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A Class of Random Variables which are not Continuous Functions of Several Independent Random Variables

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Let $(\Omega, \mathscr{F}, P)$ be a probability space. A random variable $Z\colon \Omega \to R$ is $F_{m,n}$-distributed, if it has a density

$$f_{m,n}(x) = C(m,n)\, x^{\frac{m}{2}-1} (mx+n)^{-\frac{m+n}{2}}, \qquad x>0,$$
$$=0, \qquad x \leqq 0, \tag{1}$$

where $C(m,n) = \Gamma\left(\frac{m+n}{2}\right) m^{\frac{m}{2}} n^{\frac{n}{2}} \Big/ \left(\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)\right)$. If $X\colon \Omega \to R$ and $Y\colon \Omega \to R$ are independent and χ^2-distributed, with m and n degrees of freedom respectively, then the r.v.

$$Z = (X/m)/(Y/n) \tag{2}$$

is $F_{m,n}$-distributed ([2], p. 167). Our main result shows that there are many r.v.'s having a density of the form (1) which cannot be written in the form (2) on the same space Ω. The result is more general so that it applies in an analogous way to r.v.'s having e.g. a t- or a χ^2-distribution ([2], pp. 166–167, 170–171). The result applies also to the representations of r.v.'s with infinitely divisible distributions.

Consider the space $(\Omega, \mathscr{F}, P)$. Let $g\colon \Omega \to R$ be a r.v. which is *injective* and has the property: $g(F)$ is a Borel set for all $F \in \mathscr{F}$. It can be shown that if Ω is a *Polish* space or a *Lusin* space, then the above property holds for any g which is injective and continuous (see [1], p. 135). Denote by P_g the probability measure induced by g on R.

For $i=1,2$, let I_i be an interval on the real line, and $\mathscr{B}(I_i)$ be the corresponding set of Borel sets. Consider a measurable mapping $(X,Y)\colon \Omega \to I_1 \times I_2$. This defines two r.v.'s $X\colon \Omega \to I_1$ and $Y\colon \Omega \to I_2$. Moreover, X induces a probability P_X on I_1 and Y induces a probability P_Y on I_2.

Theorem. *Assume that X and Y are independent and that both the support of P_X and of P_Y have a non-empty interior. Then there is no continuous function*

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$f\colon I_1 \times I_2 \to R$ such that

$$g(\omega) = f(X(\omega), Y(\omega)) \quad a.s.\ (P). \tag{3}$$

Proof. Assume that (3) does hold. Let $\Omega' \subset \Omega$ be a measurable set where $g(\omega) = f(X(\omega), Y(\omega))$ with $P(\Omega') = 1$. Define functions g_0, f_0 and (X_0, Y_0) as follows:

$$g_0\colon \Omega' \to g(\Omega'), \qquad g_0(\omega) = g(\omega),$$

$$(X_0, Y_0)\colon \Omega' \to (X, Y)(\Omega'), \qquad (X_0(\omega), Y_0(\omega)) = (X(\omega), Y(\omega)),$$

$$f_0\colon (X, Y)(\Omega') \to g(\Omega'), \qquad f_0(x, y) = f(x, y).$$

These are essentially the restrictions of g, f and (X, Y). Since g_0 is bijective, and (X_0, Y_0) and f_0 are surjective, we have that (X_0, Y_0) and f_0 are also bijective. By assumption the set $g(\Omega')$ is measurable.

Let μ be the probability on $I_1 \times I_2$ induced by (X, Y). The independency of X and Y implies:

$$\mu(C \times D) = P_X(C) P_Y(D) \qquad \text{for all } C \times D \in \mathscr{B}(I_1) \otimes \mathscr{B}(I_2).$$

Since the supports of P_X and P_Y both have a non-empty interior, we can find two numbers a and b such that the sets

$$C_a = \{\omega_1 \in I_1 \mid \omega_1 \geqq a\},$$

$$D_b = \{\omega_2 \in I_2 \mid \omega_2 \geqq b\}$$

satisfy: $0 < P_X(C_a) < 1$ and $0 < P_Y(D_b) < 1$. It follows that the probability of each of the sets $C_a \times D_b$, $C_a^c \times D_b$, $C_a \times D_b^c$, $C_a^c \times D_b^c$ is strictly between zero and one. Furthermore there is an $\varepsilon > 0$ such that simultaneously

$$0 < \mu([a, a+\varepsilon) \times [b, b+\varepsilon)) < \mu(C_a \times D_b),$$

$$0 < \mu((a-\varepsilon, a) \times [b, b+\varepsilon)) < \mu(C_a^c \times D_b),$$

$$0 < \mu([a, a+\varepsilon) \times (b-\varepsilon, b)) < \mu(C_a \times D_b^c),$$

$$0 < \mu((a-\varepsilon, a) \times (b-\varepsilon, b)) < \mu(C_a^c \times D_b^c).$$

In other words, the "quadrants" of the rectangular ε-neighborhood

$$N_\varepsilon(a, b) = (a-\varepsilon, a+\varepsilon) \times (b-\varepsilon, b+\varepsilon)$$

satisfy these inequalities.

Since f is continuous and $C_a \times D_b$, $C_a^c \times D_b$, $C_a \times D_b^c$, $C_a^c \times D_b^c$ are connected, the sets $f(C_a \times D_b)$, $f(C_a^c \times D_b)$, $f(C_a \times D_b^c)$, $f(C_a^c \times D_b^c)$ must be intervals. Denote them in the above order by J_1, J_2, J_3, J_4. These intervals are a.s. (P_g) disjoint, because if e.g. we would have $P_g(J_1 \cap J_2) > 0$, then we would also have $P_g(J_1 \cap J_2 \cap g(\Omega')) > 0$, which is a contradiction since

$$\begin{aligned} P_g(J_1 \cap J_2 \cap g(\Omega')) &= P(g_0^{-1}(J_1 \cap J_2)) \\ &= P((X_0, Y_0)^{-1}(C_a \times D_b) \cap (X_0, Y_0)^{-1}(C_a^c \times D_b)) = 0. \end{aligned}$$

These intervals must be in some order on the real line so that we may assume e.g. that the elements of J_1 are smaller than those of J_2 (except perhaps for the points of their intersection, which has P_g-measure zero), those of J_2 smaller than those of J_3, and those of J_3 smaller than those of J_4.

Consider now the "quadrants" of $N_\varepsilon(a,b)$. The set $[a,a+\varepsilon)\times[b,b+\varepsilon)$ is a connected subset of $C_a\times D_b$. Hence $f([a,a+\varepsilon)\times[b,b+\varepsilon))$ is an interval. Denote it by J_1'. Let $J_1''=J_1-J_1'$. With an analogous notation we decompose all J_i's:

$$J_i=J_i'\cup J_i'',\qquad i=1,\dots,4,$$

where J_i' is an image of a "quadrant". Then

$$f(N_\varepsilon(a,b))=J_1'\cup J_2'\cup J_3'\cup J_4'.$$

This is a connected set since $N_\varepsilon(a,b)$ is connected. Consider now J_1' and J_3'. With the order of J_i's that we have the set J_2'' is between them. This is a contradiction, since $P_g(J_2'')>0$. □

Remark. If X and Y have continuous densities, then the condition on the support of P_X and P_Y is satisfied.

Example 1. Take $\Omega=R_+$, $P=$measure corresponding to $F_{m,n}$-distribution and g $=$identity mapping. Let $f\colon R_+\times R_+\to R_+$ be defined by

$$f(x,y)=(x/m)/(y/n).$$

Obviously f is continuous. Let X and Y be any two independent χ^2-distributed r.v.'s on Ω. Then the theorem shows that it is not true that

$$g(\omega)=f(X(\omega),Y(\omega))\ \text{a.s.}$$

Example 2. Let $\Omega=R_+$, $g\colon\Omega\to R_+$ be strictly increasing and choose P so that g has χ^2-distribution with two degrees of freedom. Define $f\colon R^2\to R$ by

$$f(x,y)=x^2+y^2.$$

If $X\colon\Omega\to R$ and $Y\colon\Omega\to R$ are any two independent r.v.'s with $N(0,1)$-distributions then by our theorem it is not true that $g(\omega)=f(X(\omega),Y(\omega))$ a.s. Note also that this result can easily be extended to χ^2-distributions with $n>2$ degrees of freedom. Namely if g is as before but now with n degrees of freedom, and if we would have

$$g(\omega)=X_1^2(\omega)+\dots+X_n^2(\omega)\ \text{a.s.},$$

where X_i's are independent and $N(0,1)$-distributed, then by taking $X=X_1^2$, Y $=X_2^2+\dots+X_n^2$ and $f(x,y)=x+y$ we arrive at a contradiction with our theorem.

Let $g\colon\Omega\to R$ be as in the theorem and suppose that the support of P_g has a non-empty interior. Assume that r.v.'s $X\colon\Omega\to I_1$ and $Y\colon\Omega\to I_2$ are not only independent but also *identically distributed.* If the support of P_X has an empty interior, then so does the support of P_Y. In that case it is not hard to see that we cannot have a continuous function f such that (3) would hold. If the support of P_X does have a non-empty interior, then the impossibility of (3) follows from the theorem. Thus we have the following corollary.

Corollary. *Let g be as in the theorem and assume that the support of P_g has a non-empty interior. Then g cannot be represented as a finite sum of i.i.d. random variables.*

This corollary shows that in the case of infinitely divisible distributions we often have to go outside our original space to find such independent r.v.'s that their sum would have the same distribution as our original variable.

Example 3. Let g be as in the corollary and define a continuous function f: $R^2 \to R$ by $f(x, y) = x$. Suppose $X = g$, and let Y: $\Omega \to R$ be any other r.v. such that the measure P_Y has a non-empty interior. Then it follows from the theorem that X and Y cannot be independent.

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