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A Method for the Derivation of Limit Theorems for Sums of *m*-dependent Random Variables

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1. Introduction

Let $(X_k)_{k=1,2,\dots}$ be a sequence of random variables (rv's) which are defined on a probability space $(\Omega, \mathfrak{A}, P)$. For $a \leqq b$, let $\mathfrak{A}_a^b = \sigma(X_a, \dots, X_b)$ denote the σ-algebra of events generated by $X_a, \dots, X_b$. We put $EX_k = 0$, $k = 1, 2, \dots$, and write $S_N = X_1 + \dots + X_N$, $B_N^2 = ES_N^2$, $F_N(x) = P(S_N < x B_N)$, $f_N(z) = E e^{zS_N}$ and $\Delta_N(x) = |F_N(x) - \Phi(x)|$, where $\Phi(x)$ is the standard normal distribution. The sequence $(X_k)_{k=1,2,\dots}$ is called m-dependent if for $1 \leqq s < t < \infty$, $t - s > m$, the σ-algebras $\mathfrak{A}_1^s$ and $\mathfrak{A}_t^\infty$ are independent (see [12]). For example, the sequence $X_k = h(\xi_k, \dots, \xi_{k+m})$, $k = 1, 2, \dots$, where $(\xi_j)_{j=1,2,\dots}$ are independent rv's and $h | R^{m+1} \to R^1$ Borel-measurable, is m-dependent.

Sequences of m-dependent rv's are special cases of ϕ-mixing sequences (see [12, 13]). In general the estimates of the error $\Delta_N(x)$ for ϕ-mixing sequences are cruder than in the case of m-dependence (see [13]). Recently, many authors studied the error $\Delta_N(x)$ for m-dependent rv's. The reader is referred to [1, 2, 8, 9, 12, 13]. Perhaps the best published result on uniform bounds of $\Delta_N(x)$ is that of V.V. Shergin [9], who shows that under the assumption $E|X_k|^3 < \infty$, $k = 1, 2, \dots, N$, the discrepancy $\Delta_N(x)$ is less than

$$c_1 (m+1)^2 \sum_{k=1}^{N} E|X_k|^3 B_N^{-3}.$$

Here and below, $c_1, c_2, \dots$ denote positive constants (not depending on N). θ stands for a complex number with $|\theta| \leqq 1$ which may be differ from one expression to another and χ_A denotes the indicator function of the event A. □ indicates the end of a proof.

The main purpose of this paper is to provide a general method for the derivation of limit theorems for sums of m-dependent rv's. This method is based on a factorization of the characteristic (moment-generating) function $f_N(it)$, $t \in R^1$, ($f_N(z)$, $z \in C^1$) in some neighbourhood of $t = 0$ ($z = 0$). Section 4 contains some results concerning asymptotic expansions of $F_N(x)$ and large

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deviations of S_N. In Sect. 5, these theorems are proved by using the basic relations of Sect. 3 and the standard procedures from the summation theory of independent *rv*'s (see [7]). By the given factorization of $f_N(it)$ it is possible to derive further results for sums of m-dependent *rv*'s and random vectors, for example: convergence to infinitely divisible distributions, non-uniform bounds of $\Delta_N(x)$ and so on. The factorization of the characteristic function of a sum of dependent *rv*'s was first used by the author in order to investigate the limit behaviour of sums of *rv*'s connected in a Markov chain (see [3-6]).

2. Preliminary Lemmas

For a sequence of arbitrary complex-valued *rv*'s $Y_1, Y_2, \ldots$ the symbol $\hat{E} Y_1 Y_2 \ldots Y_k$ is defined recursively by:

$$\hat{E} Y_1 Y_2 \ldots Y_k = E Y_1 Y_2 \ldots Y_k - \sum_{j=1}^{k-1} \hat{E} Y_1 \ldots Y_j E Y_{j+1} \ldots Y_k \quad \text{for } k \geqq 2 \text{ [1]} \tag{2.1}$$

and $\hat{E} Y_1 = E Y_1$.

This symbol was at first defined (in another way) by V.A. Statulevičius [11]. The above equivalent definition was given by the author [3].

Applying (2.1) successively, we get

$$\hat{E} Y_1 Y_2 \ldots Y_k = \sum_{l=1}^{k} (-1)^{l-1} \sum_{\substack{k_1 + \ldots + k_l = k \\ k_i \geqq 1}} E Y_1 \ldots Y_{k_1} E Y_{k_1+1} \ldots Y_{k_1+k_2} \ldots E Y_{k_1+\ldots+k_{l-1}+1} \ldots Y_k. \tag{2.2}$$

Similarly to (2.2) we can show (see [3])

$$\hat{E}(Y_1 + a_1)(Y_2 + a_2) \ldots (Y_k + a_k) = \sum_{l=2}^{k} \sum_{1 = q_1 < q_2 < \ldots < q_l = k} \hat{E} Y_{q_1} Y_{q_2} \ldots Y_{q_l} \prod_{\substack{j=2 \\ j \neq q_2, \ldots, q_{l-1}}}^{k-1} a_j, \tag{2.3}$$

where $a_1, a_2, \ldots, a_k$ are arbitrary complex numbers.

Another result is concerned with the differentiation of $\hat{E} Y_1(t) \ldots Y_k(t)$, where $(Y_j(t))_{j=1,\ldots,k}$ are differentiable *rv*'s (with respect to the parameter t). If differentiation and expectation are exchangeable, then the following analogue to Leibniz's rule holds (see [3]): [2]

$$\begin{aligned} \frac{d^p}{dt^p} \hat{E} Y_1(t) \ldots Y_k(t) &= \sum_{\substack{p_1 + \ldots + p_k = p \\ p_k \geqq 0}} \binom{p}{p_1 \ldots p_k} \hat{E} Y_1^{(p_1)}(t) \ldots Y_k^{(p_k)}(t) \\ &= \sum_{l=1}^{\min(p,k)} \sum_{\substack{p_1 + \ldots + p_l = p \\ p_i \geqq 1}} \binom{p}{p_1 \ldots p_l} \sum_{1 \leqq i_1 < \ldots < i_l \leqq k} \hat{E} Y_1(t) \ldots Y_{i_1}^{(p_1)}(t) \ldots Y_{i_l}^{(p_l)}(t) \ldots Y_k(t). \end{aligned} \tag{2.4}$$

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[1] We suppose $E|Y_j|^k < \infty$, $j = 1, 2, \ldots, k$.

[2] $\binom{p}{p_1 p_2 \ldots p_k} = \frac{p!}{p_1! p_2! \ldots p_k!}$

Lemma 2.1. *Let* $U_1, \dots, P_p$ *and* $V_1, \dots, V_q$ *be arbitrary rv's with* $E|U_i|^p < \infty$, $i=1, \dots, p$ *and* $E|V_j|^q < \infty$, $j=1, \dots, q$. *If the* σ*-algebras* $\sigma(U_1, \dots, U_p)$ *and* $\sigma(V_1, \dots, V_q)$ *are independent, then*

$$\hat{E} U_1 \dots U_p V_1 \dots V_q = 0.$$

Proof. Because of the Definition (2.1) and the assumptions of Lemma 2.1 we have

$$\hat{E} U_1 \dots U_p V_1 \dots V_q = E V_1 \dots V_q \left(E U_1 \dots U_p - \sum_{i=1}^{p-1} \hat{E} U_1 \dots U_i E U_{i+1} \dots U_p \right)$$
$$- \hat{E} U_1 \dots U_p E V_1 \dots V_q - \sum_{j=1}^{q-1} \hat{E} U_1 \dots U_p V_1 \dots V_j E V_{j+1} \dots V_q$$
$$= - \sum_{j=1}^{q-1} \hat{E} U_1 \dots U_p V_1 \dots V_j E V_{j+1} \dots V_q.$$

In the same way we get

$$\hat{E} U_1 \dots U_p V_1 = E V_1 \left(E U_1 \dots U_p - \sum_{i=1}^{p-1} \hat{E} U_1 \dots U_i E U_{i+1} \dots U_p - \hat{E} U_1 \dots U_p \right) = 0.$$

Repeating the above procedure we obtain the assertion of Lemma 2.1. □

Corollary 2.1. *If* $Y_1, \dots, Y_k$ *are 1-dependent rv's with* $E|Y_j|^k < \infty$, $j=1, \dots, k$, *then we have*

$$\hat{E}(Y_1 + a_1)(Y_2 + a_2) \dots (Y_k + a_k) = \hat{E} Y_1 Y_2 \dots Y_k, \tag{2.5}$$

where $a_1, \dots, a_k$ *are arbitrary complex numbers.*

Proof. By Lemma 2.1 and the 1-dependence we have $\hat{E} Y_{q_1} Y_{q_2} \dots Y_{q_l} = 0$ for $1 = q_1 < q_2 < \dots < q_l = k$, $2 \leqq l < k$. Using (2.3) we immediately get (2.5). □

Lemma 2.2. *Let* $Y_1, \dots, Y_k$ *be 1-dependent rv's with* $E|Y_j|^2 < \infty$, $j=1, \dots, k$. *Then the following estimate holds:*

$$|\hat{E} Y_1 Y_2 \dots Y_k| \leqq 2^{k-1} \prod_{j=1}^{k} (E|Y_j|^2)^{1/2}. \tag{2.6}$$

Proof. We prove relation (2.6) by induction. Clearly (2.6) holds for $k=1$. By Schwarz's inequality and the 1-dependence we have

$$|E Y_{j+1} \dots Y_p| \leqq (E|Y_{j+1} Y_{j+3} \dots|^2)^{1/2} (E|Y_{j+2} Y_{j+4} \dots|^2)^{1/2}$$
$$= \prod_{i=j+1}^{p} (E|Y_i|^2)^{1/2}. \tag{2.7}$$

Let us assume that (2.6) is valid for $k=1, \dots, p-1$. Using the Definition (2.1) for $k=p$ and (2.7), it is easy to see that (2.6) also holds for $k=p$. □

Let there be given a sequence of m-dependent rv's $X_1, X_2, \ldots, X_N$. We can construct the following sequence of 1-dependent rv's:

$$Y_k = \sum_{j=1}^{m} X_{m(k-1)+j}, \quad k=1,\ldots,\left[\frac{N}{m}\right] \quad \text{and} \quad Y_{\left[\frac{N}{m}\right]+1} = S_n - \sum_{k=1}^{\left[\frac{N}{m}\right]} Y_k.^{3} \tag{2.8}$$

Therefore, it suffices to consider the case $m=1$. We use the notations of Section 1 and put $M_{pN} = \max_{1 \leqq k \leqq N} E|X_k|^p$,

$$w_N(z) = \max_{1 \leqq k \leqq N} (E|e^{zX_k} - 1|^2)^{1/2}, \qquad K_N = \{z \in C^1 \colon w_N(z) \leqq 1/6\}.$$

Lemma 2.3. *If $X_1, \ldots, X_N$ are 1-dependent rv's with $M_{pN} < \infty$, then the following estimate holds:*

$$\left|\frac{d^p}{dt^p} \hat{E}(e^{itX_j}-1)\ldots(e^{itX_k}-1)\right| \leqq \begin{cases} M_{pN}(k-j+1)^p\, 2^{3p+1}(2w_N(it))^{k-j-2p}, & k-j \geqq 2p \\ M_{pN}(k-j+1)^p\, 2^{2(k-j)}, & k-j < 2p \end{cases}, \tag{2.9}$$

where $1 \leqq j < k \leqq N$.

Proof. To prove relation (2.9) we replace the rv's $(Y_i)_{i=1,\ldots,k}$ in (2.4) by $e^{itX_l} - 1$, $l=j,\ldots,k$. Using (2.2), the well-known inequality $|e^{itx}-1| \leqq 2$, Hölder's inequality and keeping in mind the 1-dependence, it follows easily that

$$|\hat{E} \ldots Y_{i_1}^{p_1} e^{itX_{i_1}} \ldots Y_{i_l}^{p_l} e^{itX_{i_l}} \ldots| \leqq \begin{cases} M_{pN}\, 2^{3l+1}(2w_N(it))^{k-j-2l}, & k-j \geqq 2l \\ M_{pN}\, 2^{2(k-j)-l+1}, & k-j < 2l \end{cases}$$

for $j \leqq i_1 < \ldots < i_l \leqq k$.

Therefore the identity

$$\sum_{l=1}^{\min(p,k-j+1)} \sum_{\substack{p_1+\ldots+p_l=p \\ p_i \geqq 1}} \binom{p}{p_1 \ldots p_l} \binom{k-j+1}{l} = (k-j+1)^p$$

implies the desired estimate (2.9). □

3. Factorization of the Characteristic Function of a Sum of 1-dependent Random Variables

We are now in position to formulate the main lemmas of this paper.

Lemma 3.1. *Let $X_1, \ldots, X_N$ be a sequence of 1-dependent rv's. Then the product representation*

$$E e^{zS_N} = \varphi_1(z)\,\varphi_2(z) \cdot \ldots \cdot \varphi_N(z) \tag{3.1}$$

holds for each $z \in K_N$, where $\varphi_1(z) = E e^{zX_1}$ and for $k=2,\ldots,N$

$$\varphi_k(z) = \frac{E e^{zS_k}}{E e^{zS_{k-1}}} = E e^{zX_k} + \sum_{j=1}^{k-1} \frac{\hat{E}(e^{zX_j}-1)(e^{zX_{j+1}}-1)\ldots(e^{zX_k}-1)}{\varphi_j(z)\,\varphi_{j+1}(z)\ldots\varphi_{k-1}(z)}. \tag{3.2}$$

[3] $[x]$ denotes the integer part of x.

Furthermore, the following estimates are true for each $z \in K_N$ and $k=1, 2, \dots, N$:

$$|\varphi_k(z)-1| \underset{(>)}{\leqq} |E e^{zX_k}-1| \underset{(-)}{+} \frac{2(E|e^{zX_{k-1}}-1|^2 E|e^{zX_k}-1|^2)^{1/2}}{1-4w_N(z)} \tag{3.3}$$

or

$$|\varphi_k(z)-1| \underset{(>)}{\leqq} |E e^{zX_k}-1| \underset{(-)}{+} 6(w_N(z))^2.$$

Proof. Applying the elementary symmetry property $\hat{E} Y_1 Y_2 \dots Y_k = \hat{E} Y_k \dots Y_2 Y_1$ for arbitrary *rv*'s, we get from (2.1)

$$E e^{zS_k} = \hat{E} e^{zX_1} \dots e^{zX_k} + \sum_{j=1}^{k-1} E e^{zS_j} \hat{E} e^{zX_{j+1}} \dots e^{zX_k}.$$

Taking into account that $\varphi_1(z) = E e^{zX_1}$ and $\varphi_j(z) = E e^{zS_j}/E e^{zS_{j-1}}$, $j=2, \dots, k-1$, we have

$$\frac{E e^{zS_k}}{E e^{zS_{k-1}}} = E e^{zX_k} + \sum_{j=1}^{k-1} \frac{\hat{E} e^{zX_j} \dots e^{zX_k}}{\varphi_j(z) \dots \varphi_{k-1}(z)}.$$

Finally, using Corollary 2.1, we obtain the representation (3.2). Now, our problem is to estimate $|\varphi_k(z)-1|$ in certain neighbourhood of $z=0$. We shall prove (3.3) by induction.

Clearly $|\varphi_1(z)-1| = |E e^{zX_1}-1|$. By Lemma 2.2 we conclude that

$$\begin{aligned} &|\hat{E}(e^{zX_j}-1) \dots (e^{zX_k}-1)| \\ &\quad \leqq 2^{k-j} (w_N(z))^{k-j-1} (E|e^{zX_{k-1}}-1|^2 E|e^{zX_k}-1|^2)^{1/2}. \end{aligned} \tag{3.4}$$

Let us assume that (3.3) holds for $|\varphi_j(z)-1|$, $j=1, \dots, k-1$. Then $z \in K_N$ implies that

$$\max_{1 \leqq j \leqq k-1} |\varphi_j(z)-1| \leqq w_N(z) + 6(w_N(z))^2 \leqq 2 w_N(z). \tag{3.5}$$

Note that

$$\frac{2w_N(z)}{1 - \max\limits_{1 \leqq j \leqq k-1} |\varphi_j(z)-1|} \leqq \frac{2w_N(z)}{1-2w_N(z)} \leqq \frac{1}{2}$$

and hence combining (3.4) and (3.5), we get

$$\begin{aligned} |\varphi_k(z)-1| &\leqq |E e^{zX_k}-1| + \frac{2(E|e^{zX_{k-1}}-1|^2 E|e^{zX_k}-1|^2)^{1/2}}{|\varphi_{k-1}(z)|} \\ &\quad \times \sum_{j=1}^{k-1} \left(\frac{2w_N(z)}{1 - \max\limits_{1 \leqq i \leqq k-1} |\varphi_i(z)-1|} \right)^{k-j-1} \leqq |E e^{zX_k}-1| \\ &\quad + \frac{2(E|e^{zX_{k-1}}-1|^2 E|e^{zX_k}-1|^2)^{1/2} (1-2w_N(z))}{(1-|\varphi_{k-1}(z)-1|)(1-4w_N(z))} \end{aligned}$$

which proves (3.6). In the same way one can show the validity of "$\geqq$" in (3.6). □

Lemma 3.2. *Let $X_1, \ldots, X_N$ be a sequence of 1-dependent rv's. Then the estimate*

$$\left|\log E e^{zS_N} - \sum_{k=1}^{N} E(e^{zX_k}-1) - \sum_{k=2}^{N} \hat{E}(e^{zX_{k-1}}-1)(e^{zX_k}-1)\right|$$
$$\leqq 2w_N(z)\left(\sum_{k=1}^{N} |E e^{zX_k}-1| + 22\sum_{k=1}^{N} E|e^{zX_k}-1|^2\right) \tag{3.6}$$

holds for each $z \in K_N$.

Proof. Using (3.1) and (3.2) we decompose the cumulant-generating function of S_N as follows:

$$\log E e^{zS_N} = \sum_{k=1}^{N} E(e^{zX_k}-1) + \sum_{k=2}^{N} \hat{E}(e^{zX_{k-1}}-1)(e^{zX_k}-1) + R_N(z),$$

where

$$R_N(z) = \sum_{k=1}^{N} (\log \varphi_k(z) - (\varphi_k(z)-1)) + \sum_{k=2}^{N}\sum_{j=1}^{k-1}\left(\frac{1}{\varphi_j(z)\ldots\varphi_{k-1}(z)}-1\right)$$
$$\times \hat{E}(e^{zX_j}-1)\ldots(e^{zX_k}-1) + \sum_{k=3}^{N}\sum_{j=1}^{k-2} \hat{E}(e^{zX_j}-1)\ldots(e^{zX_k}-1).$$

As an immediate consequence of (3.4), (3.5) and $z \in K_N$ we obtain the following estimates:

$$\left|\frac{1}{\varphi_j(z)\ldots\varphi_{k-1}(z)}-1\right| \leqq (k-j)\frac{2w_N(z)}{1-2w_N(z)}\left(1+\frac{2w_N(z)}{1-2w_N(z)}\right)^{k-j-1}$$
$$\leqq 3(k-j)w_N(z)(1+3w_N(z))^{k-j-1},$$

$$\left|\sum_{j=1}^{k-2} \hat{E}(e^{zX_j}-1)\ldots(e^{zX_k}-1)\right| \leqq 6w_N(z)(E|e^{zX_{k-1}}-1|^2 E|e^{zX_k}-1|^2)^{1/2},$$

$$\left|\sum_{j=1}^{k-1}\left(\frac{1}{\varphi_j(z)\ldots\varphi_{k-1}(z)}-1\right)\hat{E}(e^{zX_j}-1)\ldots(e^{zX_k}-1)\right|$$
$$\leqq 6w_N(z)\sum_{j=1}^{k-1}(k-j)(3w_N(z))^{k-j-1}(E|e^{zX_{k-1}}-1|^2 E|e^{zX_k}-1|^2)^{1/2}$$
$$\leqq 24w_N(z)(E|e^{zX_{k-1}}-1|^2 E|e^{zX_k}-1|^2)^{1/2}.$$

By these estimates we get

$$\sum_{k=1}^{N} |\varphi_k(z)-1| \leqq \sum_{k=1}^{N} |E e^{zX_k}-1|$$
$$+(2+30w_N(z))\sum_{k=2}^{N}(E|e^{zX_{k-1}}-1|^2 E|e^{zX_k}-1|^2)^{1/2}.$$

Using the above estimates again and the elementary inequality

$$|\log z-(z-1)| \leqq |z-1|^2 \quad \text{for } |z-1| \leqq 1/2,$$

we have

$$|R_N(z)| \leqq 2w_N(z) \sum_{k=1}^{N} |E e^{zX_k} - 1| + 44 w_N(z) \sum_{k=2}^{N} (E|e^{zX_{k-1}} - 1|^2 E|e^{zX_k} - 1|^2)^{1/2}.$$

This yields (3.6). □

Corollary 3.1. *Let* $X_1, \ldots, X_N$ *be a sequence of* 1*-dependent rv's. If* $t \in R^1$ *satisfies* $w_N(it) \leqq 1/6$, *then*

$$\left|\log E e^{itS_N} - \sum_{k=1}^{N} E(e^{itX_k} - 1) - \sum_{k=2}^{N} \hat{E}(e^{itX_{k-1}} - 1)(e^{itX_k} - 1)\right| \leqq 90 w_N(it) \sum_{k=1}^{N} |E e^{itX_k} - 1|.$$

Proof. In fact, Lemma 3.2 and the well-known inequality

$$E|e^{itX_k} - 1|^2 \leqq 2|E e^{itX_k} - 1|$$

yield the desired result. □

Corollary 3.2. *Let* $X_1, \ldots, X_N$ *be a sequence of* 1*-dependent rv's with* $EX_k = 0$ *and* $E|X_k|^3 < \infty$. *If* $|t| \leqq B_N^3/192NM_{3N}$, *then*

$$|f_N(it/B_N)| \leqq e^{-t^2/4}. \tag{3.7}$$

Proof. Noting that $\left|e^{itx} - \sum_{k=0}^{p-1} \frac{(itx)^k}{k!}\right| \leqq \frac{|tx|^p}{p!}$, $p = 1, 2, \ldots$ and $B_N^2 \leqq 3 \sum_{k=1}^{N} EX_k^2$, we have the following estimates: $w_N(it/B_N) \leqq 1/64$,

$$w_N(it/B_N) \sum_{k=1}^{N} |E e^{itX_k/B_N} - 1| \leqq \tfrac{1}{2}|t|^3 \frac{NM_{3N}}{B_N^3},$$

$$\sum_{k=2}^{N} \left|\hat{E}(e^{itX_{k-1}/B_N} - 1)(e^{itX_k/B_N} - 1) - \frac{(it)^2}{B_N^2} \hat{E}X_{k-1}X_k\right| \leqq 2\left(\frac{|t|}{B_N}\right)^3 \sum_{k=1}^{N} E|X_k|^3,$$

$$\sum_{k=1}^{N} \left|E e^{itX_k/B_N} - 1 - it\frac{EX_k}{B_N} - (it)^2 \frac{EX_k^2}{2B_N^2}\right| \leqq \frac{1}{6}\left(\frac{|t|}{B_N}\right)^3 \sum_{k=1}^{N} E|X_k|^3.$$

We conclude from Corollary 3.1 that

$$\left|\log E e^{itS_N/B_N} + \frac{t^2}{2}\right| \leqq \frac{283}{6}\left(\frac{|t|}{B_N}\right)^3 NM_{3N} \leqq \frac{t^2}{4},$$

which implies (3.7). □

A more delailed study of the terms $\log \varphi_k(it)$ leads to the following extension of Lemma 3.2.

Lemma 3.3. *Let $X_1, \dots, X_N$ be a sequence of 1-dependent rv's. If $w_N(it) \leqq 1/6$, then*

$$\Big|\log E e^{itS_N} - \sum_{k=1}^{N} ((E e^{itX_k} - 1) - \tfrac{1}{2}(E e^{itX_k} - 1)^2 + \tfrac{1}{3}(E e^{itX_k} - 1)^3)$$
$$+ \sum_{k=2}^{N} ((E e^{itX_{k-1}} - 1) + (E e^{itX_k} - 1) - 1)\hat{E}(e^{itX_{k-1}} - 1)(e^{itX_k} - 1)$$
$$- \sum_{k=3}^{N} \hat{E}(e^{itX_{k-2}} - 1)(e^{itX_{k-1}} - 1)(e^{itX_k} - 1)\Big| \leqq c_2 (w_N(it))^2 \sum_{k=1}^{N} |E e^{itX_k} - 1|. \quad (3.8)$$

One can verify the assertion of Lemma 3.3 by simple refinements in the proof of Lemma 3.2. To avoid tedious, but not difficult calculations we omit this proof.

Lemma 3.4. *Let $X_1, \dots, X_N$ be a sequence of 1-dependent rv's with $M_{pN} < \infty$. If $w_N(it) \leqq 1/6$ then*

$$\max_{1 \leqq k \leqq N} \left| \frac{d^p \varphi_k(it)}{dt^p} \right| \leqq C(p) M_{pN}, \quad (3.9)$$

where $C(p)$ is a constant only depending on p.

Proof. For brevity we put

$$g_{jk}^{(p)} = \frac{d^p}{dt^p} (\varphi_j(it) \dots \varphi_{k-1}(it))$$

and

$$f_{jk}^{(p)} = \frac{d^p}{dt^p} \hat{E}(e^{itX_j} - 1) \dots (e^{itX_k} - 1)$$

for $p = 0, 1, 2, \dots$. It is easily seen from (2.9) and (3.5) that

$$\left| \frac{f_{jk}^{(p)}}{g_{jk}} \right| \leqq 12^{2p} M_{pN} (k - j + 1)^p 2^{-(k-j)}, \qquad 1 \leqq j \leqq k. \quad (3.10)$$

Next we prove (3.9) for $p = 1$.

Assume that $|\varphi_q^{(1)}(it)| \leqq C(1) M_{1N}$ with $C(1) = 866$ holds for $q = 1, \dots, k-1$. In view of the representation formula (3.2) the first derivative of $\varphi_k(it)$ has the form

$$\frac{d}{dt} \varphi_k(it) = i E X_k e^{itX_k} + \sum_{j=1}^{k-1} \frac{f_{jk}^{(1)}}{g_{jk}} - \sum_{j=1}^{k-1} \frac{f_{jk}}{g_{jk}} \sum_{q=j}^{k-1} \frac{\varphi_q^{(1)}(it)}{\varphi_q(it)}.$$

Then by (3.4), (3.10) and our assumption

$$\left| \frac{d}{dt} \varphi_k(it) \right| \leqq M_{1N} + \sum_{j=1}^{k-1} M_{1N} \frac{(k-j+1) 12^2}{2^{k-j}} + 3 w_N(it) \sum_{j=1}^{k-1} \frac{(k-j) C(1)}{2^{k-j+1}} M_{1N}$$
$$\leqq M_{1N} \left(1 + 144 \sum_{k=1}^{\infty} (k+1) 2^{-k} + 3 C(1) w_N(it) \sum_{k=0}^{\infty} (k+1) 2^{-k-2}\right)$$
$$\leqq M_{1N} (433 + C(1)/2) \leqq C(1) M_{1N}.$$

Thus the last relation implies $|\varphi_k^{(1)}(it)| \leqq 866\, M_{1N}$ for every $k \geqq 1$.

To prove (3.9) for $p \geqq 2$ we state the following differentiation formulae:

$$\frac{d^p \varphi_k(it)}{dt^p} = i^p E X_k^p e^{itX_k} + \sum_{j=1}^{k-1} \frac{d^p}{dt^p} \left(\frac{f_{jk}(it)}{g_{jk}(it)}\right)$$

and

$$\frac{d^p}{dt^p}\left(\frac{f_{jk}(it)}{g_{jk}(it)}\right) = \frac{f_{jk}^{(p)}}{g_{jk}} - \sum_{q=1}^{p} \binom{p}{q} \frac{f_{jk}^{(q)}}{g_{jk}} \frac{g_{jk}^{(p-q)}}{g_{jk}} - \frac{f_{jk}}{g_{jk}} \sum_{q=j}^{k-1} \frac{\varphi_q^{(p)}(it)}{\varphi_q(it)}$$

$$-\frac{f_{jk}}{g_{jk}} \sum_{q=2}^{\min(p,k-j)} \sum_{\substack{p_1+\ldots+p_q=p \\ p_i \geqq 1}} \binom{p}{p_1, \ldots, p_q} \sum_{j \leqq j_1 < \ldots < j_q \leqq k-1} \frac{\varphi_{j_1}^{(p_1)}(it) \ldots \varphi_{j_q}^{(p_q)}(it)}{\varphi_{j_1}(it) \ldots \varphi_{j_q}(it)}$$

$$-\sum_{l=3}^{p+1} (-1)^{l-1} \sum_{\substack{p_1+(p_2+1)+\ldots+(p_l+1)=p \\ p_i \geqq 0}} \binom{p}{p_1, p_2+1, \ldots, p_l+1}$$

$$\times \frac{f_{jk}^{(p_1)}}{g_{jk}} \frac{g_{jk}^{(p_2+1)}}{g_{jk}} \cdots \frac{g_{jk}^{(p_l+1)}}{g_{jk}}.$$

Now one can verify (3.9) in the following way: It is assumed that (3.9) holds for $p=1, \ldots, s-1$ and $|\varphi_q^{(s)}(it)| \leqq C(s) M_{sN}$ for $q=1, \ldots, k-1$. Then by the above formulae one can show that $|\varphi_k^{(s)}(it)| \leqq C(s) M_{sN}$ and therefore (3.9) holds for $p=s$.

This yields (3.9) for every $p \geqq 1$. We remark that in accordance with the above differentiation formulae the constants $C(p)$, $p=1, 2, \ldots$ are successively calculated. □

Corollary 3.3. *Let $X_1, \ldots, X_N$ be a sequence of 1-dependent rv's with $M_{pN} < \infty$. If $w_N(it) \leqq 1/6$, then*

$$\max_{1 \leqq k \leqq N} \left| \frac{d^p}{dt^p} \log \varphi_k(it) \right| \leqq K(p) M_{pN}, \tag{3.11}$$

where

$$K(p) = p! \sum_{q=1}^{p} \left(\frac{3}{2}\right)^q \frac{1}{q} \sum_{\substack{p_1+\ldots+p_q=p \\ p_i \geqq 1}} \frac{C(p_1)}{p_1!} \cdots \frac{C(p_q)}{p_q!}.$$

Proof. The derivatives of $\log \varphi_k(it)$ are given by

$$\frac{d^p}{dt^p} \log \varphi_k(it) = p! \sum_{q=1}^{p} \frac{(-1)^{q-1}}{(\varphi_k(it))^q} \sum_{\substack{p_1+\ldots+p_q=p \\ p_i \geqq 1}} \frac{\varphi_k^{(p_1)}(it)}{p_1!} \cdots \frac{\varphi_k^{(p_q)}(it)}{p_q!}.$$

A simple application of Hölder's inequality shows the validy of (3.11). □

Remark 3.1. If $X_1, \ldots, X_N$ are m-dependent rv's with $M_{pN} < \infty$, then using (2.8), (3.11) and the well-known inequality

$$|a_1 + \ldots + a_m|^p \leqq m^{p-1}(|a_1|^p + \ldots + |a_m|^p)$$

we get for $|t| \leqq B_N / 6m \max\limits_{1 \leqq k \leqq N} (EX_k^2)^{1/2}$

$$\left| \frac{d^p}{dt^p} \log E e^{itS_N/B_N} \right| \leqq K(p) \frac{m^{p-1} N M_{pN}}{B_N^p}. \tag{3.12}$$

4. Results

Theorem 4.1. *Let $X_1, \ldots, X_N$ be a sequence of* 1*-dependent rv's with $EX_k=0$. Suppose that*

$$E|X_k|^p \leqq H \frac{EX_k^2 p!}{\delta_N^{p-2}}, \qquad p=2,3,\ldots; \quad k=1,\ldots,N, \tag{4.1}$$

with $H \geqq 1/2$ and $0<\delta_N \leqq (18HM_{2N})^{-1/2}$.

Then in the interval $0 \leqq x \leqq c_3 \Delta_N/H_N$ the following relation for large deviations holds:

$$\frac{P(S_N \geqq xB_N)}{1-\Phi(x)} = \exp\left\{x^3 \sum_{k=0}^{\infty} \lambda_{kN} x^k\right\} \left(1+O\left(\frac{H_N(1+x)}{\Delta_N}\right)\right)^4 \quad \textit{as } N \to \infty,$$

where

$$\Delta_N = \delta_N B_N/4, \qquad H_N = 80H \sum_{k=1}^{N} EX_k^2/B_N^2$$

and

$$\lambda_{kN} = \frac{1}{(k+2)(k+3)} \sum_{l=1}^{k+1} (-1)^{l-1} \binom{k+l+1}{l} \sum_{\substack{k_1+\ldots+k_l=k+1 \\ k_i \geqq 1}} \prod_{i=1}^{l} \frac{\Gamma_{k_i+2}(S_N/B_N)}{(k_i+1)!}$$

and

$$\Gamma_k(S_N) = \frac{1}{i^k} \frac{d^k}{dt^k} \log f_N(it)|_{t=0}.$$

Corollary 4.1. *Let $X_1, \ldots, X_N$ be a sequence of m-dependent ($m \geqq 1$) rv's with $EX_k=0$. Suppose that (4.1) is satisfied. Then the assertion of Theorem 4.1 remains valid if we replace H_N and Δ_N by*

$$\bar{H}_N = 80\,Hm \sum_{k=1}^{N} EX_k^2/B_N^2 \quad \textit{and} \quad \bar{\Delta}_N = \delta_N B_N/4m,$$

respectively.

Theorem 4.2. *Let $X_1, \ldots, X_N$ be a sequence of* 1*-dependent rv's with $EX_k=0$ and $E|X_k|^3<\infty$. Put $L_{3N}=NM_{3N}/B_N^3$. Suppose that the following conditions are satisfied:*

(A) $0<c_4 \leqq M_{3N}$, $B_N \to \infty$ *and* $L_{3N} \to 0$ *as* $N \to \infty$,

(B) *there exists a real number* α, $0<\alpha<1$, *such that*

$$\max_{1 \leqq k \leqq N} E|X_k|^3 \chi_{\{|X_k| \geqq L_{3N}^{-\alpha}\}} = o(M_{3N}) \quad \textit{as } N \to \infty,$$

(C) $$\int_{|t| \geqq \frac{1}{192 L_{3N}}} \frac{|E e^{itS_N/B_N}|}{|t|} dt = o(L_{3N}) \quad \text{as } N \to \infty.$$

---

4 $\Lambda_N(x) = \sum_{k=0}^{\infty} \lambda_{kN} x^k$ is the Cramér's power series.

Then the asymptotic expansion

$$F_N(x)-\Phi(x)-\frac{(1-x^2)}{6\sqrt{2\pi}}\frac{\Gamma_3(S_N)}{B_N^3}e^{-x^2/2}=o(L_{3N}) \quad as\ N\to\infty$$

holds uniformly in x.

Corollary 4.2. *Let $X_1,\dots,X_N$ be a sequence of m-dependent rv's with $EX_k=0$ and $E|X_k|^3<\infty$. Then the whole formulation of Theorem 4.2 remains valid if we replace L_{3N} by $\bar{L}_{3N}=m^2NM_{3N}/B_N^3$.*

5. Proofs of the Results

Proof of Theorem 4.1. (2.6) implies

$$|\hat{E}X_j^{p_j}\dots X_k^{p_k}|\leqq 2^{k-j}(E|X_j|^{2p_j}\dots E|X_k|^{2p_k})^{1/2},$$

and therefore by a short calculation one can show the analyticity of

$$\hat{E}(e^{zX_j}-1)\dots(e^{zX_k}-1)=\sum_{p=k-j+1}^{\infty} z^p \sum_{\substack{p_j+\dots+p_k=p\\ p_i\geqq 1}} \frac{\hat{E}X_j^{p_j}\dots X_k^{p_k}}{p_j!\dots p_k!},$$

$1\leqq j\leqq k$, $1\leqq k\leqq N$, for $|z|<\delta_N/2$.

The analyticity of $\varphi_1(z),\dots,\varphi_N(z)$ for $|z|\leqq\delta_N/4$ can be verified inductively by the defining formula (3.2). Thus $\log Ee^{zS_N}$ is an analytic function for $|z|\leqq\delta_N/4$. Again using (2.6) we have for $|z|\leqq\delta_N/4$

$$|Ee^{zX_k}-1|\leqq\sum_{p=2}^{\infty}\frac{|z|^p}{p!}E|X_k|^p\leqq\tfrac{4}{3}HEX_k^2|z|^2$$

and

$$E|e^{zX_k}-1|^2\leqq\sum_{p=2}^{\infty}\frac{|z|^p}{p!}2^pE|X_k|^p\leqq 8HEX_k^2|z|^2.$$

δ_N was chosen in such way that each z with $|z|\leqq\delta_N/4$ belongs to K_N. Next we apply Lemma 3.2 and use the above estimates:

$$\begin{aligned}|\log Ee^{zS_N}|&\leqq\sum_{k=1}^{N}|Ee^{zX_k}-1|+2\sum_{k=1}^{N}E|e^{zX_k}-1|^2\\&\quad+2w_N(z)\left(\sum_{k=1}^{N}|Ee^{zX_k}-1|+22\sum_{k=1}^{N}E|e^{zX_k}-1|^2\right)\\&\leqq 80H\sum_{k=1}^{N}EX_k^2|z|^2 \quad \text{for } |z|\leqq\delta_N/4.\end{aligned}$$

Together with the introduced notations we can state the following result. The cumulant-generating function $\log Ee^{zS_N/B_N}$ is an analytical function for $|z|\leqq\Delta_N$ such that

$$|\log Ee^{zS_N/B_N}|_{|z|=\Delta_N}|\leqq H_N\Delta_N^2.$$

Now we are in position to apply a lemma on large deviations which is due to V.A. Statulevičius [10, 11]. This lemma yields the assertion of Theorem 4.1. □

Proof of Corollary 4.1. In order to proof Corollary 4.1 we only repeat the proof of Theorem 4.1 for the 1-dependent *rv*'s (2.8). Taking into account the inequality

$$E|Y_k|^p \leqq m^{p-1} \sum_{j=1}^{m} E|X_{(k-1)m+j}|^p, \qquad k=1,\ldots,\left[\frac{N}{m}\right] \tag{5.1}$$

and the corresponding inequality for $k=\left[\frac{N}{m}\right]+1$, it is easy to show the analyticity of $\log Ee^{zS_N}$ for $|z|\leqq \delta_N/4m$ and the estimate

$$|\log Ee^{zS_N}|_{|z|=\bar{\Delta}_N}| \leqq \bar{H}_N \bar{\Delta}_N^2. \quad \square$$

Proof of Theorem 4.2. We first prove the following

Lemma 5.1. *Let $X_1,\ldots,X_N$ be a sequence of 1-dependent rv's with $EX_k=0$ and $E|X_k|^3<\infty$. Suppose that the conditions* (A) *and* (B) *are satisfied. Then, for $|t|\leqq L_{3N}^{-\beta}$, $0<\beta<1-\alpha$,*

$$\left|\log f_N(it/B_N)-\left(-\frac{t^2}{2}+\frac{(it)^3}{6}\Gamma_3(S_N)B_N^{-3}\right)\right| \leqq \varepsilon_N L_{3N}|t|^3.$$

Here and below, ε_N denotes a positive null sequence which may be differ from one expression to another.

Proof of Lemma 5.1. Since $|L_{3N}^{-\alpha}t|\leqq \varepsilon_N B_N$ by our assumptions, we have for $p=1,2,3$

$$E(e^{itX_k/B_N}-1)=\sum_{q=1}^{p}\left(\frac{it}{B_N}\right)^q\frac{EX_k^q}{q!}+\theta\frac{|t|^p}{p!B_N^p}(E|X_k|^p\chi_{\{|X_k|\geqq L_{3N}^{-\alpha}\}}$$
$$+\frac{|t|}{(p+1)B_N}E|X_k|^{p+1}\chi_{\{|X_k|<L_{3N}^{-\alpha}\}})=\sum_{q=1}^{p}\left(\frac{it}{B_N}\right)^q\frac{EX_k^q}{q!}+\theta\varepsilon_N\frac{|t|^p(M_{3N})^{p/3}}{p!B_N^p}.$$

The following estimates may be derived in like manner:

$$\hat{E}(e^{itX_{k-1}/B_N}-1)(e^{itX_k/B_N}-1)=\left(\frac{it}{B_N}\right)^2\hat{E}X_{k-1}X_k+\tfrac{1}{2}\left(\frac{it}{B_N}\right)^3$$
$$\times(\hat{E}X_{k-1}^2X_k+\hat{E}X_{k-1}X_k^2)+\theta\varepsilon_N|t|^3M_{3N}B_N^{-3},$$

$$\hat{E}(e^{itX_{k-1}/B_N}-1)(e^{itX_k/B_N}-1)(E(e^{itX_{k-1}/B_N}-1)+E(e^{itX_k/B_N}-1))$$
$$=\left(\frac{it}{B_N}\right)^3(EX_{k-1}+EX_k)\hat{E}X_{k-1}X_k+\theta\varepsilon_N|t|^3M_{3N}B_N^{-3},$$

$$\hat{E}(e^{itX_{k-2}/B_N}-1)(e^{itX_{k-1}/B_N}-1)(e^{itX_k/B_N}-1)$$
$$=\left(\frac{it}{B_N}\right)^3\hat{E}X_{k-2}X_{k-1}X_k+\theta\varepsilon_N|t|^3M_{3N}B_N^{-3},$$

$$w_N^2\left(\frac{it}{B_N}\right)\sum_{k=1}^{N}|Ee^{itX_k/B_N}-1|\leqq \varepsilon_N\left(\frac{|t|}{B_N}\right)^3 M_{3N}.$$

A short calculation shows that

$$\begin{aligned}\Gamma_3(S_N)=&\sum_{k=1}^{N}(EX_k^3-3EX_kEX_k^2+2(EX_k)^3)+3\sum_{k=2}^{N}(\hat{E}X_{k-1}^2X_k+\hat{E}X_{k-1}X_k^2)\\&-6\sum_{k=2}^{N}(EX_{k-1}+EX_k)\hat{E}X_{k-1}X_k+6\sum_{k=3}^{N}\hat{E}X_{k-2}X_{k-1}X_k\end{aligned}$$

and by Hölder's inequality

$$|\Gamma_3(S_N/B_N)|\leqq 13\sum_{k=1}^{N}\frac{E|X_k|^3}{B_N^3}\leqq 13\,L_{3N}. \tag{5.2}$$

Using Lemma 3.3 and the above estimates, it is rapidly seen that

$$\begin{aligned}\log f_N(it/B_N)=&\frac{it}{B_N}ES_N+\tfrac{1}{2}\left(\frac{it}{B_N}\right)^2 D^2S_N+\tfrac{1}{6}\left(\frac{it}{B_N}\right)^3\Gamma_3(S_N)\\&+\theta\varepsilon_N|t|^3L_{3N}. \quad\square\end{aligned}$$

We return to the proof of Theorem 4.2.

Making use of Lemma 5.1 and (5.2) we see that for $|t|\leqq L_{3N}^{-\beta}$

$$\begin{aligned}f_N(it/B_N)=&e^{-t^2/2}\Bigg\{1+\frac{(it)^3}{6}\Gamma_3(S_N/B_N)+\theta_1\varepsilon_N L_{3N}|t|^3\\&+\theta_2\left|\frac{(it)^3}{6}\Gamma_3(S_N/B_N)+\varepsilon_N|t|^3L_{3N}\right|^2\exp\left(\frac{|t|^3}{6}|\Gamma_3(S_N/B_N)|+\varepsilon_N L_{3N}|t|^3\right)\Bigg\}\\=&e^{-t^2/2}\left(1+\frac{(it)^3}{6}\Gamma_3(S_N/B_N)\right)+\theta\varepsilon_N L_{3N}(|t|^2+|t|^5)e^{-t^2/4}.\end{aligned} \tag{5.3}$$

Now we quote a lemma which is needed for the proof of Theorem 4.2.

Lemma 5.2. (see [8], p. 13). *Let $F_1(x)$, $F_2(x)$ be functions of bounded variation on the real line,*

$$\int_{-\infty}^{\infty}|x|\,|dF_k(x)|<\infty,\qquad k=1,2.$$

Further, suppose that $R(-\infty)=R(+\infty)=0$ and $\int_{-\infty}^{\infty}\left|\frac{r(t)}{t}\right|dt<\infty$, where $R(x)=F_1(x)-F_2(x)$ and $r(t)=\int_{-\infty}^{\infty}e^{itx}\,dR(x)$. Then

$$\tfrac{1}{2}(R(x-0)-R(x+0))=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-itx}}{-it}r(t)\,dt$$

for every x.

If we take in Lemma 5.2

$$F_1(x)=F_N(x) \quad \text{and} \quad F_2(x)=\Phi(x)+\frac{(1-x^2)\Gamma_3(S_N)}{6\sqrt{2\pi}\,B_N^3}\,e^{-x^2/2},$$

it remains to show that the integral

$$I_N=\int_{-\infty}^{\infty}\left|f_N(it/B_N)-e^{-t^2/2}\left(1+\frac{(it)^3}{6}\Gamma_3(S_N/B_N)\right)\right||t|^{-1}\,dt$$

equals to the order $o(L_{3N})$ as $N\to\infty$.

Splitting the domain of integration we obtain

$$\begin{aligned} I_N \leqq & \int_{|t|\leqq L_{3N}^{-\beta}}\left|f_N(it/B_N)-e^{-t^2/2}\left(1+\frac{(it)^3}{6}\Gamma_3(S_N/B_N)\right)\right||t|^{-1}\,dt \\ &+\int_{|t|\geqq(192L_{3N})^{-1}}\left|\frac{f_N(it/B_N)}{t}\right|dt+\int_{L_{3N}^{-\beta}<|t|<(192L_{3N})^{-1}}\left|\frac{f_N(it/B_N)}{t}\right|dt \\ &+\int_{|t|>L_{3N}^{-\beta}} e^{-t^2/2}\left|1+\frac{(it)^3}{6}\Gamma_3(S_N/B_N)\right||t|^{-1}\,dt. \end{aligned}$$

By (3.7), (5.3) and (C) it is easily seen that $I_N=o(L_{3N})$ as $N\to\infty$. This completes the proof of Theorem 4.2. □.

For the proof of Corollary 4.2 we write $S_N=\sum_{k=1}^{[N/m]+1} Y_k$ with the 1-dependent rv's (2.8). Then the assertion of Corollary 4.2 can be proved by using (5.1) and the technique in the proof of Theorem 4.2.

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