\mathscr{C}(t)$ as $t\downarrow 0$ is K.

Example 5. Moving Averages. This example was proved by Chan, Csörgö, and Revesz [1]. It is an easy consequence of Theorem 1. Let a_n, n/a_n be nondecreasing functions of n. For $0\leqq x\leqq 1$, let

$$\gamma_n(x)=B(n-a_n+xa_n)-B(n-a_n)$$
$$\beta_n=(2a_n(\log n/a_n+\log\log n))^{1/2}.$$

Then, as $n\uparrow\infty$ the set of limit points of $\{\gamma_n(x)/B_n\}$ is K.

Example 6. Let

$$f_t(x)=\frac{B(t+x)-B(t)}{(2\log t)^{1/2}}.$$

Then the set of limit points of $\{f_t(x)\}$ as $t\uparrow\infty$ is K.

4. Kolmogorov's Test and the Chung-Erdös-Sirao Test

We will not use the language of Sect. 1 here; it would make Theorem 3 painfully complicated.

Definition. Given a function $R(t)$, let

$$M(l,t)=\sup\{|B(s+l)-B(s)|: s\in[0,R(t)]\}.$$

Suppose $l=l(t)$ is a function of t. The law of the iterated logarithm states that if $l(t)=t$, $R(t)=0$, then

$$\limsup_{t\uparrow\infty}\frac{M(l,t)}{(2t\log\log t)^{1/2}}=1.$$

If $l(t)=\frac{1}{t}$, $R(t)=1-\frac{1}{t}$ Levy's modulus states that

$$\limsup_{t\uparrow\infty}\frac{M(l,t)}{(2/t\log t)^{1/2}}=1.$$

We say that $\psi(t)$ is in the upper class for $l(t)$ if for almost every path there exists a constant T such that $t>T$ implies

$$\frac{M(l,t)}{\sqrt{l}}\leqq\psi(t).$$

We say that $\psi(t)$ is in the lower class if for almost all paths such a constant does not exist.

Theorem 3. *Let $R(t)$ be increasing or decreasing. Suppose that for all t,*

$$C_1(R(t)+l(t))\leqq R(2t)+l(2t)\leqq C_2(R(t)+l(t))$$

where C_1 and C_2 do not depend on t. That is, $R(t)+l(t)$ is at most exponential. Suppose that $\psi(t)$ is continuous and nondecreasing. Then

(i) *If $l(t)=t$, and if the integrand below is decreasing, then $\psi(t)$ belongs to the upper or lower class according as the integral*

$$\int^{\infty}\left(\frac{R(t)\psi^3(t)}{t^2}+\frac{\psi(t)}{t}\right)\exp(-\tfrac{1}{2}\psi^2(t))\,dt$$

converges or diverges.

(ii) *If $l(t)=1/t$ and if the integrand below is decreasing, then $\psi(t)$ belongs to the upper or lower class according as the integral*

$$\int^{\infty}\left(R(t)\psi^3(t)+\frac{\psi(t)}{t}\right)\exp(-\tfrac{1}{2}\psi^2(t))\,dt$$

converges or diverges.

The two terms in the integral correspond to the Chung-Erdös-Sirao test and to Kolmogorov's test, respectively.

Corollary 1 (Kolmogorov's test). *If $R(t)=0$ and $l(t)=t$, then $\psi(t)$ is in the upper or lower class according as*

$$\int^{\infty} \frac{\psi(t)}{t} \exp(-\tfrac{1}{2}\psi^2(t))\,dt$$

converges or diverges.

Corollary 2. *Fix $\varepsilon>0$ and let $l(t)=t$. Kolmogorov's test still applies if $R(t)=c\,t^{1-\varepsilon}$.* This follows from part (i).

Corollary 3 (Chung, Erdös, Sirao [4]). *If $R(t)=1-\frac{1}{t}$, and $l(t)=\frac{1}{t}$, then $\psi(t)$ is in the upper or lower class according as*

$$\int^{\infty} \psi^3(t) \exp(-\tfrac{1}{2}\psi^2(t))\,dt$$

converges or diverges.

We need the following lemma.

Lemma 3. *We may assume without loss of generality that*

$$\tfrac{1}{2}(2h(t))^{1/2} \leqq \psi(t) \leqq 2(2h(t))^{1/2}$$

where $h(t)=\log\left[\left(\frac{R(t)}{l(t)}+1\right)\log t\right]$.

This is proved in the same way as Lemma 1 of Chung, Erdös, and Sirao [4].

Proof of Theorem 3. We will only prove case (ii), for simplicity. Case (i) is similar; instead of intervals of length $m/2^p$ we would consider intervals of length $m2^p$, $m\leqq h(2^p)$. Let

$$\tilde{M}(l,t)=\sup\{B(s+l)-B(s): s\in[0,R(t)]\}.$$

By the symmetry of Brownian motion, it is enough to show that for almost every path, there exists a constant T such that $t>T$ implies

$$\frac{\tilde{M}(l(t),t)}{l(t)^{1/2}} \leqq \psi(t).$$

Recall that $l(t)=\frac{1}{t}$ in case (ii).

Let $E(p,k,m)$ be the event that

$$\frac{B\left(\frac{k+m}{2^p}\right)-B\left(\frac{k}{2^p}\right)}{\left(\frac{m}{2^p}\right)^{1/2}} > \psi\left(\frac{2^p}{m}\right), \qquad \begin{array}{l} k=0,\ldots,\left[2^p\left(R\left(\frac{2^p}{m}\right)+\frac{m}{2^p}\right)\right] \\ m=1,\ldots,h(2^p). \end{array}$$

For large p,

$$P\{E(p,k,m)\} \sim \frac{\exp\left(-\frac{1}{2}\psi^2\left(\frac{2^p}{m}\right)\right)}{(2\pi)^{1/2}\,\psi\left(\frac{2^p}{m}\right)}.$$

Summing $P\{E(p,k,m)\}$ over $p=1,\ldots,m=\left[\frac{h(2^p)}{3}\right],\ldots,h(2^p)$, $k=1,\ldots,\left[2^p\left(R\left(\frac{2^p}{m}\right)+\frac{m}{2^p}\right)\right]$

$$\begin{aligned}
&\sum_p\sum_m\sum_k P\{E(p,k,m)\}\\
&=O(1)\sum_p\sum_m \frac{2^p\left(R\left(\frac{2^p}{m}\right)+\frac{m}{2^p}\right)}{\psi\left(\frac{2^p}{m}\right)}\exp\left(-\frac{1}{2}\psi^2\left(\frac{2^p}{m}\right)\right)\\
&=O(1)\sum_p \frac{h(2^p)\,2^p\left(R\left(\frac{2^p}{h(2^p)}\right)+\frac{h(2^p)}{2^p}\right)}{\psi\left(\frac{2^p}{h(2^p)}\right)}\exp\left(-\frac{1}{2}\psi^2\left(\frac{2^p}{h(2^p)}\right)\right)\\
&=O(1)\int^{\infty}\frac{\left(h^2(t)R(t)+\frac{h(t)}{t}\right)}{\psi(t)}\exp(-\tfrac{1}{2}\psi^2(t))\,dt\\
&=O(1)\int^{\infty}\left(R(t)\psi^3(t)+\frac{\psi(t)}{t}\right)\exp(-\tfrac{1}{2}\psi^2(t))\,dt.
\end{aligned}$$

The last step uses Lemma 3.

First, assume that the sum converges. Let $F(p,k,m)$ be the event that

$$\sup_{0\leqq u,w\leqq\frac{1}{2^p}}\frac{\left[B\left(\frac{k+m}{2^p}+u\right)-B\left(\frac{k}{2^p}-w\right)\right]}{\left(\frac{2^p}{m}\right)^{1/2}}\geqq\psi\left(\frac{2^p}{m+2}\right),$$

$$k=0,\ldots,\left[2^p\left(R\left(\frac{2^p}{m}\right)+\frac{m}{2^p}\right)\right]$$

$$m=1,\ldots,h(2^p).$$

Now, by standard estimates,

$$
\begin{aligned}
&P\{F(p,k,m)\}\\
&\leqq P\left\{\sup_{0\leqq u\leqq\frac{1}{2^p}}\left[B\left(\frac{k+m}{2^p}+u\right)-B\left(\frac{k+m}{2^p}\right)\right]\right.\\
&\quad+\left[B\left(\frac{k+m}{2^p}\right)-B\left(\frac{k}{2^p}\right)\right]+\sup_{0\leqq s\leqq\frac{1}{2^p}}\left[B\left(\frac{k}{2^p}\right)-B\left(\frac{k}{2^p}-s\right)\right]\\
&\quad\left.\geqq\left(\frac{m}{2^p}\right)^{1/2}\psi\left(\frac{2^p}{m+2}\right)\right\}\\
&\leqq 4p\left\{B\left(\frac{k+m+1}{2^p}\right)-B\left(\frac{k+m}{2^p}\right)+B\left(\frac{k+m}{2^p}\right)-B\left(\frac{k}{2^p}\right)+B\left(\frac{k}{2^p}\right)-B\left(\frac{k-1}{2^p}\right)\right.\\
&\quad\left.\geqq\left(\frac{m}{2^p}\right)^{1/2}\psi\left(\frac{2^p}{m+2}\right)\right\}\\
&=4P\left\{B\left(\frac{k+m+1}{2^p}\right)-B\left(\frac{k-1}{2^p}\right)\geqq\left(\frac{m}{2^p}\right)^{1/2}\psi\left(\frac{2^p}{m+2}\right)\right\}.
\end{aligned}
$$

By Lemma 3, we have, for large p and m

$$
\begin{aligned}
P\{F(p,k,m)\}&\leqq\frac{4}{(2\pi)^{1/2}\psi\left(\frac{2^p}{m+2}\right)}\exp\left(-\frac{m}{m+2}\psi^2\left(\frac{2^p}{m+2}\right)\right)\\
&\sim 4P\{E(p,k,m)\}\,e^{\left(\frac{1}{m+2}\psi^2\left(\frac{2^p}{m+2}\right)\right)}.
\end{aligned}
$$

So, if $\left[\frac{h(2^p)}{3}\right]\leqq m\leqq h(2^p)$ then by our assumptions about $\psi(t)$ and $h(t)$,

$$P\{F(p,k,m)\}\leqq c\,P\{E(p,k,m)\}.$$

Because the sum of the latter terms converges, so does the sum of $P\{F(p,k,m)\}$. Thus, by the Borel-Cantelli lemma

$$P\{F(p,k,m)\text{ i.o.}\}=0.$$

Now if $s\in\left[0,R\left(\frac{1}{l}\right)\right]$, choose p so that

$$\frac{h(2^{p+1})}{2^{p+1}}<l<\frac{h(2^p)}{2^p}.$$

Choose k and m so that

$$\frac{k-1}{2^p}<s\leqq\frac{k}{2^p}<\frac{k+m}{2^p}\leqq s+l<\frac{k+m+1}{2^p}.$$

Then, $\left[\frac{h(2^p)}{3}\right] \leqq m \leqq h(2^p)$ and so for p large enough,

$$\begin{aligned}\frac{B(s+l)-B(s)}{l^{1/2}} &\leqq \sup_{0\leqq u,w\leqq \frac{1}{2^p}} \left[\frac{B\left(\frac{k+m}{2^p}+u\right)-B\left(\frac{k}{2^p}-w\right)}{l^{1/2}}\right] \\ &\leqq \psi\left(\frac{2^p}{m+2}\right) \\ &\leqq \psi(l)\end{aligned}$$

with probability 1. This establishes Theorem 3 in the convergent case.

Next suppose that the sum diverges. The hypotheses of Theorem 2 must be verified. This will show that

$$P\{E(p,k,m)\ \text{i.o.}\}=1,$$

and the proof of Theorem 3 will be complete. This verification closely follows Chung, Erdös, and Sirao [4], and will be omitted.

5. An Invariance Principle

Strassen's invariance principle [6] can be generalized in the spirit of Sect. 2, but the random variables involved must have moments of order larger than 2. The result follows easily from Theorem 1 and the results of Komlós, Major, and Tusnády [5].

Definition. If $\{X_i\}$ are random variables, let $Z(n)=X_1+\ldots+X_n$ and define $Z(t)$ for noninteger t by linear interpolation.

Fix $\alpha \geqq 1$ and let

$$h(t)=\begin{cases}(\alpha-1)\log t & \text{if } \alpha>1 \\ \log\log t & \text{if } \alpha=1\end{cases}.$$

Theorem 4. *Let $\{X_i\}$ be i.i.d. with mean 0 and variance σ^2. Suppose that*

$$E|X_1|^{2\alpha}<\infty.$$

Let

$$\mathscr{C}(t)=\left\{f(x)=\frac{Z(s+xt)-Z(s)}{\sigma(2th(t))^{1/2}}: s\leqq t^\alpha\right\}.$$

Then the set of limit points of $\mathscr{C}(t)$ as $t\uparrow\infty$ is K.

Proof. For $\alpha=1$, the theorem follows from Strassen [6].

Suppose $\alpha>1$. Theorem 2 of Komlós, Major, and Tusnády and the remark on page 58 of their paper show that $B(t)$ and $Z(t)$ can be constructed on an appropriate probability space such that

$$\lim_{n\to\infty}\left|\frac{Z(n)/\sigma-B(n)}{(2nh(n))^{1/2}}\right|=0.$$

Remark II in the proof of Theorem 1 shows that

$$\lim_{t\to\infty}\left|\frac{B(t)-B([t])}{(2th(t))^{1/2}}\right|=0.$$

Clearly,

$$\lim_{t\to\infty}\left|\frac{Z(t)-Z([t])}{\sigma(2th(t))^{1/2}}\right|\leq\lim_{n\to\infty}\frac{|X_n|}{\sigma(2nh(n))^{1/2}}=0$$

by the moment conditions.

These three results together with Theorem 1 imply Theorem 4.

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