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Harmonizable Stable Processes

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Summary. This paper examines properties of a class of complex-valued stable processes which have spectral representation by means of independent-increments processes. A representation is derived by an application of Schilder's stochastic integral. Also, another construction of harmonizable stable processes by means of generalized stochastic processes is given, and its relation to the stochastic integral is shown. Some limit theorems of the Fourier transform of a sample from harmonizable stable processes are provided. Moreover, a linear prediction theory which pertains to those processes is suggested as an extension of that of second-order stationary processes.

0. Introduction

This paper aims at exploring properties of a class of complex-valued discrete-parameter stochastic processes which are termed harmonizable stable processes, establishing a number of results paralleling to those of the second-order stationary processes, such as spectral representations, isomorphism theorems, limit theorems of the Fourier transform of observations, and an optimal linear prediction. As the related previous works, there are those by Urbanik (1967, 68, 70) and Schilder (1970) whose results are extended by Kuelb (1973). Urbanik examined the properties of stochastic process which consists of the Fourier coefficients of infinitely-divisible random measure, whereas Schilder constructed stochastic integrals with respect to an independent-increments symmetric stable process. Section 1 of the paper extends, at first, Schilder's result and constructs complex-valued stable processes which have spectral representation by means of independent-increments symmetric stable processes. Those processes are termed harmonizable. The construction differs from Urbanik's in that, though the latter deals with a wider class of probability laws, it is at the same time limited in most of its applications to the case where random measures are atomless, whereas as far as symmetric stable laws are

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concerned the present construction does not require this limitation, and could deal with processes which have harmonics of various frequencies with non-degenerate random weights. The isomorphism of Theorem 1.1 is more general than Schilder's in that it is established without the use of Schilder's concept of "length" which applies, for complex-valued stable random variables, only to the case when they are isotropic. Theorems 1.2 and 1.3 give the relationships between isotropicity and stationarity of harmonizable stable processes; in particular, the latter theorem establishes the result under a weaker condition than that given by Urbanik, as far as stable laws are concerned.

Section 2 introduces a method of constructing harmonizable processes by means of generalized stochastic process with independent values at every point, instead of using a random measure of an independent-increments process. The relationship between this construction and that of Sect. 1 is given in Theorem 2.1 and also the usefulness of this construction is exhibited. Theorem 2.3 provides an asymptotic property of the Fourier transform of a finite sample from a harmonizable stable process $\{x_t\}$. It is shown there as an extension of Gaussian case, that, under a general condition, the set $\left\{w_N \sum_{t=-N}^{N} x_t e^{i\omega_j t}\right\}$ for distinct points ω_j, $j=1,2,\ldots,p$, is asymptotically independently distributed for appropriate choice of w_N.

Section 3 deals with the linear prediction of an isotropic harmonizable stable process whose exponent is greater than or equal to 1. It is shown that, if the Schilder's length is used to measure the degree of concentration of the distribution of prediction error to the origin for the purpose of scaling the goodness of prediction of various linear predictions, there exists an optimal one-step ahead linear predictor under a condition paralleling to that in second-order stationary processes. Also a condition is given for the process to be deterministic. The optimal predictor is explicitly constructed and the distribution of the predictor error is determined. The theory of Hardy space $H^\alpha(1 \leqq \alpha \leqq 2)$ turns out to be the useful tool for this problem, as is the theory of H^2 for the second-order stationary processes. Urbanik demonstrated the isomorphism between an Orlicz space and the closure with respect to probability convergence of the linear hull of a set of random variables generated by a harmonizable process and also extended the Szegö-Kolmogorov-Krein theorem so as to apply to an Orlicz space. However, though his results are very important in themselves, thanks to his limitation of the class of predictors (see Sect. 3) they do not in effect lead to a substantial extension of the previous prediction theory as far as the construction of an optimal predictor is concerned. In his class of stationary processes admitting prediction, purely non-deterministic harmonizable processes are Gaussian and thus the prediction theory of those processes is reduced to the known one of second-order stationary processes.

As for notations and symbols used in this paper, characteristic function is abbreviated as c.f., the set of all integers is denoted as I, and the real part of a complex number is signified by Re. In representation of characteristic function, the complex plane is identified as R^2; thus a characteristic function $\phi(s_1, s_2)$ of a two-dimensional distribution is denoted as $\phi(s)$ for $s=s_1+is_2$. Sometimes the notation $e^{i\cdot t}$ is used to denote the function $e^{i\cdot t}(\omega)=e^{i\omega t}$.

1. Harmonizable Stable Processes

A complex-valued random variable x is said to have a symmetric stable distribution of exponent α $(0<\alpha\leqq 2)$ if x has the characteristic function ϕ_x of the form

$$\phi_x(s)=\exp\left\{-\int_{-\pi}^{\pi} |\mathrm{Re}(se^{-i\theta})|^{\alpha}\,\mu(d\theta)\right\}$$

for a finite measure μ on $(-\pi,\pi]$. That the function ϕ_x is a characteristic function follows from the general canonical representation of multivariate stable law [see Rvaceva (1962) or Hosoya (1978) for example], but the non-negative definiteness is proved directly as this: since the integration can be viewed as that of Stieltjes because the integrand is continuous, the function ϕ_x can be arbitrarily closely approximated by a function of the form $\exp\left\{-\sum_{j=1}^{p} |\mathrm{Re}(se^{-i\theta_j})|\, m_j\right\}$ $(m_j\geqq 0)$ for fixed s; then the necessary result follows from the facts that the function $f(x)=\exp(-m|x|^{\alpha})$ $(m>0, 0<\alpha\leqq 2)$ defined on the real line is non-negative definite and that the product of non-negative definite functions are non-negative definite. This result is also used below in Lemma 2.1. Henceforth the term symmetric is omitted since this paper deals only with symmetric stable laws. Define a weight function $F(\lambda,\theta)$ to be a non-negative, nondecreasing function on $(-\pi,\pi]\times(-\pi,\pi]$ such that if $\lambda\downarrow\lambda_0$ and $\theta\downarrow\theta_0$, then $F(\lambda,\theta)\to F(\lambda_0,\theta_0)$, $F(\pi,\theta)>0$ for all θ and also for $\lambda_1,\lambda_2,\theta_1,\theta_2$ such that $\lambda_1\leqq\lambda_2$ and $\theta_1\leqq\theta_2$

$$F(\lambda_2,\theta_2)-F(\lambda_1,\theta_2)-F(\lambda_2,\theta_1)+F(\lambda_1,\theta_1)\geqq 0.$$

Let $\{z(\lambda);\ -\pi<\lambda\leqq\pi\}$ $(z(0)=0)$ be a complex-valued stable independent-increment process such that the c.f. ϕ_λ of $z(\lambda)$ is representable as

$$\phi_\lambda(s)=\exp\left\{-\int_{-\pi}^{\pi} |\mathrm{Re}(se^{-i\theta})|^{\alpha}\,F(\lambda,d\theta)\right\}$$

and call it a process based on the weight function F. Given $-\pi=\lambda_0<\lambda_1<\ldots<\lambda_p\leqq\pi$, define $\phi_{\lambda_1,\ldots,\lambda_p}$ as

$$\phi_{\lambda_1,\ldots,\lambda_p}(s_1,\ldots,s_p)$$
$$=\exp\left[-\sum_{k=1}^{p}\int_{-\pi}^{\pi}\left|\mathrm{Re}\left(\sum_{j=k}^{p} s_j e^{-i\theta}\right)\right|^{\alpha}\{F(\lambda_k,d\theta)-F(\lambda_{k-1},d\theta)\}\right]$$

(where $F(\lambda_0,d\theta)=0\cdot d\theta$). Since $\phi_{\lambda_1,\ldots,\lambda_p}$ thus defined is a c.f. of a multivariate stable law and the class of all c.f. $\phi_{\lambda_1,\ldots,\lambda_p}$ defined for all integer $p>0$, and for all $\lambda_1,\ldots,\lambda_p$ constitutes a consistent class in Kolmogorov's sense, there exists a process $\{z(\lambda), -\pi<\lambda\leqq\pi\}$ such that the joint c.f. of $z(\lambda_1),\ldots,z(\lambda_p)$ is given as $\phi_{\lambda_1,\ldots,\lambda_p}$. Also from the definition of $\phi_{\lambda_1,\ldots,\lambda_p}$, it is seen that, for $\lambda_1<\lambda_2\leqq\lambda_3<\lambda_4$,

$$E\exp[i\,\mathrm{Re}\{s_1(z(\lambda_2)-z(\lambda_1)+s_2(z(\lambda_4)-z(\lambda_3))\}]$$
$$=\exp\left[-\int_{-\pi}^{\pi}|\mathrm{Re}\,s_1e^{-i\theta}|^{\alpha}\{F(\lambda_2,d\theta)-F(\lambda_1,d\theta)\}\right]$$
$$\cdot\exp\left[-\int_{-\pi}^{\pi}|\mathrm{Re}\,s_2e^{-i\theta}|^{\alpha}\{F(\lambda_4,d\theta)-F(\lambda_3,d\theta)\}\right]$$

and thus that the process $\{z(\lambda)\}$ is of independent-increment. Also it follows from the property of F that $z(\lambda)$ is stochastically right-continuous (i.e. $z(\lambda)\to z(\lambda_0)$ in probability when $\lambda\downarrow\lambda_0$).

Schilder (1970) constructed stochastic integral by means of a real-valued independent-increments stable process. His construction can be applied in a straightforward way to complex-valued processes as follows. Given an independent-increments stable process $z(\lambda)$ based on a weight function F. Set $G(\lambda)=\int_{-\pi}^{\pi}F(\lambda,d\omega)$; then $G(\lambda)$ is a non-negative, non-decreasing bounded function of λ such that $G(\pi)>0$. Let L^{α} be the set of the Borel measurable functions f on $(-\pi,\pi]$ such that $\int_{-\pi}^{\pi}|f(\lambda)|^{\alpha}dG(\lambda)<\infty$. If g is a step-function of $(-\pi,\pi]$ such that $g(\lambda)=g_j$ if $\lambda_{j-1}<\lambda\leqq\lambda_j$, $j=1,\dots,k$ (where $0=\lambda_0<\lambda_1<\dots<\lambda_k=\pi$), define the stochastic integral $S(g)$ of g as

$$S(g)=\int_{-\pi}^{\pi}g(\lambda)\,dz(\lambda)=\sum_{j=1}^{k}g_j[z(\lambda_j)-z(\lambda_{j-1})]. \tag{1.1}$$

Then, due to the stochastic right-continuity of the path of z, $z(\lambda_{j-1}+\varepsilon)\to z(\lambda_{j-1})$ in probability as $\varepsilon\downarrow 0$. The characteristic function $\phi_{S(g)}(s)$ of $S(g)$ is given as the limit of that of $\Sigma g_j[z(\lambda_j)-z(\lambda_{j-1}+\varepsilon)]$; namely

$$\begin{aligned}\phi_{S(g)}(s)&=\lim_{\varepsilon\to0}E\exp\left\{i\,\mathrm{Re}\left(s\sum_{j=1}^{k}g_j(z(\lambda_j)-z(\lambda_{j-1}+\varepsilon)\right)\right\}\\&=\lim_{\varepsilon\to0}\prod_{j=1}^{k}E\exp[i\,\mathrm{Re}\{sg_j(z(\lambda_j)-z(\lambda_{j-1}+\varepsilon))\}]\\&=\exp\left\{-\int_{-\pi}^{\pi}|\mathrm{Re}(sg(\lambda)\,e^{-i\theta})|^{\alpha}F(d\lambda,d\theta)\right\}.\end{aligned} \tag{1.2}$$

For general $g\in L^{\alpha}$, since the step-functions of the type above are dense in L^{α}, there is a sequence g^n of step-functions which converges to g in L^{α}-with respect to the metric induced by $\int_{-\pi}^{\pi}|\cdot|^{\alpha}G(d\lambda)$. Then given $\varepsilon>0$, since $\{g^n\}$ is Cauchy, there is an N such that for n, $m>N$, $\int_{-\pi}^{\pi}|g^n(\lambda)-g^m(\lambda)|^{\alpha}G(d\lambda)<\varepsilon$. From the inequality

$$\begin{aligned}&\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|\mathrm{Re}(s(g^n(\lambda)-g^m(\lambda))\,e^{-i\theta})|^{\alpha}F(d\lambda,d\theta)\\&\leqq|s|^{\alpha}\int_{-\pi}^{\pi}|g^n(\lambda)-g^m(\lambda)|^{\alpha}G(d\lambda),\end{aligned} \tag{1.3}$$

it follows that $\{S(g^n)\}$ is Cauchy with respect to the probability convergence in the space $\mathbb{R}$ of random variables on the probability space $(\Omega, \mathscr{B}, \Pr)$ for which the stochastic process z is defined (in $\mathbb{R}$ two random variables which are equal a.e. are identified throughout the paper), because for any compact set D and $\varepsilon>0$ there is an N such that for n, $m>N$ the c.f. $\phi_{S(g^n)-S(g^m)}(s)$ satisfies $|\phi_{S(g^n)-S(g^m)}-1|<\varepsilon$ uniformly in $s\in D$. Then there is a random variable $S(g)$ to which the sequence $\{S(g^n)\}$ converges in probability. It is evident that $S(g)$ does not depend upon the choice of sequences of simple functions. Now define the stochastic integral $\int_{-\pi}^{\pi} g(\alpha)\, dz(\lambda)$ by setting it equal to $S(g)$. The next lemma is an obvious consequence of the preceding argument.

Lemma 1.1. *Given* $g_1, \ldots, g_k \in L^\alpha$, *the c.f.* $\phi_{S(g_1),\ldots,S(g_k)}(s_1, \ldots, s_k)$ *of the joint distribution of* $S(g_1), \ldots, S(g_k)$ *is given as*

$$\phi_{S(g_1),\ldots,S(g_k)}(s_1, \ldots, s_k) = \exp\left\{-\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |\mathrm{Re}(\Sigma g_j(\lambda)\, s_j e^{-i\theta})|^\alpha F(d\lambda, d\theta)\right\}. \quad (1.4)$$

To the space $\mathbb{R}$ of complex-valued random variables on $(\Omega, \mathscr{B}, \Pr)$ a topology is endowed such that probability convergence of a sequence of random variables is equivalent to the convergence with respect to that topology, and it is termed P-topology. Specifically, for each $x\in\mathbb{R}$ let $V_x(\varepsilon, \lambda) = \{y\in\mathbb{R}: \Pr(|x-y|<\varepsilon)>1-\eta\}$ for ε, $\eta>0$; then the set $\{V_x(\varepsilon,\eta): \varepsilon>0, 0<\eta<1\}$ is a neighbourhood base of P-topology. Denote by $L^\alpha(A)$ the completion of the linear hull of an arbitrary subset A of L^α and denote by $[S(f); f\in A]$ the completion with respect to P-topology in $\mathbb{R}$ of the linear hull of $\{S(f); f\in A\}$.

Theorem 1.1. *Let S be the mapping* $S(f) = \int_{-\pi}^{\pi} f(\lambda)\, dz(\lambda)$; *then it is a topological isomorphism of* $L^\alpha(A)$ *onto* $[S(f): f\in A]$.

Proof. The continuity of S follows from the definition of the stochastic integral and from the inequality (1.3). It is subjective: For each $x\in[S(f); f\in A]$, there is a sequence $\{g^n: g^n\in \bar{A}, n=1,2,\ldots\}$ such that $\{S(g^n)\}$ converges to x in probability, where $\bar{A}$ is the linear hull of A. Since then $\{S(g^n)\}$ is Cauchy, it holds for the c.f.

$$\phi_{S(g^n)-S(g^m)}(s) = \exp\left\{-\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |\mathrm{Re}\{S(g^n-g^m)\, e^{-i\theta}\}^\alpha F(d\lambda, d\theta)\right\}$$

that, given $\varepsilon>0$, there exists N such that for n, $m>N$

$$\max_{|s|=1} \phi_{S(g^n)-S(g^m)(s)} < \varepsilon. \quad (1.5)$$

Then by setting $s_1 = e^{i\omega}$ and $s_2 = e^{i\left(\omega+\frac{\pi}{2}\right)}$ it is seen that

$$(\tfrac{1}{2})^\alpha \int_{-\pi}^{\pi} |g^n - g^m|^\alpha G(d\lambda) \leq \sum_{j=1}^{2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |\mathrm{Re}\, s_j (g^n - g^m)\, e^{-i\theta}|^\alpha F(d\lambda, d\theta) \leq 2\varepsilon.$$

Thus it is concluded that the sequence $\{g^n\}$ is Cauchy, and that g^n converges to $g \in L^\alpha(A)$. It is evident that $S(g)=x$. Since S is continuous and bijective the Banach homomorphism theorem implies that S is an open-mapping [see Schaefer (1971)]. □

A discrete-parameter stochastic process $\{y_t\}$ $(t \in I)$ is said to be stationary if for any finite subset $(t_1, \ldots, t_k)$ of I and any integer l $(y_{t_1+l}, \ldots, y_{t_k+l})$ has the same distribution as $(y_{t_1}, \ldots, y_{t_k})$. An independent-increments stable process $z(\lambda)$ is said to be isotropic if for any λ and $\omega(-\pi<\lambda, \omega \leqq \pi)$, $z(\lambda)$ and $z(\lambda)e^{i\omega}$ have the same distributions; in other words, if the distribution of $z(\lambda)$ is preserved under rotational transformations. A discrete-parameter process $\{x_t: t \in I\}$ which is representable as $x_t = \int_{-\pi}^{\pi} e^{it\lambda}\, dz(\lambda)$ for an independent-increments stable process $z(\lambda)$ based on a weight function is called harmonizable.

Theorem 1.2. *The process $z(\lambda)$ is isotropic if and only if the harmonizable process $x_t = \int e^{i\lambda t}\, dz(\lambda)$ is stationary and has the c.f.'s which are representable as*

$$\phi_{t_1, \ldots, t_j}(s_1, \ldots, s_j) = \exp\left\{-\int_{-\pi}^{\pi} \left|\sum_{l=1}^{j} s_l e^{i\omega t_l}\right|^\alpha G(d\omega)\right\} \tag{1.6}$$

for a non-negative, bounded, non-decreasing function G.

Proof. Suppose $z(\lambda)$ is isotropic; then for a weight function F, the c.f. $\phi_\lambda(t)$ of $z(\lambda)$ is representable as

$$\phi_\lambda(s) = \exp\left\{-|s|^\alpha \int_{-\pi}^{\pi} |\cos(\theta-\psi)|^\alpha F(\lambda, d\theta)\right\} \tag{1.7}$$

for any $\psi(-\pi<\psi \leqq \pi)$. Consequently ϕ_λ is expressed as $\phi_\lambda(s) = \exp\{-|s|^\alpha G(\lambda)\}$ where $G(\lambda) = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |\cos\psi|^\alpha d\psi\right) F(\lambda, \pi)$. Then in view of the construction of stochastic integrals $x_t = \int_{-\pi}^{\pi} \exp(it\omega)\, dz(\omega)$ and of Lemma 1.1, the c.f. of $x_{t_1}, \ldots, x_{t_k}$ is representable as given in (1.7). Conversely suppose a harmonizable process x_t has the c.f. of the form (1.6). Given $\lambda(-\pi<\lambda<\pi)$ and $\varepsilon\left(0<\varepsilon<\frac{\pi-\lambda}{2}\right)$, denote by h_λ and $h_{\lambda,\varepsilon}$ the functions defined on $(-\pi, \pi]$ respectively such that $h_\lambda(\omega)=1$ for $\omega \in (-\pi, \lambda]$ and $h_\lambda(\omega)=0$ otherwise; $h_{\lambda,\varepsilon}(\omega)=1$ for $\omega \in (-\pi, \lambda]$ $h_{\lambda,\varepsilon}(\omega) = 1 - \frac{\omega-\lambda}{\varepsilon}$ for $\omega \in (\lambda, \lambda+\varepsilon]$, $h_{\lambda,\varepsilon}(\omega)=0$ for $\omega \in (\lambda+\varepsilon, \pi-\varepsilon)$, and $h_{\lambda,\varepsilon}(\omega) = \frac{\omega-\pi+\varepsilon}{\varepsilon}$ for $\omega \in (\pi-\varepsilon, \pi]$. Since $h_{\lambda,\varepsilon}$ is continuous, the sequence of the Cesaro means converges uniformly to it. In other words, there exists a sequence $\{a_{j,n}: j=-n, \ldots, 0, \ldots, n; n=1,2,\ldots\}$ such that $p_n(\omega) = \sum_{j=-n}^{n} a_{jn} e^{i\omega j}$ converges uniformly to $h_{\lambda,\varepsilon}$. Since the c.f. of $y_n = \int_{-\pi}^{\pi} P_n(\omega)\, dz(\omega)$ is given by

$$\phi_{p_n}(s) = \exp\left\{-|s|^\alpha \int_{-\pi}^{\pi} |\Sigma a_{jn} e^{i\omega j}|^\alpha G(d\omega)\right\},$$

it is concluded that the c.f. of $S(h_{\lambda,\varepsilon})$ is given by

$$\phi_{S(h_{\lambda,\varepsilon})}(s)=\exp\left\{-|s|^{\alpha}\int_{-\pi}^{\pi}|h_{\lambda,\varepsilon}|^{\alpha}\,G(d\omega)\right\}.$$

In view of the fact that $h_{\lambda,\varepsilon}(\omega)$ converges monotonically to $h_{\lambda}(\omega)$ for all ω and that $S(h_{\lambda})=z(\lambda)$, the c.f. ϕ_{λ} of $z(\lambda)$ is given by

$$\phi_{\lambda}(s)=\exp\left\{-|s|^{\alpha}\int_{-\pi}^{\pi}G(d\omega)\right\},$$

and thus $z(\lambda)$ is isotropic. As for the isotropicity of $z(\pi)$, it is a straightforward consequence that $z(\pi)=\int_{-\pi}^{\pi}dz(\omega)=x_0$. □

As in the preceding theorem, let F be a weight function and let

$$G(\lambda)=\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|\cos\psi|^{\alpha}\,d\psi\right)F(\lambda,\pi).$$

The next theorem is a little stronger version of that given by Urbanik (1968, p. 80) as far as harmonizable stable processes are concerned in the sense that his proof applies to the case where $F(\cdot,\theta)$ has no jumps for all θ whereas the next one requires only that $F(\cdot,\theta)$ has no jumps for all θ on those points $\{\lambda\}$ such that $\lambda/2\pi$ is rational.

Theorem 1.3. *Suppose an independent-increment process $\{z(\lambda)\}$ is based on F such that for all $\theta, F(\lambda,\theta)$ has no jumps on points λ such that $-\pi<\lambda\leqq\pi$ and $\lambda/2\pi$ is rational, then a harmonizable stable process $x_t=\int_{-\pi}^{\pi}\exp(i\omega t)\,dz(\omega)$ is stationary if and only if the process $\{z(\omega)\}$ is isotropic.*

Proof. The sufficiency is evident. The necessity is proved as follows. Because of the stationarity of $\{x_t\}$, it holds that for every integer k

$$\begin{aligned}&\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left|\sum_{j=1}^{l}s_j e^{i\omega t_j}\right|^{\alpha}|\cos(\psi(\omega)-\theta)|^{\alpha}\,F(d\omega,d\theta)\\ &=\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left|\sum_{j=1}^{l}s_j e^{i\omega t_j}\right|^{\alpha}|\cos(\psi(\omega)-k\omega-\theta)|^{\alpha}\,F(d\omega,d\theta)\end{aligned}\tag{1.8}$$

where $\psi(\omega)$ is the argument of the complex number $\Sigma s_j e^{i\omega t_j}$. Now it follows from Weyl's theorem (see Breinan (1968) p. 117, for example) that, for ω such that $\omega/2\pi$ is irrational,

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}|\cos(\psi(\omega)-k\omega-\theta)|^{\alpha}&=\int_{-\pi}^{\pi}|\cos(\psi(\omega)-\lambda-\theta)|^{\alpha}\,d\lambda\\ &=\int_{-\pi}^{\pi}|\cos\lambda|^{\alpha}\,d\lambda.\end{aligned}\tag{1.9}$$

Hence in view of the bounded-convergence theorem, it follows from (1.8) that

$$\begin{aligned}
&\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |\Sigma s_j e^{i\omega t_j}|^\alpha |\cos(\psi(\omega)-\theta)|^\alpha F(d\omega, d\theta) \\
&= \left(\int_{-\pi}^{\pi} |\cos\lambda|^\alpha d\lambda\right) \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |\Sigma s_j e^{i\omega t_j}|^\alpha F(d\omega, d\theta) \qquad (1.10)\\
&= \int_{-\pi}^{\pi} |\Sigma s_j e^{i\omega t_j}|^\alpha G(d\omega).
\end{aligned}$$

This equation implies that the c.f. of $(x_{t_1}, \ldots, x_t)$ is written as in the form of the right-hand side of (1.6) in Theorem 1.2.

Remark 1. An example of a harmonizable stationary stable process $\{x_t\}$ with a non-isotropic independent-increments process is this. Suppose the function F is such that it gives the unit masses to each of the lattice points

$$\left(-\pi+\frac{\pi}{2}j,\ -\pi+\frac{\pi}{2}k\right), \qquad j, k=1, \ldots, 4.$$

Then the c.f. of $x_{t_1}, \ldots, x_{t_k}$ is given as

$$\phi_{t_1,\ldots,t_k}(s_1, \ldots, s_k)=\exp\left\{-\sum_{m=-1}^{2}\sum_{n=-1}^{2}\left|\sum_{j=1}^{k} \operatorname{Re} s_j e^{i\left(\frac{\pi m}{2}t_j-\frac{\pi n}{2}\right)}\right|^\alpha\right\}.$$

It is evident that $\phi_{t_1,\ldots,t_k}(s_1, \ldots, s_k)=\phi_{t_1+l,\ldots,t_k+l}(s_1, \ldots, s_k)$ for all integer l: thus $\{x_t\}$ is stationary. On the other hand, since the c.f. of $z(\lambda)$ is given as

$$\phi_\lambda(s)=\exp\left\{-\sum_{m=-1}^{l}\sum_{n=-1}^{2} |\operatorname{Re} s e^{i\left(\frac{\pi m}{2}-\frac{\pi n}{2}\right)}|^\alpha\right\}$$

where $l=\left[\frac{\lambda}{(\pi/2)}\right]$ and consequently ϕ_λ is not invariant under orthogonal transformations $s\to se^{i\omega}$.

Remark 2. An independently identically distributed (i.i.d.) sequence of Gaussian random variables is harmonizable and plays the basic role in the family of Gaussian stationary processes in that a stationary process with given spectral structure can be reduced to and also can be constructed from an i.i.d. sequence through appropriate linear filter. However, this is not the case for stable processes with exponent less than 2. In this connection, it is important to note the fact that any i.i.d. sequence of isotropic random variables whose exponent is less than 2 is not harmonizable except for the degenerate case.

The c.f. of a complex-valued isotropic random variable x which has a symmetric stable distribution with exponent α is expressed as $\exp\{-b|s|^\alpha\}$ $(b>0)$. Following Schilder (1970), define the length $\|x\|$ of x as $\|x\|=b^{1/\alpha}$ for $1\leqq\alpha\leqq 2$ and $\|x\|=b$ for $0<\alpha<1$. The following properties concerning to this length are derived as a straightforward extension of his result. Given a family $\mathbb{R}_\alpha$ of random variables such that any linear combination of elements in $\mathbb{R}_\alpha$ is isotropically, stably distributed with exponent α, then the length $\| \ \|$ is a metric on the family $\mathbb{R}_\alpha$; namely, if $\|x\|=0$, $x=0$ (note that two random variables which are identical a.e. are identified) and $\|x_1+x_2\| \leqq \|x_1\|+\|x_2\|$. If x_1 and x_2 are independent,

$$\|x_1+x_2\|=\|x_1\|+\|x_2\| \quad \text{if } 0<\alpha<1$$

and

$$\|x_1+x_2\|^\alpha=\|x_1\|^\alpha+\|x_2\|^\alpha \quad \text{if } 1\leqq\alpha\leqq 2.$$

Given $\{z(\lambda)\}$ an isotropic independent increments process with exponent α, $\|z(\lambda)\|$ is a bounded, nondecreasing function $(-\pi,\pi]$. Denote by L^α the space of α-th power integrable Borel functions with respect to $d\|z(\lambda)\|$ for $0<\alpha<1$ and to $d\|z(\lambda)\|^\alpha$ for $1\leqq\alpha\leqq 2$. Suppose $f\in L^\alpha$, then

$$\begin{aligned} \left\| \int_{-\pi}^{\pi} f(\omega)\, dz(\omega) \right\| &= \int_{-\pi}^{\pi} |f(\omega)|^\alpha\, d\,\|z(\lambda)\|, \qquad \text{for } 0<\alpha<1, \\ &= \int_{-\pi}^{\pi} |f(\omega)|^\alpha\, d(\|z(\lambda)\|^\alpha), \qquad \text{for } 1\leqq\alpha\leqq 2. \end{aligned} \tag{1.11}$$

Let J be a set of integers and denote by L^α $(e^{i\cdot t}\colon t\in J)$ the completion of the linear hull of the set $\{e^{i\cdot t}\colon t\in J\}$ in the space L^α of α-th power integrable functions with respect to $d\|z\|$ for $0<\alpha<1$ and to $d(\|z\|^\alpha)$ for $1\leqq\alpha\leqq 2$. Moreover let $[x_t\colon t\in J]$ be the closure with respect to probability convergence of the linear hull of $\{x_t\colon t\in J\}$. It is obvious that the length $\| \ \|$ defined on $[x_t\colon t\in J]$ is a metric. The next theorem is a special form of Theorem 1.1 and will be used later in Sect. 3.

Theorem 1.4. *There is an isometric isomorphism S from $L^\alpha(e^{i\cdot t}\colon t\in J)$ to $[x_t\colon t\in J]$ such that $S(e^{i\cdot t})=x_t$ for $t\in J$.*

Proof. Evident in view of Theorem 1.1.

2. Stochastic Integral with Respect to a Generalized Stochastic Process

Let $\mathscr{D}$ be the space of complex-valued infinitely differentiable periodic functions (modulo 2π) defined on the real line and define norms in this space by

$$\|\psi\|_n^2=\sum_{q=0}^{n} \int_{-\pi}^{\pi} \left|\frac{d^q\psi(\omega)}{d\omega^q}\right|^2 d\omega, \qquad n=0,1,2,\ldots \tag{2.1}$$

for $\psi\in\mathscr{D}$. Endow the space $\mathscr{D}$ with the topology generated by the countable family $\{\|\psi\|_n\}_{n=0}^\infty$ of the above norms; then it is a separable, nuclear, countably Hilbert space [see Gel'fand and Vilenkin (1968) pp. 80–84]. Denote by F a weight function as in the previous section and define a real-valued functional C as:

$$C(\xi)=\exp\left\{-\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |\operatorname{Re}\xi(\lambda)\, e^{-i\theta}|^\alpha\, F(d\lambda, d\theta)\right\} \tag{2.2}$$

where $\xi\in\mathscr{D}$ and $0<\alpha\leqq 2$.

Lemma 2.1. *The functional C is a characteristic functional; namely it has the properties:* (i) $C(0)=1$, (ii) *C is continuous,* (iii) *C is non-negative definite.*

Proof. (i) and (iii) are obvious. In order to establish (ii), it is sufficient to prove C is continuous at 0. Let α_j be the j-th Fourier coefficient of $(\xi \in \mathscr{D})$. Since ξ is infinitely-differentiable, $\xi(\omega)$ and $\frac{d^2\,\xi(\omega)}{d\omega^2}$ have the expansion respectively such that

$$\xi(\omega)=\sum_{j=-\infty}^{\infty} \alpha_j e^{i\omega j} \quad \text{and} \quad d^2\,\xi(\omega)/d\omega^2=\sum_{j=-\infty}^{\infty} \alpha_j(-ij)^2\, e^{i\omega j}$$

where each sum converges uniformly. Now given $\varepsilon(>0)$, choose ξ such that

$$\left\{\left(2\eta \sum_{j=1}^{\infty} \frac{1}{j^2}\right)^{1/2}+\sqrt{2}\,\eta\right\}^{\alpha} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} F(d\omega, d\theta)<\varepsilon. \tag{2.3}$$

Then for ξ such that $\|\xi\|_2<\eta$,

$$\int_{-\pi}^{\pi} \left|\frac{d^2\,\xi(\omega)}{d\omega^2}\right|^2 d\omega=\sum_{j=-\infty}^{\infty} |\alpha_j|^2 j^2<\eta^2. \tag{2.4}$$

Hence in view of the relation (2.4) and the inequalities

$$|\xi(\omega)| \leqq \Sigma\,|\alpha_j| \leqq \sqrt{(\Sigma\,|\alpha_j|^2 j^2)}\left(\sum_{j\neq 0} \frac{1}{j^2}\right)+|\alpha_0| \tag{2.5}$$

it follows that

$$\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |\mathrm{Re}\,\xi(\omega)\, e^{-i\theta}|^{\alpha}\, dF(\omega, \theta) \leqq \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |\xi(\omega)|^{\alpha}\, dF(\omega, \theta)<\varepsilon. \tag{2.6}$$

Since $\|\xi\|_2<\eta$ is a neighbourhood of $\mathscr{D}$ at 0, (2.6) implies that C is continuous at 0. □

Denote the dual space of $\mathscr{D}$ as $\mathscr{D}^*$, and μ be the probability measure on $\mathscr{D}^*$ provided by the Minlos theorem [Minlos (1959)]. The next result is the counterpart of Lemma 1.1.

Lemma 2.2. *Given $f_1, \ldots, f_k \in \mathscr{D}$, the joint c.f. $\phi_{f_1,\ldots,f_k}(s_1, \ldots, s_k)$ of the random variables $T(f_1), \ldots, T(f_k)$ is representable as*

$$\phi_{f_1,\ldots,f_k}(s_1, \ldots, s_k) = \exp\left\{-\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left|\mathrm{Re} \sum_{j=1}^{k} s_j^*\, f_j(\omega)\, e^{-i\theta}\right|^{\alpha} F(d\omega, d\theta)\right\}, \tag{2.7}$$

where s_j^* is the conjugate of s_j.

Proof. This is a straightforward consequence of the equalities:

$$\begin{aligned}\phi_{f_1,\ldots,f_k}(s_1, \ldots, s_k) &= \int \exp\{i\,\mathrm{Re}\,\Sigma s_j^*\, T(f_j)\}\, d\mu \\ &= \int \exp\{i\,\mathrm{Re}\langle \Sigma s_j^*\, f_j, x\rangle\}\, d\mu(x)=C(\Sigma s_j^*\, f_j). \quad \square\end{aligned}$$

The sequence $z_t = \langle e^{i\cdot t}, \mathbf{z}\rangle$ $(t \in I)$, for $\mathbf{z}$ as defined in the above, is equivalent in distribution to a harmonizable stable process $x_t = \int e^{i\omega t}\, dz(\omega)$ for an independent-increments process z. Moreover if a generalized stochastic process $\mathbf{x}$ is termed isotropic when $\mathbf{x}$ and $e^{i\lambda}\mathbf{x}$ have the same distribution for all λ, it is easily shown that if $\mathbf{x}$ is isotropic, the process $z_t = \langle e^{i\cdot t}, \mathbf{x}\rangle$ is stationary and any subset $z_{t_1}, \ldots, z_{t_k}$ of $\{z_t\}$ has the c.f. which is expressed as

$$\exp\left\{-\int_{-\pi}^{\pi} |\Sigma s_j^* e^{i\omega t_j}|^\alpha\, dG(\omega)\right\}$$

for a nonnegative nondecreasing bounded function G.

The relation between the spectral representation by means of generalized stable process and the one by independent-increments stable process is exhibited in the next theorem:

Theorem 2.1. *Let $\{x_t\}$ be a stable process defined as $x_t = \langle e^{i\cdot t}, \mathbf{x}\rangle$ for a generalized stable process whose characteristic functional is given as (2.2); then there is an independent-increments process $\{z(\lambda)\}$ such that $x_t = \int_{-\pi}^{\pi} e^{i\omega t}\, dz(\omega)$ a.e., for all $t \in I$.*

This theorem is a consequence of the following more general proposition. Namely,

Lemma 2.3. *Suppose a discrete-parameter stable process $\{x_t\}$ has finite dimensional c.f.'s such that*

$$\phi_{t_1,\ldots,t_k}(s_1,\ldots,s_k) = \exp\left\{-\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |\mathrm{Re}\,\Sigma s_j e^{i(\omega t_j - \theta)}|^\alpha\, F(d\omega, d\theta)\right\}$$

for a weight function F. Then there exists an independent-increments stable process $\{z(\lambda)\}$ such that $x_t = \int_{-\pi}^{\pi} e^{i\omega t}\, dz(\omega)$, a.e.

Proof. Given $\lambda(-\pi < \lambda < \pi)$, let h_λ and $h_{\lambda,\varepsilon}$ be functions on $[-\pi, \pi]$ defined as in the proof of Theorem 1.2. Denote by $\Sigma a_{j,n,\varepsilon} e^{i\omega j}$ a Cesaro sum for $h_{\lambda,\varepsilon}$. Let $\{\varepsilon_n\}$ be a sequence of positive numbers which nonotonically tend to 0. Then it is seen in view of Theorem 1.2 that

$$\int |\Sigma a_{j,n,\varepsilon_n} e^{i\omega j} - h(\omega)|^\alpha\, G(d\omega) \to 0 \quad \text{as } n \to \infty. \tag{2.8}$$

Set $y_n = \sum_{j=-n}^{n} a_{j,n,\varepsilon_n} x_j$; then the y_n's converge in probability to a random variable $z(\lambda)$ whose c.f. is expressed as

$$\exp\left\{-\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |\mathrm{Re}\, s h(\lambda)\, e^{-i\theta}|^\alpha\, F(d\lambda, d\theta)\right\}.$$

Set $z(\pi) = x_0$. Then the set $\{z(\lambda)\}$ of random variables constitutes an independent-increments stable process, and thus the stochastic integral $x_t' = \int_{-\pi}^{\pi} e^{i\omega t}\, dz(\omega)$ is able to be defined by means of Schilder's method. In view of the construction of the process $\{z(\lambda)\}$, however, the joint c.f. $\phi(s_1, s_2, \dots, s_k)$ of x_t,

$$z(\lambda_2)-z(\lambda_1), \dots, z(\lambda_k)-z(\lambda_{k-1})\,(-\pi<\lambda_1<\lambda_2<\dots<\lambda_k\leqq\pi)$$

is given as

$$\phi(s_1, \dots, s_k) \tag{2.9}$$
$$=\exp\left[-\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|\mathrm{Re}(s_1 e^{it\omega}-\Sigma s_j(h_{\lambda_j}(\omega)-h_{\lambda_{j-1}}(\omega))\,e^{-i\theta}|^{\alpha}\,F(d\omega, d\theta)\right].$$

On the other hand, there exists a sequence of sums of the form

$$s_n=\Sigma b_{j,n}(h_{j,n}-h_{j-1,n})\,(-\pi<\lambda_{1,n}<\dots<\lambda_{n,n}\leqq\pi)$$

such that for all $s_1, \dots, s_k$

$$\lim_{h\to\infty}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|\mathrm{Re}\, s\{e^{it\omega}-\Sigma b_{j,n}(h_{\lambda_{j,n}}(\omega)-h_{\lambda_{j-1,n}}(\omega))\}|^{\alpha}\,F(d\omega, d\theta)=0. \tag{2.10}$$

Since the difference $d_n=x_n-\Sigma b_{j,n}\{z(\lambda_{j,n})-z(\lambda_{j-1,n})\}$ converges in probability to x_t-x_t' and the c.f. $\phi(s)$ of d_n is given as

$$\phi(s)=\exp\left[-\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|\mathrm{Re}\, s\{e^{i\omega t}-\Sigma b_{j,n}(h_{\lambda_{j,n}}(\omega)-h_{\lambda_{j-1,n}}(\omega))\,e^{-i\theta}|^{\alpha}\,F(d\omega, d\theta)\}\right],$$

it follows from that $x_t-x_t'=0$ a.e. □

Though stronger convergences than that in probability are hard to argue for the class of random variables which are represented by the stochastic integral using independent-increments process, since those integrals are only defined as probability limit, it is possible for the class $\{\langle f, \mathbf{x}\rangle : f\in\mathscr{D}\}$. For instance it is evident in view of the definition of duality that if f_n, $n=1, 2, \dots$, converge to f in $\mathscr{D}$ with respect to the topology generated by the countable norms (2.1), then $T(f_n)$ converges to $T(f)$ a.e.

As another result pertaining to the process $z_t=\langle e^{i\cdot t}, \mathbf{x}\rangle$ $(t\in I)$, the following theorem established a weak convergence of generalized processes. Given samples $z_{-n}, \dots, z_n$ of a harmonizable stable process $z_t=\langle e^{i\cdot t}, \mathbf{x}\rangle$, the Fourier transform $\sum_{j=-n}^{n} e^{i\cdot t} z_j$ defines a linear form l on $\mathscr{D}$ by means of identification such that for $\xi\in\mathscr{D}$,

$$l(\xi)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\xi(\omega)\sum_{j=-n}^{n}(z_j e^{i\omega j})\,d\omega. \tag{2.11}$$

Lastly a type of point-wise convergence is shown. For a fixed $\alpha(1<\alpha\leqq 2)$, let

$$K_n(\omega)=(2n+1)^{1-\alpha}\left|\sin\left\{\left(n+\frac{1}{2}\right)\omega\right\}\Big/\sin\left(\frac{\omega}{2}\right)\right|^{\alpha}, \quad \text{if } \omega\neq 0;$$
$$=2n+1, \quad \text{if } \omega=0,$$

and define a constant P_n as $1/P_n=\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(\omega)\,d\omega$. Then define $D_n(\lambda)$ as

$$D_n(\lambda)=(2\pi)^{-1/\alpha}P_n^{1/\alpha}(2n+1)^{(1-\alpha)/\alpha}\sum_{j=-1}^{n} e^{i\lambda j}, \qquad -\pi\leqq\lambda\leqq\pi.$$

Suppose that $\{x_t\}$ is a harmonizable stationary stable process and that the characteristic function $\phi(s_{-n},\ldots,s_n)$ of the sample $x_{-n},x_{-n+1},\ldots,x_n$ is represented as

$$\phi(s_{-n},\ldots,s_n)=\exp\left\{-\int_{-\pi}^{\pi}\left|\sum_{l=-n}^{n} s_l e^{-i\omega l}\right|^{\alpha} dF(\omega)\right\} \tag{2.12}$$

and F has a density f with respect to the Lebesque measure. Now construct statistics $I_n(\lambda)$, $-\pi\leqq\lambda\leqq\pi$, as this:

$$I_n(\lambda)=(2\pi)^{-1/\alpha}P_n^{1/\alpha}(2n+1)^{(1-\alpha)/\alpha}\sum_{j=-1}^{n} x_j\exp(i\lambda j). \tag{2.13}$$

Theorem 2.3. *Let $\lambda_i, i=1,2,\ldots,p$, be distinct Lebesgue points of f. Then $I_n(\lambda_i)$, $i=1,2,\ldots,p$, are asymptotically independent and the c.f. $\phi_i(s)$ of the asymptotic distribution of $I_n(\lambda_i)$ is provided as $\phi_i(s)=\exp\{-f(\lambda_i)\,|s|^{\alpha}\}$.*

Proof. Denote by $\phi_n(\mu_1,\ldots,\mu_p)$ the joint c.f. of $I_n(\lambda_i)$. Then,

$$\begin{aligned}\phi_n(\mu_1,\ldots,\mu_p)&=E\exp\left\{i\,\mathrm{Re}\sum_{j=1}^{n}\mu_j I_n(\lambda_j)\right\}\\ &=\exp\left\{-\int_{-\pi}^{\pi}\left|\sum_{j=1}^{n}\mu_j D_n(\lambda_j-\lambda)\right|^{\alpha} f(\lambda)\,d\lambda\right\}.\end{aligned}$$

The theorem is a straightforward consequence of Theorem 4.1 of Hosoya (1978). □

3. An Optimal Linear Prediction

In the theory of linear prediction of discrete-parameter strictly stationary processes, Urbanik examines for those processes various concepts of the theory of second-order stationary processes such as deterministic or non-deterministic processes, or Wold's decomposition [see Urbanik (1967, 68, 70)]. However, it is important to note that his theory is framed essentially on his particular way of defining prediction. Namely, he calls a stationary process to admit a prediction if there exists a continuous linear operator A from $[x_t: t\in I]$ to $[x_t: t\leqq -1]$ such that

(i) $Ax=x$ whenever $x\in[x_t: t\leqq -1]$;
(ii) if a random variable x is independent with every $y\in[x_t: t\leqq -1]$;
(iii) for very $x\in[x_t: t\in I]$ and $y\in[x_t: t\leqq -1]$, $x-Ay$ and y are independent. (3.1)

His concept of linear prediction, however, turns out to be of limited use except for Gaussian stationary processes. Actually he shows that if $\{x_t\}$ is a harmonizable stationary process such that $x_t = \int_{-\pi}^{\pi} e^{i\omega t}\, dM(\omega)$ for an atomless isotropic random measure M, $\{x_t\}$ is completely non-deterministic only if either $M \equiv 0$ or M is Gaussian [see Urbanik (1968) p. 86]. The purpose of this section is to consider a prediction problem of harmonizable stationary stable process from a less restrictive view, giving a criterion of optimality of prediction and constructing an optimal one-step head predictor. For that purpose, Schilder's idea is useful [see Schilder (1970) p. 420]. He considers the problem of minimizing the quantity $\left\| y_1 - \sum_{j=2}^{n} \alpha_j y_j \right\|$ with respect to the α_j for a finite set of real-valued stable random variables y_j such that $y_j = \int_{-\pi}^{\pi} f_j(\omega)\, dz(\omega)$ $(j=1, \ldots, p)$ where z is a real-valued independent-increments stable process and $\| \;\|$ is the length as was defined in Sect. 1, and he gives a necessary and sufficient condition of the α_j minimizing $\|y_1 - \Sigma \alpha_j y_j\|$ where $\| \;\|$ is the length as defined in Sect. 1. Since, as was shown in that section, the length $\|x\|$ is defined for a complex-valued isotropic stable random variable, it is able to be employed for the purpose of comparing the goodness of prediction of various predictors. Given a harmonizable stationary stable process $\{x_t\}$ such that $x_t = \int_{-\pi}^{\pi} e^{i\omega t}\, dz(\omega)$ for an isotropic independent-increments stable process z, a one-step a head predictor of x_0 is an element of $[x_t : t \leqq -1]$ and the predictor error of $y \in [x_t : t \leqq -1]$ is measured in terms of the length $\|x_0 - y\|$, and if there is an element z in $[x_t : t \leqq -1]$ such that $\|x_0 - z\| \leqq \|x_0 - y\|$ for all $y \in [x : t \leqq -1]$, the random variable z is called optimal (the symbol $[x_t : t \leqq -1]$ was defined in the last paragraph of Sect. 1). It is to be noted that if there is an optimal predictor, it will usually be not a predictor in Urbanik's sense except for in the case of Gaussian process, since his conditions (ii) and (iii) are violated. Now as the next theorem states, an optimal one-step ahead predictor in the above sense exists under appropriate conditions on the process z, and as the proof of the theorem will show, it is actually able to be constructed by an extension of Kolmogorov and Wiener's result for second-order stationary processes. The result is summarized in the following.

Theorem 3.1. *Let $x_t = \int_{-\pi}^{\pi} e^{it\omega}\, dz(\omega)$ be a harmonizable stationary stable process for an isotropic independent-increments process $\{z(\lambda)\}$ such that the measure induced by $\|z(\lambda)\|^\alpha$ has a density $f(\lambda)$ with respect to the Lebesque measure. If $\int_{-\pi}^{\pi} \log f(\omega)\, d\omega > -\infty$, there exists an optimal predictor x^* in $[x_t : t \leqq -1]$ of x_0 and then*

$$\|x^* - x_0\|^\alpha = 2\pi \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \log f(\omega)\, d\omega\right).$$

If $\int_{-\pi}^{\pi} \log f(\omega)\, d\omega = -\infty$, there exists a predictor y in $[x_t : t \leqq -1]$ such that $y^* = x_0$ a.e.

The rest of the section is for the proof of this theorem and the proof is broken into steps where results are given as lemmas. First, suppose that $\int_{-\pi}^{\pi} \log f(\omega)\,d\omega > -\infty$. Then an optimal predictor is constructed as follows: Since $\frac{1}{\alpha}\log f$ is then integrable, it has the formal Fourier series

$$\frac{1}{\alpha}\log f(\lambda) \sim \sum_{-\infty}^{\infty} a_n e^{in\lambda} \quad \text{where} \quad a_n = \frac{1}{2\pi\alpha}\int_{-\pi}^{\pi} \log f(\lambda)\, e^{-in\lambda}\, d\lambda.$$

By means of these coefficients, construct a function g defined on the open unit disk of the complex plane as $g(z) = a + 2\sum_{i=1}^{\infty} a_n z^n$. Since $\{a_n\}$ is bounded, g is analytic. Set

$$c(z) = \sum_{j=0}^{\infty} c_j z^j \quad \text{where} \quad \sum_{j=0}^{\infty} c_j z^j = (2\pi)^{1/\alpha} \exp\{g(z)\}, \quad |z| < 1.$$

Note that $c(z)$ is analytic on the open unit disk. For r such that $0 < r < 1$,

$$\begin{aligned}\int_{-\pi}^{\pi} |c(\mathrm{re}^{-i\lambda})|^\alpha\, d\lambda &= 2\pi \int_{-\pi}^{\pi} \exp\{\alpha \operatorname{Re} g(\mathrm{re}^{i\lambda})\}\, d\lambda \\ &= 2\pi \int_{-\pi}^{\pi} \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-\lambda) \log[f(\theta)]^{\alpha/2}\, d\theta\right\} d\lambda \\ &\leqq 2\pi \int_{-\pi}^{\pi} [f(\theta)]^{\alpha/2}\, d\theta < \infty\end{aligned}$$

where $P_r(\theta-\lambda) = (1-r^2)/(1-2r\cos(\theta-r)+r^2)$. Therefore it is seen that $c(z)$ belongs to the Hardy space H^α, and the boundary value $c(e^{-i\lambda}) = \lim_{r\to 1} c(\mathrm{re}^{-i\lambda})$ exists a.e. Also it follows from the construction of c that

$$\frac{1}{2\pi}|c(e^{-i\lambda})|^\alpha = f(\lambda) \text{ a.e.} \quad \text{and} \quad c_0 = 2\pi \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log f(\lambda)\, d\lambda\right\}.$$

Now set $p(\lambda) = \sum_{j=1}^{\infty} c_j e^{-j\lambda}/c(e^{-i\lambda})$.

Lemma 3.1. $p \in L^\alpha_f(e^{i\cdot t};\, t \leqq 1)$.

Proof. The Fourier coefficients b_j of p satisfy that $b_j = 0$ for $j \geqq 0$. Also it holds that

$$\int_{-\pi}^{\pi} |p(\lambda)|^\alpha f(\lambda)\, d\lambda \leqq 2\int_{-\pi}^{\pi} f(\lambda)\, d\lambda + 2\pi |c_0|^\alpha < \infty. \quad \square$$

In view of Theorem 1.4, there exists an element x^* of $[x_t : t \leqq -1]$ such that $S(p) = x^*$, and $x^* - x_0$ has the distribution whose characteristic function $\phi(u)$ is given as

$$\phi(u)=\exp\left\{-|u|^\alpha \int_{-\pi}^{\pi} |1-\phi(\lambda)|^\alpha f(\lambda)\,d\lambda\right\}=\exp\{-c_0|u|^\alpha\}$$
$$=\exp\left\{-2\pi|u|^\alpha \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\log f(\lambda)\,d\lambda\right)\right\}.$$

Lemma 4.2. *If* $x\in[x_t;\, t\leqq -1]$, *then* $\|x^*-x_0\|\leqq\|x-x_0\|$.

Proof. In view of Theorem 1.4, for x, there exists $g\in L^\alpha(\{t\leqq -1\};f)$ such that the characteristic function of $x-x_0$ is written as

$$\exp\left\{-|u|^\alpha\int_{-\pi}^{\pi}|1-g(\omega)|^\alpha f(\omega)\,d\omega\right\}$$

and moreover that there exists a sequence $\{a_1^n,\dots,a_n^n\}$, $n=1,2,\dots$ and

$$\int_{-\pi}^{\pi}\left|1+\sum_{j=1}^{n}a_j^n e^{ij\omega}\right|^\alpha f(\omega)\,d\omega$$

converges to $\int_{-\pi}^{\pi}|1-g(\omega)|^\alpha f(\omega)\,d\omega$ as n tends to infinity. On the other hand, Szego's theorem [see Achiezer (1956), p. 262] maintains that if $w(\omega)$ be an integrable function such that $\int_{-\pi}^{\pi}\log w(\omega)\,d\omega>-\infty$,

$$\lim_{n\to\infty}\min_{A_k}\int_{-\pi}^{\pi}|1+A_1e^{-i\omega}+\dots+A_ne^{-in\omega}|^\alpha w(\omega)\,d\omega$$
$$=2\pi\exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\log w(\omega)\,d\omega\right\}.$$

Therefore

$$\int_{-\pi}^{\pi}|1-g(\omega)|^\alpha f(\omega)\,d\omega\geqq 2\pi\exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\log f(\omega)\,d\omega\right\}.\quad\square$$

Lemma 3.3. *In case* $\int_{-\pi}^{\pi}\log f(\omega)\,d\omega=-\infty$, *there exists a predictor* z^* *in* $[x_t;\, t\leqq -1]$ *such that* $z^*-x_0=0$ *a.e.*

Proof. Let l be a positive number and define f_l as: $f_l(\omega)=f(\omega)$ if $f(\omega)\geqq l$ and $f_l(\omega)=l$ if $f(\omega)<l$. Let $x(l)$ be the optimal predictor constructed according to the foregoing argument when the spectral density is given as f_l. Let p_l be an element in $L^\alpha(\{t\leqq -1\},m)$ corresponding to $x(l)$ $\left(\text{note that }\int_{-\pi}^{\pi}|p_l|^\alpha\,d\omega<\infty\right)$; then it follows from the already established result that

$$P(p_l,f_l)=\int_{-\pi}^{\pi}|1-p_l(\omega)|^\alpha f_l(\omega)\,d\omega=2\pi\exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\log f_l(\omega)\,d\omega\right\}.\tag{3.2}$$

Let $\{l_n\colon n=1,2,\dots\}$ be a sequence of positive constants such that $\lim l_n=0$. It can be shown that $\{x(l_n)\}$ is a Cauchy sequence in probability. For that purpose, note that $P(p_l, f_l)$ monotonically decreases to 0 as $l \to 0$, and also that

$$\begin{aligned}&\int_{-\pi}^{\pi} |1-p_l(\omega)|^\alpha f(\omega)\,d\omega\\&=\int_{-\pi}^{\pi} |1-p_l(\omega)|^\alpha f_l(\omega)\,d\omega+\int_{-\pi}^{\pi} |1-p_l|^\alpha (f(\omega)-f_l(\omega))\,d\omega\\&\leqq (2\pi+1)\,P(p_l, f_l).\end{aligned}$$

For any positive m and n, $x(l_m)-x(l_n)$ has the characteristic function given as

$$\exp\left\{-|u|^\alpha \int_{-\pi}^{\pi} |p_{l_m}-p_{l_n}|^\alpha f(\omega)\,d\omega\right\}.$$

Then according to the inequality (3.3),

$$\begin{aligned}&\int_{-\pi}^{\pi} |p_{l_m}(\omega)-p_{l_n}(\omega)|^\alpha f(\omega)\,d\omega\\&=2\left\{\int_{-\pi}^{\pi} |1-p_{l_m}(\omega)|^\alpha f(\omega)\,d\omega+\int_{-\pi}^{\pi} |1-p_{l_n}(\omega)|^\alpha f(\omega)\,d\omega\right\}\end{aligned}$$

can be made arbitrarily small for sufficiently large n and m. Thus it follows that $\{x(l_n)\}$ is Cauchy in probability. Thus it converges to a random variable z^* in $[x_t; t \leqq -1]$. It is evident that z^*-x_0 is equal to 0 a.e. □

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