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# Dirichlet Forms and Diffusion Processes on Rigged Hilbert Spaces*

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## 1. Introduction

This paper is concerned with the extension to a real separable Hilbert space $K$ of the finite dimensional theory of Dirichlet diffusion forms (energy forms) $\int \nabla f \nabla f \, d\mu$, of the associated infinitesimal generators of symmetric Markov processes and of the processes themselves. In our case $\mu$ is a (non necessarily Gaussian) measure associated with the Hilbert space $K$ through a rigging $Q \subset K \subset Q'$, where $Q$ is a locally convex complete real vector space densely contained in $K$, with dual $Q'$. The Dirichlet form defines a self-adjoint operator in $L_2(d\mu)$, which generates a symmetric stationary Markov process satisfying a stochastic differential equation with drift coefficient in $L_2(d\mu)$. Besides their intrinsic probabilistic and functional analytic interest, such processes and the correspondent potential theory have also interest from the point of view of quantum mechanics. In the finite dimensional case the relation is immediate through the well known connection between the heat equation and the Schrödinger equation, and Dirichlet forms permit e.g. to define the Hamiltonians of quantum mechanical systems with very singular potentials. In the infinite dimensional case the Dirichlet forms and the associated self-adjoint operators and diffusion processes are related to quantum fields, e.g. in the case of certain measures $\mu$ on $Q' = \mathscr{S}'(R^n)$. These processes can be looked upon as a particularly interesting class of homogeneous generalized random fields with values in $\mathscr{S}'(R^{n+1})$, namely they are such as to be symmetric stationary Markov looked upon as processes indexed by the real line and with state space $\mathscr{S}'(R^n)$. Although our present study has its roots in our previous work [1], the knowledge of that work is not necessary for reading the present one.[1] Some references concerning work from some other points of view on stochastic differential equations,

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[1] The present paper constitutes, together with the paper under Ref. [88], the reference number 4 in Reference [1]

stochastic processes and their relations to differential operators in infinite dimensional spaces are e.g. [2–14] (and references therein).

We shall now briefly discuss the content of the different sections of our present paper.

In Section 2 we study Dirichlet forms on a rigged separable real Hilbert space $K$, the rigging $Q \subset K \subset Q'$ being as above. We recall that for any probability measure $\mu$ on $Q'$ which is quasi invariant under translations by elements of $Q$ two strongly continuous unitary representations $q \to U(q)$ and $q \to V(q)$ of $Q$ in $L_2(d\mu)$ are defined, such that $U$ and $V$ satisfy the Weyl commutation relations. Such representations have been studied intensively before, see e.g. [15–23]. We have

$$(V(q)f)(\xi) = \left(\frac{d\mu(\xi+q)}{d\mu(\xi)}\right)^{\frac{1}{2}} f(\xi+q),$$

for any $f \in L_2(d\mu)$, $q \in Q$, $\xi \in Q'$.

Let $i\pi(q)$ be the infinitesimal generator of the unitary group $V(tq)$, $t \in R$. Let $\mathscr{P}_1(Q')$ be the space of all quasi invariant probability measures on $Q'$ with the property that the function 1 is in the domain of $\pi(q)$ for all $q \in Q$. $\mathscr{P}_1(Q')$ is the space of measures considered henceforth. The gradient $q \cdot \nabla$ in the direction $q$ is defined in a natural way, hence also the closed map $f \to \nabla f$ from a dense subset $W^1$ of $L_2(d\mu)$ into $K \hat{\otimes} L_2(d\mu)$. The Dirichlet form we consider is then defined as the closed positive form $\int \nabla \bar{f} \cdot \nabla f \, d\mu$ in $L_2(d\mu)$.[2]

We study the correspondent self-adjoint operator $H = \nabla^* \bar{\nabla}$. In particular we exhibit its $Q$-ergodic decomposition induced by the ergodic decomposition of $\mu$ with respect to translations by elements in $Q$. We also give a definition of a Laplacian on $L_2(d\mu)$, for $\mu$ in a subspace $\mathscr{P}_1(Q')$, as a self-adjoint positive operator.

We start Section 3 by proving that the semigroup $e^{-tH}$, $t \geqq 0$ generated by $H$ in $L_2(d\mu)$ is positivity preserving, i.e. it is a Markov semigroup. Although a direct proof could also be given, we do the proof by reduction to finitely many dimensions, followed by application of the general theory of symmetric processes in $R^n$ generated by Dirichlet forms given in a series of papers by Fukushima, extending classical work of Beurling and Deny, see [27–29] and references therein. We do the reduction to finitely many dimensions because it provides useful additional information, and in fact we shall use this reduction also further on. For this reduction we assume that $Q'$ is such that regular conditional probability measures with respect to subalgebras generated by finite codimensional subspaces exist (which is e.g. the case when $Q'$ is a Suslin space). We then associate with the Markov semigroup $e^{-tH}$ in $L_2(d\mu)$ a homogeneous Markov process $\xi(t)$ with invariant measure $\mu$ and infinitesimal generator $H$. This process can be realized, e.g. with state space a suitable compactification $\bar{Q}'$ of $Q'$ ($L_2(d\mu)$ being separable, we can take $\bar{Q}'$ as the Gelfand spectrum of the uniform closure of a separable subalgebra of smooth cylinder functions on $Q'$, containing the unit and separating points). This realization is discussed in [64] and [62]. Another realization with state space $Q'$ is constructed from the

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[2] Some contexts where related Dirichlet forms appear are [1, 17, 22–26]

transition probability functions given by $e^{-tH}$ and the initial distribution $d\mu$. Here assumptions on $(Q',\mu)$ are needed, e.g. $\mu$ a Radon measure on the Suslin space $Q'$ ([58]). We then show that $\xi(t)$ solves, in the sense of weak processes on $Q'$ ([30–32]) the stochastic differential equation of a diffusion process

$$d\xi(t)=\beta(\xi(t))\,dt+dw(t),$$

where $w(t)/\sqrt{2}$ is the standard Wiener process on $K$ and the drift coefficient $\beta$ (the osmotic velocity $\beta(\cdot)$, in the sense of [1]) is such that $q\cdot\beta=2i\pi(q)1$. In the proof a suitable characterization of the standard Wiener process on $R$ is used. Note that $\beta$ is, in general, neither Lipschitz nor bounded.[3] We continue Section 3 by giving the time ergodic decomposition of the process $\xi$ and of its generator $H$. We also compare the time-ergodic and $Q$-ergodic decompositions and show that the former is in general strictly finer than the latter. We give a sufficient condition for the measure $\mu$ in order for the two ergodic decompositions to be equivalent. The condition, called strict positivity, is that the conditional measures obtained from $\mu$ by conditioning with respect to closed subspaces of codimension one be bounded away from zero on compacts of the corresponding one-dimensional subspaces. Two simple criteria for strict positivity of $\mu$ are then given. The first requires 1 to be an analytic vector for $\pi(q)$ and that $\pi(q)^n\cdot 1\in D(q\cdot V)$ for all $q\in Q$. The second requires a gap at the bottom of the spectrum of $H$ and a simple estimate involving the multiple commutators of $\pi(q)$ with $H$.

We also prove in Section 3 continuity properties of the paths $\xi(t)$ in natural Banach norms, for a class of measures $\mu$ in $\mathscr{P}_1(Q')$. We use here results from Gross theory of abstract Wiener spaces (see e.g. [6, 7]). Our results on continuity properties give an extension of the corresponding ones of Stroock-Varadhan [35] to processes with infinite dimensional state space.

We end Section 3 with a general sufficient condition for a measure to be quasi invariant, a result whose usefulness is illustrated by the subsequent sections.

In Section 4 we study the behaviour of Dirichlet forms and operators under weak convergence of measures. If the adjoint of a natural gradient operator in $L_2(d\mu)$ is densely defined, we call $\mu$ admissible, and a criterium for the admissibility of weak limits of measures is given. A concept of analytic and equi analytic probability measures is defined and the latter are shown to yield analytic measures in the weak limit which are $Q$-quasi invariant and, if admissible, are also such that $1\in D(\pi(q))$, for all $q\in Q$. Note that analytic measures, conditioned with respect to cofinite subspaces, have densities which are analytic in a strip.

In Section 5 we apply the results of the previous sections to the case of two space-time dimensional quantum field theoretical models. For these applications the rigging is given by the real spaces $Q=\mathscr{S}(R)$, $K=L_2(R)$, $Q'=\mathscr{S}'(R)$. We show in particular for the weakly coupled $P(\varphi)_2$ models ([38, 39]), the $P(\varphi)_2$ model

[3] In fact the drift-coefficient $\beta$ is, even for $K$ finite dimensional, more singular (just $L_2(d\mu)$) than the ones usually considered in the theory of stochastic differential equations. See e.g. [33–34]. The finite dimensional case is studied in more details from our point of view in [63]

with Dirichlet boundary conditions and isolated vacuum ([40–43]) and the exponential interaction models [44, 45], that the physical vacuum measure restricted to the $\sigma$-algebra generated by the time zero fields is an admissible analytic quasi invariant measure $\mu$ on $\mathscr{S}'(R)$ such that 1 is in the domain of the canonical momentum $\pi(\varphi)$, $\varphi\in\mathscr{S}(R)$ in $L_2(\mathscr{S}'(R), d\mu)$.

The restriction to time zero fields of the physical Hamiltonian of above Wightman field models coincides as a form on the dense domain $FC^2$ of finitely based twice continuously differentiable functions with $\frac{1}{2}\times$ the Dirichlet operator $H$ given by the Dirichlet from $\int \nabla\bar{f}\cdot\nabla f\, d\mu$.[4] One has the ergodic decompositions as well as the stochastic differential equation for the Markov process $\xi(t,\cdot)$ with state space $\mathscr{S}'(R)$, infinitesimal generator $H$ and invariant measure $\mu$:

$$d\xi(x,t)=\tfrac{1}{2}\beta(\xi(t))(x)\,dt+dw(x,t),$$

$w(\cdot,t)$ being the standard Wiener process on $\mathscr{S}'(R)$ and $\beta(\cdot)=2i\pi(\cdot)\cdot 1$ the osmotic velocity corresponding to the measure $\mu$.

## 2. The Dirichlet Form and the Dirichlet Operator

In this section we give an extension to the infinite dimensional case of the construction of an Hamiltonian by Dirichlet forms in finite dimensions.

The extension is such as to give, as in the finite dimensional case, the infinitesimal generator of a Markov diffusion semigroup, yielding a diffusion process.

We shall say that a real separable Hilbert space $K$ is rigged if there exists a real locally convex complete vector space $Q$ such that

$$Q\subset K\subset Q', \tag{2.1}$$

where $Q'$ is the dual space of $Q$ and such that $Q$ is densely contained in $K$ and $Q'$ respectively and the inner product $(\ ,\ )$ in $K$ coincides on $Q\times K$ with the dualization between $Q$ and $Q'$.

In this case the inner product $(\ ,\ )$ on $Q\times K$ extends by continuity in the last variable to $Q\times Q'$ and this extension coincides with the dualization between $Q$ and $Q'$. Hence we shall denote the dualization between $Q$ and $Q'$ by $(q,\xi)$, $q\in Q$, $\xi\in Q'$.

Let $\mathscr{P}(Q')$ be the space of probability measures defined on the $\sigma$-algebra generated by the weak*-topology. We shall say that $\mu\in\mathscr{P}(Q')$ is quasi invariant if, for any $q\in Q$, $d\mu(\xi)$ and $d\mu(\xi+q)$ are equivalent as measures, and we shall let $\mathscr{P}_0(Q')$ denote the subset of quasi invariant probability measures.

Let now $\mu\in\mathscr{P}(Q')$, then on $L_2(d\mu)$ we have a representation $U(q)$ of $Q$ by unitary operators with the cyclic vector $\Omega(\xi)\equiv 1$, given by

$$(U(q)f)(\xi)=e^{i(q,\xi)}f(\xi). \tag{2.2}$$

[4] It also coincides with $\frac{1}{2}\times$ the "diffusion operator given by $\mu$", in the terminology of Theorem 2.7 in Reference [1]

We have easily that $q \to U(q)$ is a strongly continuous representation of $Q$, because for $f \in L_\infty(d\mu)$ we have

$$\|(U(q)-1)f\|_2^2 \leqq 2\|f\|_\infty^2 (1 - Re \int_{Q'} e^{i(q,\xi)} d\mu(\xi)), \tag{2.3}$$

which shows that $U(q)$ is strongly continuous since $L_\infty(d\mu)$ is dense in $L_2(d\mu)$.

If moreover $\mu \in \mathscr{P}_0(Q')$, then we also have another representation of $Q$. Since $d\mu(\xi+q)$ and $d\mu(\xi)$ are equivalent we know that

$$\alpha(\xi, q) = \frac{d\mu(\xi+q)}{d\mu(\xi)} \tag{2.4}$$

is a non negative $L_1$-function, and if we define

$$(V(q)f)(\xi) = \alpha^{\frac{1}{2}}(\xi, q) f(\xi+q) \tag{2.5}$$

then $q \to V(q)$ is again a unitary representation of $Q$ on $L_2(d\mu)$, which is not necessarily continuous. However, it is always ray continuous i.e. $V(tq)$ is strongly continuous in $t$, $t \in R$.

*Remark.* If $Q$ is a Fréchet space or a strict inductive limit of Fréchet spaces then $q \to V(q)$ is also strongly continuous i.e. for any $f \in L_2(d\mu)$ the mapping $q \to V(q)f$ is strongly continuous. For this result see [20].

One sees easily (in the general case) that $U$ and $V$ satisfy the Weyl-commutation relation

$$V(p)\, U(q) = e^{i(p,q)}\, U(q)\, V(p). \tag{2.6}$$

We have obviously that $(q, \xi) \equiv \xi(q)$ is the infinitesimal generator for the one parameter unitary group $U(tq)$. Let $\pi(q)$ be the infinitesimal generator of the unitary group $V(tq) = e^{it\pi(q)}$, and let $\Omega \in L_2(d\mu)$ be the function $\Omega(\xi) = 1$.

We shall say that $\mu \in \mathscr{P}_0(Q')$ is $n$-times differentiable if $\Omega$ is in the domain of $\pi(q_1) \circ \cdots \circ \pi(q_n)$ for all $n$-tuples $q_1, \ldots, q_n$ in $Q$, and the subset of $n$-times differentiable probability measures will be denoted by $\mathscr{P}_n(Q')$.[5] We shall also say that $\mu \in \mathscr{P}_0(Q')$ is in $\mathscr{P}_\omega(Q')$ if $\Omega$ is an analytic vector for $\pi(q)$, for all $q \in Q$. Let now $\mu \in \mathscr{P}_1(Q')$ then

$$\beta \cdot q = 2i\pi(q)\Omega \tag{2.7}$$

is a linear mapping from $Q$ into $L_2(d\mu)$, and we denote by $\beta(\xi) \cdot q$ the value of the image function at the point $\xi \in Q'$. We call $\beta$ the drift coefficient or osmotic velocity given by $\mu$.

*Remark 1.* The mapping $q \to \beta \cdot q$ is not necessarily continuous. We have though by Prospositions 2.3 and 2.5 of Reference [1] that if $Q$ is a countabley normed space then $q \to \beta \cdot q$ is continuous, and if $Q$ is a nuclear space then $\beta$ is actually

[5] The measures in $\mathscr{P}_1(Q')$ were called "measures with regular first order derivatives" in Reference [1]. Let us also take the opportunity to correct a misprint in Reference [1]: in the Remark following Definition 2.2, $U$ should be replaced by $(U, V)$

given by a measurable mapping $\beta(\xi)$ from $Q'$ to $Q'$ so that $(q, \beta(\xi))$ is the value of $\beta \cdot q$ at the point $\xi$.

Let now $R$ be a finite dimensional subspace of $Q$. Then the orthogonal projection $P_R$ in $K$ with range $R$ extends by continuity to a continuous projection from $Q'$ into $Q$ with range $R$. This because if $r_1, \ldots, r_n$ is an orthonormal base in $R$ then for any $k \in K$ we have that

$$P_R k = \sum_{i=1}^{n} (r_i, k) r_i,$$

which obviously extends by continuity.

We shall say that a measurable function $f$ on $Q'$ is finitely based if there is a finite dimensional subspace $R$ of $Q$ such that $f(\xi) = f(P_R \xi)$. Moreover we shall say that a finitely based function $f$ is in $FC^n(Q')$ if its restriction to its base $R$ is in $C_b^n(R)$ i.e. $n$-times continuously differentiable with bounded derivatives of order $j = 0, 1, \ldots, k$. This definition is obviously independent of the choice of $R$.

We shall say that a function $f \in C(Q')$ is in $C^n(Q')$ if, for any $\xi \in Q'$ and any $q \in Q$, $f(\xi + tq)$ is $n$-times continuously differentiable functions of $t$ and at $t = 0$ all the derivatives are in $C(Q')$. If $f \in C^1$ we define

$$(q \cdot \nabla f)(\xi) = \frac{d}{dt} f(\xi + tq)\bigg|_{t=0}, \tag{2.8}$$

so that the operator $q \cdot \nabla$ in $L_2(d\mu)$ has a dense domain, containing $C_b^1$.

We see that if $\mu \in \mathscr{P}_1(Q')$ then $C_b^1(Q')$ is contained in the domain of $\pi(q)$ for all $q \in Q$ and for $f \in C_b^1$ we have

$$\pi(q) = \frac{1}{i}(q \cdot \nabla f + \tfrac{1}{2} \beta \cdot q f). \tag{2.9}$$

The operator $q \cdot \nabla$ has a densely defined adjoint, namely $-q \cdot \nabla - \beta \cdot q$, whose domain contains $C_b^1$ and is therefore dense in $L_2(d\mu)$. Hence $q \cdot \nabla$ is closable i.e. the associated form $\|q \cdot \nabla f\|^2$ is closable. We shall henceforth denote the closure of $q \cdot \nabla$ by the same symbol.

We remark incidentally in the next Lemma that $D(q \cdot \nabla) \cap FL_\infty$ is a core for $q \cdot \nabla$ i.e. it is dense in $D(q \cdot \nabla)$ in the graph norm, where $FL_\infty$ denotes the finitely based functions which belong to $L_\infty(d\mu)$.

**Lemma 2.1.** *If $\mu \in \mathscr{P}_1(Q')$ then $FL_\infty \cap D(q \cdot \nabla)$ is a core for $q \cdot \nabla$ and for the corresponding quadratic from $\|q \cdot \nabla f\|^2$, for any $q \in Q$.*

*Proof.* See Appendix.

Let now $f \in D(q \cdot \nabla)$ for all $q \in Q$. For any finite dimensional subspace $R \subset Q$ we define

$$(f, f)_1^R = \sum_{i=1}^{n} \|e_i \cdot \nabla f\|_2^2, \tag{2.10}$$

where $e_1, \ldots, e_n$ is an orthonormal basis for $R$. Actually (2.10) is independent of the particular basis $e_1, \ldots, e_n$ chosen. Since the forms $\|e_i \cdot \nabla f\|_2^2$ are closed, so is

$(f,f)_1^R$ a closed quadratic form. We have obviously that if $R \subset R'$ then $(f,f)_1^R \leqq (f,f)_1^{R'}$.

Consider now the net $\mathcal{N}$ of finite dimensional subspaces of $Q$. The function $R \to (f,f)_1^R$ is a monotone function from the net to the real line, hence the supremum of $(f,f)_1^R$ over all $R \in \mathcal{N}$ is the limit of $(f,f)_1^R$ along the net $\mathcal{N}$. Call $(f,f)_1$ this supremum. Note that $(f,f)_1 = \infty$ is also allowed. By taking the limit over the subspaces spanned by $\{e_1, \ldots, e_n\}$, $n = 1, 2, \ldots$, where $\{e_i\}_1^\infty$ is an orthonormal base in $K$ of elements in $Q$, we have that

$$(f,f)_1 = \sum_{i=1}^{\infty} \|e_i \cdot \nabla f\|_2^2. \tag{2.11}$$

Note however that $(f,f)_1$, by the above definition, is basis independent. We shall also use the notation

$$(f,f)_1 = \int \nabla f \cdot \nabla f \, d\mu. \tag{2.12}$$

Let us now define the gradient $\nabla_R$ in the direction of the finite dimensional subspace $R$ as follows. Let $e_1, \ldots, e_n$ be a complete orthonormal system in $R$. For $f$ in $D(q \cdot \nabla)$, for all $q \in Q$, we define $\nabla_R f$ as the element in the Hilbert tensor product $R \hat{\otimes} L_2(d\mu)$ given by

$$\nabla_R f = \sum_{i=1}^{n} e_i \otimes (e_i \cdot \nabla f). \tag{2.13}$$

One has

$$(f,f)_1^R = \|\nabla_R f\|^2, \tag{2.14}$$

where $\| \ \|$ is the Hilbert norm in $R \hat{\otimes} L_2(d\mu)$.

**Lemma 2.2a.** *Let $\mu \in \mathcal{P}_1(Q')$. Then $\nabla_R$ is a closable operator from the dense domain $\bigcap_{q \in Q} D(q \cdot \nabla) \subset L_2(d\mu)$ into $R \hat{\otimes} L_2(d\mu)$. The closure $\bar{\nabla}_R$ of $\nabla_R$, independent of the chosen basis in $R$ in the expression* (2.13), *satisfies*

$$\|\bar{\nabla}_R f\|^2 = (f,f)_1^R.$$

*Proof.* Let $h \in R \hat{\otimes} L_2(d\mu)$, with components

$$h_i \in \bigcap_{q \in R} D(q \cdot \nabla) \cap L_\infty(d\mu)$$

with respect to the basis vectors $e_i$, $i = 1, \ldots, n$ in $R$. Such functions are obviously dense in $R \hat{\otimes} L_2(d\mu)$. We now see that the adjoint $\nabla_R^*$ of $\nabla_R$ is well defined as a linear map from a dense domain of $R \hat{\otimes} L_2(d\mu)$ into $L_2(d\mu)$, since on elements $h$ of the above form it is given by

$$\nabla_R^* h = -\sum_{i=1}^{n} (e_i \cdot \nabla h_i + (\beta \cdot e_i) h_i), \tag{2.15}$$

and the right hand side is in $L_2(d\mu)$, due to the assumption $\mu\in\mathscr{P}_1(Q')$ and the choice of functions $h_i$. Thus $\nabla_R$ has a densely defined adjoint, hence it is closable as a map from the dense domain $\bigcap_{q\in Q} D(q\cdot\nabla)\subset L_2(d\mu)$ into $R\hat{\otimes}L_2(d\mu)$. This is equivalent with the form $\|\nabla_R f\|^2$ being closable. The closure of this form is given by $\|\bar{\nabla}_R f\|^2$ and by (2.14) it coincides with $(f,f)_1^R$. The latter form being independent of the basis in $R$ the Lemma is proven.

Consider now the operator $\nabla$ defined as a map from a dense domain of $L_2(d\mu)$ into $K\hat{\otimes}L_2(d\mu)$ by its action on elements $f$ in

$$\{f\in\bigcap_{q\in Q} D(q\cdot\nabla)|(f,f)_1<\infty\}\equiv W_0^1(d\mu)$$

given by

$$\nabla f=\sum_{i=1}^{\infty} e_i\otimes e_i\cdot\nabla f, \tag{2.16}$$

where $\{e_i\}$ is a complete orthonormal system in $Q\subset K$. Note that

$$\|\nabla f\|^2\equiv\int\nabla\bar{f}\cdot\nabla f\,d\mu=\sum_{i=1}^{\infty}\|e_i\cdot\nabla f\|_2^2<\infty \tag{2.17}$$

for such $f$, where $\|\ \|$ is the Hilbert norm in $K\hat{\otimes}L_2(d\mu)$.

**Lemma 2.2b.** *Let $\mu\in\mathscr{P}_1(q')$. $\nabla$ is a closable map from the domain $W_0^1(d\mu)$ into $K\hat{\otimes}L_2(d\mu)$. The closure $\bar{\nabla}$ of $\nabla$ is independent of the basis in $Q$ used in* (2.16) *and one has*

$$\|\bar{\nabla}f\|^2\equiv\int\bar{\nabla}\bar{f}\cdot\bar{\nabla}f\,d\mu=(f,f)_1$$

*for all $f\in D(\bar{\nabla})$, where $(f,f)_1$ denotes the closure of the form* (2.12), *as defined on $D(\nabla)\cdot(f,f)_1$ is called the Dirichlet form given by $\mu$.*[6]

*Proof.* Consider as in the proof of Lemma 2.2a elements in $R\hat{\otimes}L_2(d\mu)$ of the form $h=\{h_1,\dots,h_n\}$, with $h_i\in D(q\cdot\nabla)\cap L_\infty(d\mu)$. For such $h$ we have $\nabla^*h=\nabla_R^*h$ and, when $R$ runs over the net of finite dimensional subspaces of $Q$, the set of such $h$ is dense in $K\hat{\otimes}L_2(d\mu)$. This then shows that $\nabla^*$ is densely defined, hence $\nabla$ is closable. The rest follows analogously as in the proof of Lemma 2.2a.

Let us from now on denote by $\nabla_R$ resp. $\nabla$ the closures of the operators denoted by the same symbols in Lemma 2.2a and Lemma 2.2b. Define $H^R=\nabla_R^*\nabla_R$. $\nabla_R$ being closed in $L_2(d\mu)$ we have that $H^R$ is densely defined positive and self-adjoint as an operator from a dense domain $D(H^R)\subset L_2(d\mu)$ into $L_2(d\mu)$. One has $D((H^R)^{\frac{1}{2}})=D(\nabla_R)$ and

$$(f,f)_1^R=\|\nabla_R f\|^2=\|(H^R)^{\frac{1}{2}}f\|_2^2, \tag{2.18}$$

for all $f\in D(\nabla_R)$. The map $R\to H^R$ from the net of finite dimensional subspaces of $Q$ to the ordered set of positive self adjoint operators $H^R$ is monotone non

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[6] The restriction of this form to $FC^2(Q')$ is what was called Dirichlet form in [1]. Note that in [1] we used the notation $F^n$ for $FC^n$ and $(f,f)_1$ was denoted by $(f,f)_\mu$

decreasing. Define $H=\nabla^* \nabla$. $\nabla$ being closed we have that $H$ is densely defined positive and self-adjoint as an operator from a dense domain $D(H)\subset D(\nabla)\subset L_2(d\mu)$ into $L_2(d\mu)$. One has $D(H^{\frac{1}{2}})=D(\nabla)$ and

$$(f,f)_1=\|\nabla f\|^2=\|H^{\frac{1}{2}}f\|_2^2 \tag{2.19}$$

for all $f\in D(\nabla)$.

We now note that $D(\nabla)=W^1(d\mu)$, where

$$W^1(d\mu)=\{f\in D(\nabla_R)\ \forall R\in Q \mid \lim_{R\uparrow Q}\|\nabla_R f\|^2<\infty\},$$

where $R\uparrow Q$ along the net $\mathcal{N}$.

In fact $D(\nabla)\supset W^1(d\mu)$ is evident and one has

$$\|\nabla f\|^2=\lim_{R\uparrow Q}\|\nabla_R f\|^2=\sup_{R\subset Q}\|\nabla_R f\|^2 \tag{2.20}$$

for any $f\in W^1(d\mu)$.

To show $D(\nabla)\subset W^1(d\mu)$ it suffices to use that $\nabla$ is closed and that $FC^1$ is dense in $D(\nabla)$ in the norm $\|\cdot\|+\|\nabla\cdot\|$.

Finally we note that for $f\in FC^2$ with base spanned by the orthonormal complete system $\{e_1,\dots,e_n\}$, $e_i\in Q$ we have

$$\nabla f=\sum_{i=1}^n e_i\otimes(e_i\cdot\nabla f).$$

Hence $\nabla f\in D(\nabla^*)$ and thus, by (2.16)

$$\nabla^*\nabla f=-\sum_{i=1}^n [(e_i\cdot\nabla)(e_i\cdot\nabla)f+(\beta\cdot e_i)(e_i\cdot\nabla)f]=-\Delta f-\beta\cdot\nabla f,$$

where

$$\Delta f\equiv\sum_{i=1}^n (e_i\cdot\nabla)^2 f \tag{2.21}$$

and

$$\beta\cdot\nabla f\equiv\sum_{i=1}^n(\beta\cdot e_i)(e_i\cdot\nabla)f. \tag{2.22}$$

Thus for $g\in D(\nabla)$, we have

$$(H^{\frac{1}{2}}g,H^{\frac{1}{2}}f)=(\nabla g,\nabla f)=(g,\nabla^*\nabla f)=(g,-\Delta f-\beta\cdot\nabla f),$$

which shows that $Hf=-\Delta f-\beta\cdot\nabla f$, for all $f\in FC^2$. We have thus proven the following

**Theorem 2.1.** *For any measure $\mu$ in $\mathcal{P}_1(Q')$ the Dirichlet form*

$$(f,f)_1\equiv\int\nabla\bar{f}\cdot\nabla f\,d\mu$$

*is a closed positive form in $L_2(Q',d\mu)$, with domain*

$$D(\nabla)=\{f\in\bigcap_R D(\nabla_R)\mid\lim_{R\uparrow Q}\|\nabla_R f\|^2<\infty\}.$$

*The associated operator is the positive self-adjoint operator*

$$H = \nabla^* \nabla.$$

*One has* $D(H^{\frac{1}{2}}) = D(\nabla)$ *and* $D(H)$ *is a core for* $\nabla$. *The restriction* $H$ *to the dense subset* $FC^2$ *of* $L_2(Q', d\mu)$ *is given by*

$$Hf = -\Delta f - \beta \cdot \nabla f,$$

*where* $f$ *is arbitrary in* $FC^2$ *and*

$$\Delta f = \sum_{i=1}^{n} (e_i \cdot \nabla)^2, \qquad \beta \cdot \nabla = \sum_{i=1}^{n} (\beta \cdot e_i)(e_i \cdot \nabla),$$

*when* $\{e_1, \ldots, e_n\}$ *is an orthonormal set of vectors spanning the base of* $f$.

Let now $f \in D(\nabla)$. From the fact that the restriction of $\nabla$ to the domain

$$\left\{ f \in \bigcap_{q \in Q} D(q \cdot \nabla) \,\middle|\, \sum_{i=1}^{\infty} \|e_i \cdot \nabla f\|_2^2 < \infty \right\}$$

is closable, with closure $\nabla$, we have that the inequality

$$\|(H^R)^{\frac{1}{2}} f\| \leqq \|H^{\frac{1}{2}} f\| \tag{2.23}$$

extends from the above domain to the whole of $D(\nabla)$.

By the Theorem on monotone convergence of lower bounded forms (see e.g. [51] and [47]) we have then from (2.23) and (2.20), recalling $D(\nabla) = W^1(d\mu)$, that $H^R$ converges strongly in the generalized sense, as $R$ increases to $Q$ along the net $\mathcal{N}$, to the positive self-adjoint operator $H$. It is well known, see e.g. [51], that strong convergence in the generalized sense implies strong convergence of the corresponding semigroups. We have thus the following

**Theorem 2.2.** *Let* $\mu$ *be in* $\mathscr{P}_1(Q')$. *Then* $R \to H^R$ *is a monotone map from the directed set of finite dimensional subspaces of* $Q$ *into the ordered set of positive self-adjoint operators.*

*One has* $(f, f)_1^R \uparrow (f, f)_1$ *for all* $f \in D(\nabla)$ *and* $e^{-tH^R} \downarrow e^{-tH}$ *strongly in* $L_2(Q', d\mu)$, *uniformly on finite* $t$ *intervals, as* $R \uparrow Q$ *through the net of finite dimensional subspaces of* $Q$.

We shall call the operator $H$ of Theorems 2.1 and 2.2 the *Dirichlet operator* given by $\mu$.[7]

We shall say that a measure $\mu \in \mathscr{P}(Q')$ is $Q$-ergodic iff the only measurable subsets of $Q'$ which are $Q$-invariant, i.e. invariant under translations by arbitrary $q \in Q$, have $\mu$-measure zero or one.

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[7] The Friedrichs extension of the restriction of the Dirichlet operator $\nabla^* \nabla$ to the dense domain $FC^2(Q')$ of $L_2(d\mu)$ is what was called "the diffusion operator given by $\mu$" in Reference [1] (Th. 2.7). It is an open question whether $FC^2(Q')$ is a core for the Dirichlet operator, in which case Dirichlet operator and diffusion operator would coincide on their whole domain

Consider now the closed subspace $L_\infty^{\text{inv}}(d\mu)$ of $L_\infty(d\mu)$ consisting of functions $f$ such that

$$f(\xi+q)=f(\xi) \quad \text{for all } q\in Q \cdot$$

$L_\infty^{\text{inv}}(d\mu)$ is obviously closed under multiplication, so that it is a commutative $C^*$-algebra with identity in the $L_\infty(d\mu)$-norm. Hence by the Gelfand representation theorem $L_\infty^{\text{inv}}(d\mu)$ is isomorphic (as a $C^*$-algebra) with the space $C(Z)$ of continuous complex-valued functions on a compact Hausdorff space $Z$, unique up to homeomorphisms. $Z$ is the spectrum space of the $C^*$-algebra $L_\infty^{\text{inv}}(d\mu)$ i.e. the space of all maximal ideals of the $C^*$-algebra $L_\infty^{\text{inv}}(d\mu)$. The restriction of the measure $d\mu$ to $L_\infty^{\text{inv}}(d\mu)$ gives a positive continous linear functional, hence an element in the topological dual of $L_\infty^{\text{inv}}(d\mu)$, thus by the above isomorphism we get a positive linear functional on $C(Z)$. By the Riesz-Markov theorem this functional gives a bounded positive regular Borel measure $dz$ on $Z$. Since $L_\infty^{\text{inv}}(d\mu)$ is weakly closed in $L_2(d\mu)$ we have by the isomorphism that $C(Z)$ is weakly closed in $L_2(Z,dz)$. This gives us that $C(Z)=L_\infty(Z,dz)$, so that in particular all measurable sets in $Z$ are open.

$(Z,dz)$ is then a standard Borel space. By the well known central decomposition (e.g. [50]) to the standard measure $dz$ there exists a measurable field $z\to\mathscr{H}_z$ of Hilbert spaces over $Z$ and an isomorphism of $L_2(d\mu)$ with the direct integral $\int^{\oplus}\mathscr{H}_z\,dz$, so that the $C^*$-algebra $L_\infty^{\text{inv}}(d\mu)$ is irreducibly represented in $\mathscr{H}_z$. Let us now assume that $Q'$ is such that regular conditional probability measures with respect to the $\sigma$-algebra generated by $L_\infty^{\text{inv}}(d\mu)$ exist.

*Remark 2.* Conditions yielding regular conditional probability measures with respect to subalgebras are well studied, both from the abstract measure theoretical point of view, see e.g. [52–56], and the topological point of view, see e.g. [55–59]. We can take e.g. $Q'$ to be a Suslin space.

Under the above regularity assumption we can identify the central decomposition $L_2(d\mu)=\int^{\oplus}\mathscr{H}_z\,dz$ with the decomposition

$$L_2(d\mu)=\int^{\oplus} L_2(d\mu_z)\,dz \tag{2.24}$$

where $d\mu_z$ is the conditional probability measure on $Q'$ given the $\sigma$-algebra generated by $L_\infty^{\text{inv}}(d\mu)$, so that

$$\mu(\xi)=\int\mu_z(\xi)\,dz. \tag{2.25}$$

$dz$ is here simply the restriction of $\mu$ to the $\sigma$-algebra generated by $L_\infty^{\text{inv}}(d\mu)$.

For any measurable subset $A$ of $Q'$ with $\mu(A)>0$, $\mu_z(A)$ is the density with respect to $dz$ of the $dz$-absolutely continuous measure obtained by restricting $\mu(A)$ to the $\sigma$-algebra generated by $L_\infty^{\text{inv}}(d\mu)$. If $A$ is invariant by translations by elements in $Q$ then by construction $\mu_z(A)=\chi_{\hat{A}}(z)$, where $\chi_{\hat{A}}(z)$ is the characteristic function of the image of $A$ under the isomorphism of $L_\infty^{\text{inv}}(d\mu)$ and $C(Z)$, which shows that $\mu_z(\cdot)$ are $Q$-ergodic measures.

Moreover we have the following

**Lemma 2.3.** *If* $z_1 \neq z_2$ *then* $\mu_{z_1} \perp \mu_{z_2}$.

*Proof.* If $z_1 \neq z_2$, since $Z$ is Hausdorff, there are two open non intersecting sets $A_1$ and $A_2$ such that $z_1 \in A_1$ and $z_2 \in A_2$. Now for any $\mu$-measurable set $B \subset Q'$ and any $A$ open in $Z$ we have by definition that

$$\int_A \mu_z(B)\, dz = \mu(\tilde{A} \cap B)$$

where $\tilde{A}$ is a $Q$-invariant measurable set such that its characteristic function is represented on $Z$ by the characteristic function of $A$. So that if $z \in A$ then, by (2.25), $\mu_z$ has support in $\tilde{A}$. Since $A_1 \cap A_2 = \emptyset$ we have that $\tilde{A}_1$ and $\tilde{A}_2$ may be chosen such that $\tilde{A}_1 \cap \tilde{A}_2 = \emptyset$.

**Lemma 2.4.** *If* $\mu$ *is* $Q$*-quasi invariant then* $\mu_z$ *is* $Q$*-quasi invariant, for* $dz$*-almost every* $z \in Z$.

*Proof.* Let $A$ be an invariant set so that its characteristic function $\chi_A \in L_\infty^{\text{inv}}(d\mu)$. Then for any $f \in C_b(Q')$ and any $q \in Q$

$$\int f(\xi+q)\, \chi_A(\xi)\, d\mu(\xi) = \int f(\xi)\, \chi_A(\xi) \frac{d\mu(\xi-q)}{d\mu(\xi)}\, d\mu(\xi) = \int_A \int f(\xi) \frac{d\mu(\xi-q)}{d\mu(\xi)}\, d\mu_z(\xi)\, dz.$$

This then gives, the left hand side being $\int_A \int f(\xi+q)\, d\mu_z(\xi)\, dz$, that

$$\int f(\xi+q)\, d\mu_z(\xi) = \int f(\xi) \frac{d\mu(\xi-q)}{d\mu(\xi)}\, d\mu_z(\xi),$$

for $dz-$a.e. $z$. The exceptional set can be chosen independent of $f$, since the equality holds for a countable set of functions dense in the bounded convergence norm. This then implies $d\mu_z(\xi-q) = \frac{d\mu(\xi-q)}{d\mu(\xi)}\, d\mu_z(\xi)$ for $dz$-a.e. $z$, proving the quasi invariance of $\mu_z$. Thus we have

**Theorem 2.3.** *Let* $\mu$ *be a probability measure on* $Q'$. *Suppose* $Q'$ *is such that the conditional probability measure with respect to the* $\sigma$*-algebra generated by* $L_\infty^{\text{inv}}(d\mu)$ *exists, which is e.g. the case when* $Q'$ *is Suslin. Then* $\mu$ *has the* $Q$*-ergodic decomposition*

$$\mu = \int_Z \mu_z\, dz,$$

*with* $\mu_{z_1} \perp \mu_{z_2}$ *for* $z_1 \neq z_2$. *If* $\mu$ *is quasi invariant then* $\mu_z$ *is quasi invariant, for* $dz$*-a.e.* $z$.

**Theorem 2.4.** *Let* $\mu \in \mathscr{P}_1(Q')$ *and zero a simple eigenvalue of* $H$. *Then* $\mu$ *is* $Q$*-ergodic. In fact the eigenspace of eigenvalue zero contains the subspace of* $L_2(d\mu)$ *consisting of* $Q$*-invariant functions.*

*The decomposition* (2.25) *gives a direct decomposition of* $L_2(d\mu)$ *of the form*

$$L_2(d\mu) = \int_Z L_2(d\mu_z)\, dz$$

*and with respect to this decomposition $H$ decomposes as*

$$H=\int H_z\,dz,$$

*where $H_z$ is the self-adjoint operator associated with $\mu_z$.*

*Proof.* If $f$ is $Q$-invariant, then obviously $(f,f)_1=0$ so that $f\in W^1=D(H^{\frac{1}{2}})$ and $H^{\frac{1}{2}}f=0$, which implies that $f\in D(H)$ and $Hf=0$.

The direct decomposition of $L_2(d\mu)$ follows from the fact that $\mu_{z_1}\perp\mu_{z_2}$ for $z_1\neq z_2$. That $H$ decomposes follows from the corresponding decomposition of $W^1$.

We shall say that a $\mu\in\mathscr{P}_1(Q')$ is in $\mathscr{P}_D(Q')$ if $\beta\in K\hat{\otimes}L_2(d\mu)$ i.e. if

$$\|\mu\|_D^2\equiv\|\beta\|^2\equiv\sum_{i=1}^{\infty}\|\beta_i\|_2^2 \tag{2.26}$$

is finite, where $\beta_i=\beta\cdot e_i$ and $\{e_i\}_{i=1}^{\infty}$ is an orthonormal base in $K$ of elements in $Q$. Similarly as in Lemma 2.1 we have

**Lemma 2.4.** *If $\mu\in\mathscr{P}_D(Q')$ then $FL_\infty\cap W^1$ is dense in $W^1=D(\nabla)$ in the graph norm of $\nabla$, i.e. in the Dirichlet norm $(\|f\|^1)^2=(f,f)_1+(f,f)$.*

*Proof.* Let $f\in W^1$ and set $f^k(\xi)=f(\xi)$ if $|f(\xi)|\leqq k$ and equal to $+k$ (resp. $-k$) if $f(\xi)$ is larger than $k$ (smaller than $-k$). Then $f^k\to f$ in $L_2(d\mu)$. Moreover $f-f^k\in D(\nabla)$, since $f-f^k=0$ for $|f(\xi)|\leqq k$ and $f-f^k=f-k$ for $|f(\xi)|>k$. We have

$$(f-f^k,f-f^k)_1=\int_{|f(\xi)|>k}\nabla\bar{f}\cdot\nabla f\,d\mu$$

and $f^k=f-(f-f^k)\in W^1$, because both $f$ and $f-f^k$ are in $W^1$. But $(f-f^k,f-f^k)_1$ goes to zero since $\nabla\bar{f}\cdot\nabla f\in L_1$. So that $W^1\cap L_\infty$ is dense in $W^1$. Let now $f\in W^1\cap L_\infty$ and let $R$ be a finite dimensional subspace of $Q$ with its corresponding conditional expectation $E_R$. It is proven in Appendix that

$$\nabla E_R f=E_R\nabla f+E_R[(\beta-E_R\beta)f]. \tag{2.27}$$

So from the triangle inequality in $K\hat{\otimes}L_2(d\mu)$ we have

$$\|\nabla E_R f\|\leqq\|E_R\nabla f\|+\|E_R[(\beta-E_R\beta)f]\|$$

and thus, since $E_R$ is a projection in $L_2$,

$$\|\nabla E_R f\|\leqq\|\nabla f\|+\|f\|_\infty\|\beta\|$$

so that $E_R f\in W^1$. Consider now $f-E_R f$, which obviously goes to zero in $L_2(d\mu)$ as $R\to Q$. On the other hand

$$\begin{aligned}\|\nabla(f-E_R f)\|&\leqq\|\nabla f-E_R\nabla f\|+\|E_R[(\beta-E_R\beta)f]\|\\&\leqq\|\nabla f-E_R\nabla f\|+\|f\|_\infty\|\beta-E_R\beta\|.\end{aligned}$$

Since $E_R\to 1$ in $L_2(d\mu)$ we have that $1\otimes E_R\to 1$ in $K\hat{\otimes}L_2(d\mu)$, hence the right hand side of the previous inequality goes to zero. This proves the lemma.

Let now $\mu \in \mathscr{P}_2(Q')$ and let us also assume that, for an orthonormal basis $\{e_n\}$ in $K$ of elements in $Q$,

$$V \equiv -\sum_{n=1}^{\infty} \pi(e_n)^2 \Omega \tag{2.28}$$

converges in $L_2(d\mu)$, where $\Omega(\xi) \equiv 1$. In that case the operator

$$\pi^2 \equiv \sum_{i=1}^{\infty} \pi(e_i)^2 \tag{2.29}$$

is defined on $FC^2$, it is obviously non negative and we shall denote by $\pi^2$ also its Friedrichs extension. Although (2.29) looks basis dependent, we may see in the following way that it is not. Let $R$ be a finite dimensional subspace of $Q$ and

$$\pi_R^2 = \sum_{i=1}^{n} \pi(r_i)^2, \tag{2.30}$$

where $r_1, \ldots, r_n$ is an orthonormal base in $R$. We denote also by $\pi_R^2$ its Friedrichs extension. It is easy to see that (2.30) is basis independent. Moreover $R \to \pi_R^2$ is monotone from the directed set of finite dimensional subspaces into the directed set of non negative operators. $\pi^2$ is then simply the limit, by Theorem 3.13 of Chapter VIII of Reference [51], of $\pi_R^2$ as $R \to Q$. This shows that the Friedrichs extension of (2.29) is basis independent.

We have obviously that on $FC^2(Q')$

$$\pi^2 + V = H, \tag{2.31}$$

where $H$ is the Dirichlet operator. We can also give the $L_2$-function $V$ directly in terms of $\beta$ if we assume $\|\mu\|_D < \infty$. Since

$$i\pi(q)\Omega = \tfrac{1}{2}\beta \cdot q$$

we see that

$$-\pi^2 \Omega = \sum_{i=1}^{\infty} \left(\tfrac{1}{2} \nabla_i \beta_i + \tfrac{1}{4}\beta_i \cdot \beta_i\right). \tag{2.32}$$

Now, if $\|\mu\|_D < \infty$, $\sum_{i=1}^{\infty} \beta_i^2$ converges in $L_2$ so, by the assumption that (2.28) converges, we get that

$$V = \tfrac{1}{2} \operatorname{div} \beta + \tfrac{1}{4} \beta \cdot \beta, \tag{2.33}$$

where

$$\beta \cdot \beta = \sum_{i=1}^{\infty} \beta_i^2 \quad \text{and} \quad \operatorname{div} \beta = \sum_{i=1}^{\infty} \nabla_i \beta_i \tag{2.34}$$

and $\beta_i = \beta \cdot e_i$ and $\nabla_i = e_i \cdot \nabla$, $\{e_i\}$ being an orthonormal base in $K$ of elements in $Q$.

In the finite dimensional case, i.e. in the case where $\dim K < \infty$, the fact that $\pi^2 + V$ on functions of the form $f\Omega$ with $f \in C_b^2(K)$ is given by

$$(\pi^2 + V) f\Omega = (-\Delta - \beta \cdot \nabla) f\Omega$$

gives us, since we know that $-\Delta - \beta \cdot \nabla$ on $C_b^2$ in $L_2(d\mu)$ is actually equivalent with $-\Delta + V$ on $C_b^2$ in $L_2(dx)$, that $-\pi^2$ is the image, in $L_2(d\mu)$, of the Laplacian in $L_2(dx)$. $V$ is in this interpretation the potential. In this sense it is natural to call $-\pi^2$ and $V$ the Laplacian resp. potential given by $\mu$ in $L_2(d\mu)$, and to extend these names also to the infinite dimensional case.

*Remark 3.* The Laplacian given by an arbitrary quasi invariant measure in $\mathscr{P}_2(Q')$ is an $L_2$-concept linked to the existence of a measure $\mu$ and is thus different from the Laplacians defined from other points of view in other works, in particular by Gross and Lévy ([6, 60, 61, 65]).

*Remark 4.* It is not immediately obvious that the class of quasi invariant measures so that (2.28) converges in $L_2(d\mu)$ is non empty. So we shall therefore give a simple example.

*Example.* Let $A$ be a positive invertible trace class operator on a real separable Hilbert space $K$. Consider now the Gaussian measure $d\mu_A$ with covariance $A^{-1}$, it is, for any $x, y \in K$

$$\int (\xi, x)(\xi, y)\, d\mu_A(\xi) \equiv E_A[(\cdot, x)(\cdot, y)] = (x, A^{-1} y).$$

Let $Q$ be the Hilbert space $Q = D(A^{-1})$ with its natural norm $\|A^{-1} \cdot\|$. Then we have that $Q'$ is the completion of $K$ in the norm $\|A \cdot\|$. It is well known that, since $A$ is of trace class, $d\mu_A$ is a measure on $Q'$, which is quasi invariant under translations by all $q \in Q$, in fact by all $q \in Q'$ such that $(q, Aq) < \infty$. In this case we have

$$\beta q = -Aq \tag{2.35}$$

so that $\mu_A \in \mathscr{P}_D(Q')$ namely, from (2.26)

$$\|\mu_A\|_D = \left(\int \|A\xi\|^2\, d\mu_A(\xi)\right)^{\frac{1}{2}} = \operatorname{tr} A \tag{2.36}$$

and

$$(\pi^2 \Omega)(\xi) = -\|A\xi\|^2 + \operatorname{tr} A. \tag{2.37}$$

So that with

$$V = \|A\xi\|^2 - \operatorname{tr} A$$

we have that

$$\begin{aligned}\|V\|_2^2 &= E_A[(\xi A^2 \xi)^2 - 2 \operatorname{tr} A (\xi A^2 \xi) + (\operatorname{tr} A)^2] \\ &= \operatorname{tr} A^2 + (\operatorname{tr} A)^2 - 2(\operatorname{tr} A)^2 + (\operatorname{tr} A)^2,\end{aligned}$$

i.e.

$$\|V\|_2^2 = \operatorname{tr} A^2, \tag{2.38}$$

which is finite since $A$ is of trace class. We see in fact that we may do with the weaker condition that $A$ is a Hilbert-Schmidt operator, because (2.38) still holds and also in this case $d\mu_A$ is a measure in $Q'$.

A particular example is given by the Gaussian measure corresponding to non interacting (i.e. free) Euclidean fields in a closed interval of $R$ with e.g. Dirichlet boundary conditions, i.e. the measure with covariance $(-\Delta_D+m^2)^{-1}$, where $\Delta_D$ is the Laplacian with Dirichlet boundary conditions.

## 3. The Diffusion Process Generated by the Dirichlet Operator

We have from the previous section that the Dirichlet operator $H=\nabla^*\nabla$ is a self adjoint operator in $L_2(d\mu)$ which is the limit in the strong resolvent sense of the operators

$$H_R=\nabla_R^*\nabla_R \tag{3.1}$$

where $\nabla_R$ is the gradient in the direction of the finite dimensional subspace $R$. The limit is to be taken over the filter of all finite dimensional subspaces. From the strong resolvent convergence we then have that $e^{-tH_R}\to e^{-tH}$ strongly.

We say that a contraction semigroup $T_t$ in $L_2(d\mu)$ is a Markov semigroup if for any $f\in L_2(d\mu)$ with $f\geqq 0$ we have that $T_t f\geqq 0$. From the strong convergence above we get that if $e^{-tH_R}$ is Markov, then so is $e^{-tH}$. We shall now see that $e^{-tH_R}$ is Markov if $\mu\in\mathscr{P}_1(Q')$.

We have seen in the previous section that since $R$ is finite dimensional $P_R$ extends by continuity to a continuous projection defined on all of $Q'$ and with range $R$. We shall denote this extension still by $P_R$. The decomposition of the identity on $Q'$ given by

$$I=P_R+(I-P_R) \tag{3.2}$$

gives a direct decomposition of $Q'$ of the form

$$Q'=R\oplus R^\perp, \tag{3.3}$$

where $R^\perp$ is the annihilator of $R$ in $Q'$. Since $P_R$ is continuous on $Q'$, so is $I-P_R$, hence for $x\in R$ and $\eta\in R^\perp$ we have that $(x,\eta)\to x\oplus\eta$ is one to one and bicontinuous. Hence $Q'$ and $R\times R^\perp$ are equivalent as measure spaces. Therefore we may consider $\mu$ as a measure on the product space $R\times R^\perp$.

Let us now assume that $Q'$ is such that the conditional probability measure obtained on $R$ form $\mu$ by conditioning with respect to $R^\perp$ exists. We shall say shortly, if this is the case for any finite dimensional subspace $R$ of $Q$ that $(Q',\mu)$ is regular. For this condition of the regularity the same comment as in Remark 2 holds. In particular it suffices to assume that $Q'$ is a Suslin space.

Thus assume $(Q',\mu)$ is regular and let $\mu(x|\eta)$ be the conditional probability measure on $R$ conditioned with respect to $R^\perp$, assumed to exist, as in Remark 2 of Section 2. Then for any measurable set $A\subset R$ we have that $\mu(A|\eta)$ is a positive measurable function on $R^{\perp}$ such that, for any measurable set $B$ in $R^{\perp}$,

$$\int_B \mu(A|\eta)\,d\nu(\eta)=\mu(A\times B), \tag{3.4}$$

where $\nu$ is the projection of $\mu$ on $R^{\perp}$. Let now $A\subset R$ and $B\subset R^{\perp}$. The quasi invariance of $\mu$ under translations by elements in $Q$ gives us that $\mu(A\times B)$ as a function of $A$ for fixed $B$ is a quasi invariant measure on $R$, and therefore by (3.4) we get that $\mu(A|\eta)$ is a quasi invariant measure on $R$ for $\nu$-almost all $\eta\in R^{\perp}$. Thus we have

$$\mu(A|\eta)=\int_A \rho(x|\eta)\,dx=\int_A \varphi^2(x|\eta)\,dx \tag{3.5}$$

with $\varphi(x|\eta)$ and $\rho(x|\eta)$ different from zero almost everywhere in the sense of Lebesgue. From (3.4) we now easily get

$$L_2(d\mu)=\int_{R^{\perp}}^{\oplus} L_2(d\mu(\cdot|\eta)\,d\nu(\eta) \tag{3.6}$$

where the integral is taken in the sense of a direct integral of Hilbert spaces. We see that the operator $H_R$ of (3.1) is reducible with respect to the direct integral decomposition (3.6) and in fact with respect to that decomposition we have

$$H_R=\int_{R^{\perp}} H_\eta\,d\nu(\eta), \tag{3.7}$$

where $H_\eta$ is the Dirichlet operator in $L_2(R;d\mu(\cdot|\eta))$. Hence

$$e^{-tH_R}=\int_{R^{\perp}} e^{-tH_\eta}\,d\nu(\eta). \tag{3.8}$$

Therefore if we can prove that $e^{-tH_\eta}$ is a Markov semigroup, then $e^{-tH_R}$ is a Markov semigroup. Hence we have reduced the problem of whether $e^{-tH}$ is Markov or not to a corresponding finite dimensional problem.

Let now $\{X,dm\}$ be a $\sigma$-finite measure space. Let $\varepsilon$ be a closed non negative symmetric form on the real $L^2$-space $L^2(X,dm)$ with domain of definition $D(\varepsilon)$ which is dense in $L^2(X,dm)$.

We shall say that *every unit contraction operates on* $\varepsilon$ if for any $u\in D(\varepsilon)$ the function $v=(0\vee u)\wedge 1$ is again in $D(\varepsilon)$ and

$$\varepsilon(v,v)\leqq\varepsilon(u,u). \tag{3.9}$$

The following theorem is proved in Section 3 of Reference [29].

**Theorem 3.1** [Fukushima]. *Let $X$ be a locally compact separable Hausdorff space with a Radon measure $dm$. Let $\varepsilon$ be a closed non negative symmetric form on real $L^2(X,dm)$ with a dense domain of definition $D(\varepsilon)$. If every unit contraction operates on $\varepsilon$, then the semigroup $e^{-tH_\varepsilon}$ generated by the self adjoint operator $H_\varepsilon$ associated with the closed form $\varepsilon$ is a Markov semigroup. Moreover if $e^{-tH_\varepsilon}$ is a Markov semigroup, then every unit contraction operates on $\varepsilon$.*

Since $H_\eta$ in (3.7) is the Dirichlet operator in $R$ and $R$ is finite dimensional and $H_\eta$ is the operator associated with the Dirichlet form in $L_2(R,\rho(x|\eta)\,dx)$ we have only to check that every unit contraction operates on the corresponding Dirichlet form. However with $v=(0 \vee u) \wedge 1$ we have that

$$(v,v)_1 = \int\limits_{0 \leqq u \leqq 1} |\nabla u|^2 \rho(x|\eta)\,dx \leqq \int |\nabla u|^2 \rho(x|\eta)\,dx = (u,u)_1. \tag{3.10}$$

Hence we see that the condition Theorem 3.1 is satisfied so that $e^{-tH_\eta}$ is Markov. Thus we have proved the following theorem.[8]

**Theorem 3.2.** *Let $\mu \in \mathscr{P}_1(Q')$, then the corresponding Dirichlet operator $H$ generates a contraction semigroup $e^{-tH}$ which is Markov.*

*Remark 1.* This theorem has been proven under the assumption that $(Q',\mu)$ is regular, used for exploiting the finite dimensional approximations. However the result holds also without the regularity assumption, see [62]. As mentioned before results of this form are central the work of Fukushima, continuing the classical work of Beurling and Deny. For cases particularly studied in connection with quantum field theory see Section 5.

Since $e^{-tH}$, $t \geqq 0$ is a Markov semigroup in $L_2(d\mu)$ and $e^{-tH}1=1$ we have by duality and the usual interpolation theorems that $e^{-tH}$ is a positivity preserving contraction in all $L_p(Q',d\mu)$, $1 \leqq p \leqq \infty$, strongly continuous for all $1 \leqq p < \infty$. By a standard technique used e.g. by Fukushima and Silverstein, selecting a dense separable subspace of the bounded functions in $D(H^{\frac{1}{2}})$ forming an algebra and taking the maximal ideal space for the uniform closure of this algebra, we can realize the Dirichlet form $\int \nabla f \nabla f\, d\mu$, of which $H$ is the associated self-adjoint Dirichlet operator, as a Dirichlet form on the $L_2(Z,d\lambda)$ space over this maximal ideal space $Z$, where $d\lambda$ is the measure corresponding to $\mu$. The Dirichlet form becomes then a regular Dirichlet form in the sense of Fukushima and to the corresponding Markov semigroup in $L_2(d\lambda)$, over the compact space $Z$, there exists an associated Markov process (in fact a Hunt process, after elimination of sets of zero capacity). Thus $e^{-tH}$ gives always rise to a Markov process. We shall however remark, under a general regularity assumption on $Q'$, that the process can actually be realized as the canonical process with state space $Q'$. Let us again assume that $(Q',\mu)$ is regular. Then for any measurable set $A$ in a finite dimensional subspace of $Q$ we have that $P(t,\xi,A)=(e^{-tH}\chi_A)(\xi)$ is well defined and $P(t,\xi,\cdot)$ is a positive finite measure. We extend easily by approximation $P(t,\xi,\cdot)$ to a measure on $Q'$, and we have the Chapman-Kolmogorov equation $P(t+s,\xi,A)=\int\limits_{Q'} P(t,\xi,d\eta)P(s,\eta,A)$. $P(t,\xi,A)$ is then a transition probability function. Consider now the cylinder measure on $Q'^{[0,\infty)}$ determined by the projective system defined by the transition probability functions and the initial $d\mu$. If $Q'$ is such that a Kolomogorov type theorem applies (e.g. $Q'$ the dual of a nuclear space, or a polish space, see e.g. [58]) then we get a measure space $(X,d\omega)$ and a canonical Markov process $t \to \xi(t)$, $t \in [0,\infty)$, with values in $Q'$, such that the finite dimensional distributions of $\xi(t)$ are given by above cylinder measure. In the usual way, the process being time homogeneous with invariant measure $d\mu$ it is extended to a symmetric time homogeneous process for all

[8] This is the correspondent for the Dirichlet operator of Theorem 2.7 in [1]

$t \in R$, with invariant initial measure $d\mu$, in such a way that for any $f \in L_2(d\mu)$ we have

$$e^{-tH} f = E_0 f(\xi(t)), \tag{3.11}$$

where $E_0$ is the conditional expectation with respect to the subalgebra generated by the linear functions $(q, \xi(0))$ for $q \in Q$.

We have the natural inclusion $L_2(d\mu) \subset L_2(X, d\omega)$ as the subspace of $L_2$-functions measurable with respect to the subalgebra generated by $q \cdot \xi(0)$. Moreover the time translation $\xi(s) \to \xi(s+t)$ induces in a natural way a strongly continuous unitary group $T_t$ in $L_2(X, d\omega)$, and with this notation (3.11) takes the form

$$e^{-tH} = E_0 T_t E_0 \tag{3.12}$$

where $E_0$ is the projection onto the $\xi(0)$ measurable functions, i.e. onto $L_2(d\mu)$. Let now $f \in L_2(d\mu)$, then of course $f(\xi(t)) = T_t f(\xi(0)) T_{-t}$ so that $f(\xi(t)) \in L_2(X, d\omega)$ and depends strongly continuously on $t$.

Since $\mu \in \mathscr{P}_1(Q')$ we have that $q \cdot \beta \in L_2(d\mu)$ so that $q \cdot \beta(\xi(t)) \in L_2(X, d\omega)$, and this depends strongly continuously on $t$. Hence it is strongly integrable and $\int_0^t q \cdot \beta(\xi(\tau))\, d\tau \in L_2(X, d\omega)$ is actually strongly differentiable with respect to $t$. Consider now the real valued process

$$q \cdot w(t) = q \cdot \xi(t) - \int_0^t q \cdot \beta(\xi(\tau))\, d\tau. \tag{3.13}$$

We have obviously that $q \cdot w(t)$ is well defined for all $q \in Q$ and as a function on the probability space $(X, d\omega)$ it is linear in $q$. In short $w(t)$ is a weak process on $Q'$ [30–32]. We shall see that it is actually the restriction to $Q$ of $\sqrt{2} \times$ the standard weak Wiener process on $K$.[9] Consider for this

$$e^{i\alpha q \cdot w(t)} = e^{i\alpha q \cdot \xi(t)} \cdot e^{-i\alpha \int_0^t q \cdot \beta(\xi(\tau))\, d\tau}. \tag{3.14}$$

From (3.12) we get that if $f \in L_2(d\mu)$ is in the domain of definition of the Dirichlet operator $H$ then $E_0 f(\xi(t))$ is strongly differentiable in $L_2(X, d\omega)$ with respect to $t$ and for all $t \geqq 0$

$$\frac{d}{dt} E_0 f(\xi(t)) = -E_0 (Hf)(\xi(t)).$$

By homogeneity we therefore get that

$$\frac{d}{dt} E_s f(\xi(t)) = -E_s (Hf)(\xi(t)) \tag{3.15}$$

[9] For the standard weak Wiener process on a real separable Hilbert space $K$ see e.g. [6–8]

for all $s \geqq 0$ and $t \geqq s$. For $t=s$ the derivative above is the one sided derivative. Since $e^{i\alpha q\cdot\xi} \in D(H)$ we have by (3.14) that $E_s e^{i\alpha q\cdot w(t)}$ is strongly differentiable with respect to $t$ for $t \geqq s$ and since $H e^{i\alpha q\cdot\xi} = (\alpha^2 q^2 - i\alpha q\cdot\beta(\xi))\, e^{i\alpha q\cdot\xi}$ we have for $t \geqq s$

$$\frac{d}{dt} E_s e^{i\alpha q w(t)} = -\alpha^2 q^2 E_s e^{i\alpha q\cdot w(t)}. \tag{3.16}$$

Hence for any function $f \in S(R)$ we get that $E_s f(q\cdot w(t))$ is strongly $L_2(X, d\omega)$-differentiable and

$$\frac{d}{dt} E_s f(q\cdot w(t)) = q^2 E_s(\Delta f(q\cdot w(t))) \tag{3.17}$$

where $q^2 = (q, q)$. By Lemma 3.1 below we then have that $q\cdot w(t)$ is the Wiener process with diffusion $2q^2$ on $R$. Hence we have proven that $w(t)/\sqrt{2}$ given by (3.13) is the standard weak Wiener process on $K$. We have thus proven the following theorem.[10]

**Theorem 3.3.** *Let $\xi(t)$ be the Markov process given by the Markov semigroup of Theorem 3.2. Then $\xi(t)$ satisfies the following stochastic differential equation, with initial distribution $d\mu$,*

$$d(q\cdot\xi)(t) = q\cdot\beta(\xi(t))\,dt + d(q\cdot w)(t),$$

*where $w(t)/\sqrt{2}$ is the standard weak Wiener process on $K$.*

In the proof above we made use of the following lemma.

**Lemma 3.1.** *Let $\eta(t)$, $t \geqq 0$ be a real valued stochastic process, i.e. a real valued measurable function $\eta(t, \omega)$ from $([0, \infty) \times X, d\lambda \times d\omega)$ into $R$, where $(X, d\omega)$ is a probability space and $\lambda$ is the Lebesgue measure. For any measurable function $f$ on $R$ we define the forward derivative*

$$(D_+ f)(\eta(t)) = \lim_{h \searrow 0} \frac{1}{h} E_t[f(\eta(t+h) - f(\eta(t))]$$

*where $E_t$ is the conditional expectation with respect to the subalgebra generated by $\eta(\tau)$ for $0 \leqq \tau \leqq t$, whenever this limit exists in the strong $L_2(X, d\omega)$ sense. $D_+ f(\eta(t))$ is thus a function in $E_t L_2(X, d\omega)$ whenever it exists.*

*If for any $f \in S(R)$, the Schwartz test function space we have that $f(\eta(t))$ is strongly $L_2(X, d\omega)$ differentiable and*

$$(D_+ f)(\eta(t)) = \frac{\sigma}{2}(\Delta f)(\eta(t)),$$

[10] Note that about the osmotic velocity $\beta$ we only used what follows from the assumption $\mu \in \mathscr{P}_1(Q')$, namely $q\cdot\beta(\xi(t)) \in L_2$. Thus, the remark in footnote 3 applies. Cases where $\beta$ is linear, Lipschitz continuous or smooth are considered e.g. in [3–13, 79]. One reason for our interest in results of above generality, with singular $\beta$, is that in the applications to interacting quantum fields such cases actually arise, see Section 5 below

*then $\eta(t)$ is a Wiener process on $R$ with diffusion $\sigma$, i.e. $\eta(t)$ is a Markov process and if $v$ is the distribution of $\eta(0)$, then the distribution of $\eta(t)$ is*

$$(2\pi\sigma t)^{-\frac{1}{2}}\int e^{-\frac{1}{2\sigma t}(x-y)^2}\,d v(y).$$

*Proof.* Since obviously $E_s\cdot E_{s+t}=E_s$ for $s$ and $t$ positive, we have by the assumptions of the lemma that $E_s f(\eta(t+s))$ is strongly $L_2(X,d\omega)$ differentiable in $t$ for $t\geqq 0$, since $E_s$ is a strongly $L_2(X,d\omega)$ continuous projection, and

$$\frac{d}{dt}E_s f(\eta(t+s))=E_s\left(\frac{\sigma}{2}\Delta f(\eta(t+s))\right). \tag{3.18}$$

Therefore since $\Delta f$, $\Delta^2 f,\ldots$ are again in $S(R)$ we get from (3.18) that, for all $t\geqq 0$,

$$\frac{d^n}{dt^n}E_s f(\eta(t+s))=E_s((\tfrac{1}{2}\sigma)^n\Delta^n f(\eta(t+s))) \tag{3.19}$$

where we must remember that for $t=0$ the derivatives are the one sided derivatives. Now, for $f\in S(R)$ with $\hat{f}$ of bounded support we easily get by Sobolev inequalities that there is a constant $c$ such that

$$\|\Delta^n f\|_\infty\leqq c^n. \tag{3.20}$$

But then $\|\Delta^n f(\eta(t+s)))\|_\infty\leqq c^n$ so that

$$\left\|\frac{d^n}{dt^n}E_s f(\eta(t+s))\right\|_\infty\leqq(\tfrac{1}{2}\sigma c)^n. \tag{3.21}$$

From this it follows that $E_s f(\eta(t+s))$ is strongly $L_\infty(X,d\omega)$ analytic in $t$ so that for all $t\geqq 0$ we have

$$E_s f(\eta(t+s))=\sum_{n=0}^{\infty}\frac{t^n}{n!}(\tfrac{1}{2}\sigma)^n E_s\Delta^n f(\eta(s)). \tag{3.22}$$

Since $\frac{1}{2}\Delta$ is the infinitesimal generator of the semigroup $e^{\frac{1}{2}t\Delta}$ with kernel

$$e^{\frac{1}{2}t\Delta}(x,y)=(2\pi t)^{-\frac{1}{2}}e^{-\frac{1}{2t}(x-y)^2} \tag{3.23}$$

we have that

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}(\tfrac{1}{2}\sigma)^n\Delta^n f(x)=(2\pi\sigma t)^{-\frac{1}{2}}\int e^{-\frac{1}{2t\sigma}(x-y)^2}f(y)\,dy, \tag{3.24}$$

where the sum is strongly $L_\infty$ convergent. From (3.22) and the strong $L_\infty$ convergence of (3.24) we get

$$E_s f(\eta(t+s))=(T_t f)(\eta(s)), \tag{3.25}$$

where

$$(T_t f)(x)=(2\pi\sigma t)^{-\frac{1}{2}}\int e^{-\frac{1}{2t\sigma}(x-y)^2}f(y)\,dy. \tag{3.26}$$

In particular

$$E_0 f(\eta(t)) = (T_t f)(\eta(0)). \tag{3.27}$$

Since $T_t$ is a semigroup (3.27) proves that $\eta(t)$ is a Markov process and from (3.26) we get that the conditional distribution of $\eta(t)$ given the condition $\eta(0)=0$ is

$$(2\pi\sigma t)^{-\frac{1}{2}} e^{-\frac{1}{2t\sigma}(x-y)^2} dx. \tag{3.28}$$

This then proves the lemma.

In what follows we shall also need the following lemma of Frobenius type.[11]

**Lemma 3.2.** *Let $A$ be a bounded operator on an $L_2$-space such that $\|A\| \leqq 1$ and $A$ is positivity preserving, i.e. $f \geqq 0 \Rightarrow Af \geqq 0$. If 1 is an eigenvalue for $A$, then 1 is a simple eigenvector if the only multiplication operators that commute with $A$ are the constants. Moreover if 1 is a simple eigenvalue, then the corresponding eigenfunction may be taken non negative, and if the only multiplication operators that commutes with $A$ are the constants, then the corresponding eigenfunction is positive almost everywhere.*

*Proof.* Let us assume that 1 is an eigenvalue of $A$ with a corresponding eigenfunction $\varphi$. Since $A$ is positivity preserving, we have, if $\|\varphi\|=1$, that $1 = (\varphi, A\varphi) \leqq (|\varphi|, A|\varphi|)$ so that $|\varphi|$ is an eigenfunction to the eigenvalue 1, since $\|A\| \leqq 1$. Hence if 1 is simple, we may take $|\varphi|$ as the corresponding eigenfunction. On the other hand if 1 is not simple, we have at least another one, $\psi$, which is orthogonal to $|\varphi|$. Since $A$ is positivity preserving, the real and imaginary parts of $\psi$ will also be eigenfunctions and both will be orthogonal to $|\varphi|$, so we may for this reason take $\psi$ to be real. If $\psi = \pm|\psi|$, then $|\varphi|$ and $|\psi|$ are orthogonal, and if $\psi$ and $|\psi|$ are not proportional, then $|\psi| \pm \psi$ are two positive orthogonal eigenfunctions. Hence if 1 is not simple, we can always find a non negative eigenfunction $v$ corresponding to the eigenvalue 1 such that the characteristic function $\chi$ of its support is not a constant.

As a multiplication operator $\chi$ is a projection of $L_2(X, d\omega)$ onto $L_2(\chi X, d\omega)$. Obviously the functions $f \in L_2(X, d\omega)$ such that $|f| \leqq c \cdot v$ for some constant $c$ are dense in the range of $\chi$. Since $Av = v$ and $A$ is positivity preserving we have, for any $-cv \leqq f \leqq cv$, that $-cv \leqq Af \leqq cv$, so that $A$ takes a dense subspace of the range of $\chi$ into itself. By continuity $A$ then takes the range of $\chi$ into itself, i.e. $A$ commutes with $\chi$. Suppose now that the only multiplication operators that commute with $A$ are the constants. Then 1 is a simple eigenvalue and it follows from above that the characteristic function to its support commutes with $A$. If this characteristic function is to be constant, then the eigenfunction must be positive almost everywhere, q.e.d.

Let now $\mu \in \mathscr{P}_1(Q')$ and let $H$ be the corresponding Dirichlet operator in $L_2(Q', d\mu)$. By $L_\infty(V)$ we shall understand the subalgebra of $L_\infty(Q', d\mu)$ of multiplication operators which commute with $e^{-tH}$ for all $t>0$. Since $L_\infty(V)$ is a

[11] This is well known, but we give nevertheless a proof for introducing methods also used later on. For references to the theorem, see e.g. §10 of Reference [47]

commutative $C^*$-algebra, we have that it is equal to all the continuous functions on some compact space which we shall denote $V$. Let $dv$ be the measure induced on $V$ by the integral induced on $L_\infty(V)$ by $d\mu$. It is then easy to see that $L_\infty(V)$ is also isomorphic with $L_\infty(V, dv)$. The spectral decomposition of $L_2(d\mu)$ with respect to the commutative algebra of operators $L_\infty(V)$ is then given by

$$L_2(Q', d\mu) = \int_V^{\oplus} L_2(d\mu(\cdot|v))\, dv, \tag{3.29}$$

where $d\mu(\cdot|v)$ is the conditional probability measure conditioned with respect to the $\sigma$-subalgebra generated by the functions in $L_\infty(V)$. Since all the elements in $L_\infty(V)$ commute with $e^{-tH}$ we have that $H$ is reduced by the direct decomposition (3.29) and

$$H = \int_V H_v\, dv. \tag{3.30}$$

Thus $H_v$ is a self adjoint operator for almost all $v$. By the corresponding reduction of the Dirichlet form we get that

$$(f, H_v f)_v = \int_{Q'} \nabla f \cdot \nabla f\, d\mu(\xi|v). \tag{3.31}$$

Hence we get that the Dirichlet form in $L_2(d\mu(\cdot|v))$ is closed, and the corresponding Dirichlet operator is $H_v$. We should here bear in mind that $d\mu(\xi|v)$ is not necessarily quasi invariant under translations by elements in $Q$, but nevertheless the corresponding Dirichlet form (3.31) is closed.

By the decomposition (3.29) we have that the only multiplication operators which commute with all $e^{-tH_v}$ in $L_2(d\mu(\cdot|v))$ are the constants. Hence, by Lemma 3.2, 0 is a simple eigenvalue of $H_v$. We have thus proved the following theorem.

**Theorem 3.4.** *Let $\mu \in \mathscr{P}_1(Q')$ and let*

$$L_2(d\mu) = \int_V L_2(d\mu(\cdot|v))\, dv$$

*be the spectral decomposition with respect to the subalgebra $L_\infty(V)$ of multiplication operators which commutes with $e^{-tH}$ for all $t>0$, then $\mu(\cdot|v)$ is the conditional probability measure conditioned with respect to the $\sigma$-subalgebra generated by $L_\infty(V)$ and the Dirichlet forms in $L_2(d\mu(\cdot|v))$ are closed for almost all $v$. If*

$$H = \int_V H_v\, dv$$

*is the corresponding direct decomposition of $H$, then $H_v$ are the self adjoint operators in $L_2(d\mu(\cdot|v))$ given by the Dirichlet forms in $L_2(\cdot|v))$. Zero is a simple eigenvalue for $H_v$ and the corresponding eigenfunction is positive almost everywhere, for almost all $v$. Moreover the zero eigenspace for $H$ is the closure of $L_\infty(V)$ in $L_2(d\mu)$.*

*Proof.* That the zero eigenfunction for $H_v$ is positive almost everywhere follows from the fact that the only multiplication operators that commute with $e^{-tH_v}$ for all $t>0$ are the constants, in a similar way as in Lemma 3.2. Now obviously $L_\infty(V)$ is in the zero eigenspace for $H$ since it is invariant under $e^{-tH}$. Suppose now $e^{-tH}f=f$ for all $t$, and let us assume $f$ real. Then of course we have also that $e^{-tH}(f-\lambda)=f-\lambda$ and by the proof of Lemma 3.2 $|f-\lambda|\pm(f-\lambda)$ is also invariant. In the same way as in Lemma 3.2 we then also get that the support of $|f-\lambda|\pm(f-\lambda)$ has a characteristic function which is invariant. Hence the characteristic function of any set of form $\lambda_1\leqq f<\lambda_2$ is invariant under $e^{-tH}$. But then $f$ is obviously in the $L_2$-closure of $L_\infty(V)$. This proves the theorem.

The Markov semigroup $e^{-tH}$ is said to be ergodic if the only multiplication operators that commute with $e^{-tH}$ are the constant. We see from above that this is equivalent with 1 being a simple eigenvalue which again is equivalent with the condition that if $f\geqq 0$ and $g\geqq 0$, then $(f, e^{-tH}g)=0$ for all $t$ implies that $f=0$ or $g=0$. Take $f$ and $g$ to be characteristic functions for measurable sets $A$ and $B$. Then for $s\leqq t$

$$\Pr\{\xi(s)\in A \,\&\, \xi(t)\in B\}=(\chi_A, e^{-(t-s)H}\chi_B). \tag{3.32}$$

Now we have that if (3.32) is zero for all $t$, then either $A$ or $B$ has measure zero which is to say that the stochastic process $\xi(t)$ is ergodic. We also get that if $\xi(t)$ is ergodic, then $e^{-tH}$ is ergodic.

Since in the decomposition

$$e^{-tH}=\int_V e^{-tH_v}\,dv \tag{3.33}$$

the semigroup $e^{-tH_v}$ is ergodic, (3.33) gives the ergodic decomposition of the Markov semigroup $e^{-tH}$. But by what is said above we then have that

$$\mu(A)=\int_V \mu(A|v)\,dv \tag{3.34}$$

is the ergodic decomposition of the measure $\mu$ with respect to the action of the Markov process $\xi(t)$.

*Example 3.1.* Let $K$ be one dimensional, i.e. $Q=K=Q'=R$ (the real line) and let $d\mu=(\pi)^{-\frac{1}{2}}P_2(x)^2e^{-x^2}dx$ where $P_2(x)$ is the properly normalized second Hermite polynomial, i.e.

$$d\mu=\tfrac{1}{2}\pi^{-\frac{1}{2}}(2x^2-1)^2e^{-x^2}dx$$

and $\mu\in\mathscr{P}_1$. We then have that $d\mu=\varphi^2dx$, where $\varphi$ is the third lowest eigenfunction of the operator $-\Delta+x^2$. In fact $(-\Delta+x^2)\varphi=5\varphi$ so that by (2.6)

$$H=-\Delta+x^2-5$$

when applied to functions of the form $f\cdot\varphi$ with smooth $f$. Since $\varphi$ has simple zeros at $x=\pm\frac{1}{2}\sqrt{2}$, we actually find also that $H=-\Delta_0+x^2-5$ in $L_2(dx)$, where

$\Delta_0$ is the Laplacian with Dirichlet boundary conditions on $x=\pm\frac{1}{2}\sqrt{2}$. Hence

$$d\mu=\theta(-\tfrac{1}{2}\sqrt{2}-x)\,\varphi^2\,dx+\theta(x+\tfrac{1}{2}\sqrt{2})\,\theta(\tfrac{1}{2}\sqrt{2}-x)\,\varphi^2\,dx+\theta(\tfrac{1}{2}\sqrt{2}-x)\,\varphi^2\,dx \tag{3.35}$$

is the ergodic decomposition of $d\mu$ given by (3.33) in this case. The corresponding decomposition of $H$ and $e^{-tH}$ is given by

$$L_2(dx)=L_2(-\infty,-\tfrac{1}{2}\sqrt{2})\oplus L_2(-\tfrac{1}{2}\sqrt{2},\tfrac{1}{2}\sqrt{2})\oplus L_2(\tfrac{1}{2}\sqrt{2},\infty), \tag{3.36}$$

where in each component $H=-\Delta_0+x^2-5$, $\Delta_0$ being the Laplacian with Dirichlet boundary conditions for each component.

We shall call the ergodic decomposition (3.34) of $\mu$ with respect to the action of the Markov process $\xi(t)$ "the $T$-ergodic decomposition". Thus we have that the $T$-ergodic decomposition of $\mu$ is just the decomposition of $\mu$ into its conditional probability measures $\mu(\cdot|v)$ conditioned with respect to the $\sigma$-algebra generated by the functions which are eigenfunctions with eigenvalue zero for $H$.

Since we know already that the $Q$-invariant functions are eigenfunctions with eigenvalue zero for $H$, we see that the $T$-ergodic decomposition is a finer decomposition than the $Q$-ergodic decomposition given in (2.24), and the Example 3.1 indicates that normally the $T$-ergodic decomposition is strictly finer than the $Q$-ergodic decomposition.

Let now $\mu$ be a $Q$-quasi invariant probability measure on $Q'$. Let $P_q$ be the orthogonal projection onto $q$ in $K$, and let $\rho(x|\eta)$ for $x\in P_q K$ and $\eta\in(1-P_q)Q'$ be the conditional probability density in (3.5). We may identify $P_q K$ with the real line $R$. So that for $A\subset P_q K$ and $B\subset(1-P_q)Q'$ we have

$$\mu(A\times B)=\int_B\Big(\int_A\rho(x|\eta)\,dx\Big)\,d\nu(\eta). \tag{3.37}$$

We shall say that $\mu$ is *strictly positive* if $\rho(x|\eta)$ are bounded away from zero on compacts in $R$ for $\nu$-almost all $\eta$.

**Theorem 3.5.** *If $\mu$ is strictly positive and $\mu\in\mathscr{P}_1(Q')$ then the $T$-ergodic decomposition and the $Q$-ergodic decompositions are identical.*

*Proof.* Let $A\subset Q'$ be a subset that is measurable with respect to the $\sigma$-subalgebra generated by the eigenfunctions corresponding to the eigenvalue zero of $H$. Then as we have seen the characteristic function $\chi_A$ is an eigenfunction of eigenvalue zero of $H$. Since $H=\nabla^*\nabla$ we therefore have that $\chi_A\in D(\nabla)$ and $\nabla\chi_A=0$. In particular $q\cdot\nabla\chi_A=0$, so that

$$\int|q\cdot\nabla\chi_A|^2\,d\mu=0. \tag{3.38}$$

Let now $q^2=1$ and $\chi_A(\xi)=\chi_A(xq\oplus\eta)$ with $(q,\eta)=0$. Since

$$\int|q\cdot\nabla f|^2\,d\mu=\int_{R^\perp}\left(\int_R\left|\frac{d}{dx}f(xq+\eta)\right|^2\rho(x|\eta)\,dx\right)d\nu(\eta), \tag{3.39}$$

we see by (3.38) and the fact that $\rho(x|\eta)$ is bounded away from zero on compacts that $\chi_A(xq+\eta)$ is independent of $x$ for $v$-almost all $\eta$. Since $q$ was arbitrary we have that $\chi_A$ and therefore $A$ is invariant under translations by elements in $Q$. Hence we have proved that the $\sigma$-algebra generated by the zero eigenfunctions is contained in the $\sigma$-algebra of $Q$-invariant subsets. The other direction was proved in Theorem 2.4. This proves the theorem.

We recall that a quasi invariant probability measure $\mu$ was said to be in $\mathscr{P}$ if 1 is an analytic vector for $\pi(q)$ for any $q \in Q$. We have the following criteria for the strict positivity of $\mu$.[12]

**Theorem 3.6.** *Let $\mu$ be in $\mathscr{P}_\omega$, and such that $\pi(q)^n \cdot 1$ is in the domain of $q \cdot V$. Then we have that $\mu$ is strictly positive.*

*Proof.* That 1 is an analytic vector for $\pi(q)$ is by definition to say that there are some $r>0$ depending on $q$, such that

$$\|\pi(q)^n \cdot 1\| \leqq r^{-n} n!. \tag{3.40}$$

Let now $q \in Q$ with $q^2=1$ and $\rho(x|\eta)$ for $(\eta, q)=0$ be given by (3.37). Then we have the direct decomposition

$$L_2(Q', d\mu) = \int_{R^\perp} L_2(\rho(x|\eta)\,dx)\,dv(\eta), \tag{3.41}$$

where $R^\perp$ is the subspace of $Q'$ orthogonal to $q$. This decomposition reduces $V(tq)$ and therefore also $\pi(q)$, so that

$$\pi(q) = \int_{R^\perp} \pi_\eta(q)\,dv(\eta). \tag{3.42}$$

We shall see that 1 is an analytic function also for $\pi_\eta(q)$, inasmuch as

$$(f, V(tq)1) = \int_{R^\perp} \left[\int_R \bar{f}(xq+\eta)\,\varphi(x|\eta) \cdot \varphi(x+t|\eta)\,dx\right] dv(\eta) \tag{3.43}$$

where $\varphi(x|\eta) = \rho(x|\eta)^{\frac{1}{2}}$.

(3.43) is analytic in $t$ for $|t|<r$, so let $\Gamma$ be any smooth closed curve in the disk $|z|<r$. Then the integral of (3.43) with respect to $t$ around $\Gamma$ is zero. So by the Fubini theorem

$$\int_{R^\perp} \left[\int_R \bar{f}(xq+\eta)\,\varphi(x|\eta)\left(\int_\Gamma \varphi(x+t|\eta)\,dt\right) dx\right] dv(\eta) = 0. \tag{3.44}$$

Since $f$ is arbitrary in $L_2(d\mu)$ and $\varphi(x|\eta)>0$ for almost all $x$ and $v$-almost all $\eta$ we have that

$$\int_\Gamma \varphi(x+z|\eta)\,dz = 0 \tag{3.45}$$

[12] These (and related) criteria find applications e.g. to quantum fields, see e.g. [66, 67] and Section 5

for almost all $x$ and $\nu$-almost all $\eta$. Hence $\varphi(x+z|\eta)$ is analytic for $|z|<r$ for all $x$ and $\nu$-almost all $\eta$, where $r$ is given in (3.40). So that $\varphi(z|\eta)$ is analytic in a strip of width $2r$ around the real $z$ axis.

Furthermore we have that $\pi(q)^n \cdot 1$ is in the domain of $q \cdot \nabla$. Using now the direct decomposition (3.42) we have

$$\int \|q \cdot \nabla_\eta \pi_\eta(q)^n \cdot 1\|^2 \, d\nu(\eta) = \|q \cdot \nabla \cdot \pi(q)^n 1\|^2, \tag{3.46}$$

so that, for $\nu$-almost all $\eta$, $\|q \cdot \nabla_\eta \pi_\eta(q)^n \cdot 1\| < \infty$.

However

$$\begin{aligned} \|q \nabla_\eta \pi_\eta(q)^n \cdot 1\|^2 &= \int_R \left(\left(\frac{\varphi^{(n)}(x|\eta)}{\varphi(x|\eta)}\right)'\right)^2 \varphi^2(x|\eta)\, dx \\ &= \int_R \left|\frac{\varphi^{(n+1)}(x|\eta)}{\varphi(x|\eta)} - \frac{\varphi'(x|\eta)}{\varphi(x|\eta)} \cdot \frac{\varphi^{(n)}(x|\eta)}{\varphi(x|\eta)}\right|^2 \varphi^2(x|\eta)\, dx. \end{aligned} \tag{3.47}$$

Now we have that $\dfrac{\varphi^{(n+1)}}{\varphi} \in L_2(\varphi^2(\cdot|\eta))$ for $\nu$-almost all $\eta$ since

$$\int_{R^\perp} \int_R \left(\frac{\varphi^{(n+1)}(x|\eta)}{\varphi(x|\eta)}\right)^2 \varphi(x|\eta)^2 \, dx \, d\nu(\eta) = \|\pi(q)^{n+1} \cdot 1\|^2, \tag{3.48}$$

which is finite by assumption. Since by (3.47) and (3.46) the difference is in $L_2(\varphi^2(x|\eta)\,dx)$ for $\nu$-almost all $\eta$ we have that

$$\frac{\varphi'(x|\eta)}{\varphi(x|\eta)} \cdot \frac{\varphi^{(n)}(x|\eta)}{\varphi(x|\eta)} \in L_2(\varphi^2(x|\eta)\,dx) \tag{3.49}$$

for $\nu$-almost all $\eta$. Now since $\varphi(x|\eta)$ is analytic in $x$ we have that the zeros of $\varphi$ are isolated and of finite order.

Suppose $\varphi$ has a zero $a$, so that $\varphi(a|\eta)=0$, then there is an $n$ such that $\varphi^{(n)}(a|\eta) \neq 0$. By (3.49) $\dfrac{\varphi'(x|\eta)}{\varphi(x|\eta)} \cdot \varphi^{(n)}(x|\eta) \in L_2(dx)$ and since $\varphi^{(n)}(x|\eta) \neq 0$ in a neighborhood of $a$, we get that $\dfrac{\varphi'(x|\eta)}{\varphi(x|\eta)}$ is in $L_2(dx)$ near $a$. Let $n$ be the lowest value such that $\varphi^{(n)}(a|\eta) \neq 0$, then $\varphi(x|\eta) \sim c(x-a)^n$ near $a$. From this we get that

$$\frac{\varphi'(x|\eta)}{\varphi(x|\eta)} \notin L_2(dx)$$

near $x=a$, a contradiction. Hence we have that $\varphi(x|\eta)>0$ for all $x$, and this proof goes for $\nu$-almost all $\eta$. Since $\varphi(x|\eta)$ is analytic in $x$, it is therefore bounded away from zero on compacts. This proves the theorem.

**Theorem 3.7.** *Let $\mu \in \mathscr{P}_1(Q')$ and assume that zero is separated from the rest of the spectrum of $H$ by a positive distance $m>0$, where $H$ is the corresponding Dirichlet operator.*

*Let* $\operatorname{ad}\pi(q)(H)=[(q),H]$ *and let us assume that for any* $q\in Q$ *there is a constant* $c_q>0$ *depending only on* $q$ *such that*

$$\|(H+1)^{-\frac{1}{2}}\operatorname{ad}^n\pi(q)(H)(H+1)^{-\frac{1}{2}}\|\leqq c_q^n$$

*for all* $n=1,2,3,\ldots$. *Then for any vector* $v$ *such that* $Hv=0$ *we have that*

$$\|(H+1)^{\frac{1}{2}}\pi(q)^n v\|\leqq n!\left(\frac{m+1}{m}e\,c_q\right)^n\|v\|.$$

*In particular we get that* $v$ *is an analytic vector for* $\pi(q)$, *and* $\mu$ *is in* $\mathscr{P}_\omega$ *and strictly positive.*

*Proof.* For any $n$ we have the following algebraic relation

$$H\pi(q)^n=\pi(q)^n H-\sum_{j=1}^n\binom{n}{j}\operatorname{ad}^j\pi(q)(H)\,\pi(q)^{n-j}. \tag{3.50}$$

So if $Hv=0$ we get by the assumptions of the theorem that

$$\|(H+1)^{-\frac{1}{2}}H\pi(q)^n v\|\leqq\sum_{j=1}^n\binom{n}{j}c_q\|(H+1)^{\frac{1}{2}}\pi(q)^{n-j}v\|. \tag{3.51}$$

Let $m>0$ be the separation of zero from the rest of the spectrum of $H$, then, since $H\pi(q)^n v$ is in the subspace orthogonal to the zero eigenspace and $(H+1)H^{-1}$ is norm bounded by $\frac{m+1}{m}$ on that subspace, we have

$$\|(H+1)^{\frac{1}{2}}\pi(q)^n v\|\leqq\frac{m+1}{m}\sum_{j=1}^n c_q^j\binom{n}{j}\|(H+1)^{\frac{1}{2}}\pi(q)^{n-j}v\|. \tag{3.52}$$

Let us now assume that

$$\|(H+1)^{\frac{1}{2}}\pi(q)^k v\|\leqq k!\left(\frac{m+1}{m}e\,c_q\right)^k\|v\|, \tag{3.53}$$

which is obviously true for $k=0$. Then by (3.52) we have that

$$\|(H+1)^{\frac{1}{2}}\pi(q)^{k+1}v\|\leqq\frac{m+1}{m}c_q^{k+1}\left(\frac{m+1}{m}\right)^k e^k(k+1)!\sum_{j=1}^{k+1}\frac{1}{j!}\|v\|,$$

$$\|(H+1)^{\frac{1}{2}}\pi(q)^{k+1}v\|\leqq\left(c_q\frac{m+1}{m}\cdot e\right)^{k+1}(k+1)!\,\|v\|.$$

Hence the inequality of the theorem is proved by induction. Take $v=1$, then the inequality gives us that $\mu$ is analytic and $\pi(q)^n 1$ is in the domain of $H^{\frac{1}{2}}$ hence also in the domain of $q\cdot V$. This proves the theorem.

Let $dn$ be the normal pro measure (i.e. normal distribution) associated with the real separable Hilbert space $K$, i.e. the integral with respect to $dn$ is defined for all functions on $K$ which are continuous bounded and for which $f(x)=f(Px)$

for some finite dimensional projection $P$, and

$$\int e^{i(x,y)}\,dn(x)=e^{-\frac{1}{2}(y,y)}. \tag{3.54}$$

It is well known that the integral above is not given by a countable additive measure on $K$, but however there exist suitable compactifications of $K$ such that the finitely based continuous functions can be continued onto the compactification and the integral (3.54) is given by a countably additive measure $dn$ on this compactification. However there is no natural choice of such a compactification, and a class of compactifications were given by Leonard Gross in the following way. For reference see [6, 7, 68, 69].

A seminorm $p(x)$ on $K$ is said to be measurable if for any $\varepsilon>0$ there is a finite dimensional orthogonal projection $P_0$ such that, for any finite dimensional projection $P$ orthogonal to $P_0$, we have that

$$\int\limits_{p(Px)>\varepsilon} dn(x)<\varepsilon. \tag{3.55}$$

It is a consequence of (3.55) that $p(x)$ is a continuous seminorm on $K$. Moreover Gross proves that if $B'$ is the completion of $K$ with respect to a measurable norm, then the integral (3.54) is given by a regular measure $n$ on $B'$. In fact we have the following theorem due to Gross.

**Theorem 3.8.** *Let $B$ be the completion of $K$ with respect to a measurable norm. Then the integral* (3.54) *is given by a regular measure on $B'$. Moreover if $|x|$ is any measurable norm on $K$, then $|x|$ is a continuous norm on $K$ and if $\hat{w}(t)$ is the standard weak Wiener process on $K$ and if $B'$ is the completion of $K$ with respect to $|x|$, then $\hat{w}(t)$ may be realized as a stochastic process on $B'$ with continuous paths.*

For the proof of this theorem and more details about the Wiener process associated with a real separable Hilbert space see References [6, 7, 68, 69]. We also remark that in fact it follows from the proof of Theorem 3.8 that the standard weak Wiener process on $K$ is continuous with respect to any measurable seminorm.

Let us now consider a Banach rigging

$$B\subset K\subset B' \tag{3.56}$$

of the real separable Hilbert space $K$ where $B$ is a real separable Banach space dense in its dual $B'$ such that $B'$ is the completion of $K$ with respect to a measurable norm on $K$. We shall refer to the rigging (3.56) shortly as a measurable Banach rigging of $K$. Let now $\mu\in\mathscr{P}_1(Q')$ where $Q\subset K\subset Q'$ is the original rigging of $K$, and let $q\cdot\beta(\xi)$ be the corresponding osmotic velocity, i.e.

$$i\pi(q)\cdot 1=\tfrac{1}{2}q\cdot\beta(\xi). \tag{3.57}$$

Let $|\ |'$ be the norm in $B'$, then since $B$ is separable

$$|\beta(\xi)|'=\sup_n\frac{|q_n\cdot\beta(\xi)|}{|q_n|} \tag{3.58}$$

is a measurable function, where $\{q_n\}$ is a dense countable set in $Q$ that is dense in $B$, and $|\;|$ is the norm in $B$.

We then have the following theorem:

**Theorem 3.9.** *Let $B \subset K \subset B'$ be a measurable Banach rigging of $K$ such that $Q \subset B \subset K \subset B' \subset Q'$. Let $\mu \in \mathscr{P}_1(Q')$ and $\beta(\xi)$ and $\xi(t)$ the corresponding osmotic velocity and Markov process and let us assume that $|\beta(\xi)|'$ is bounded, where $|\;|'$ is the norm in $B'$. Then $\xi(t)$ is continuous in the $B'$ norm, i.e., for any $t$ and $s$, $\xi(t) - \xi(s)$ is in $B'$ and $|\xi(t)-\xi(s)|' \to 0$ as $t \to s$, for almost all paths.*

*Remark.* We may conclude that $\xi(t)$ is a continuous process with values in $B'$ only in the case where $B'$ has $\mu$-measure 1.

*Proof.* From Theorem 3.3 we know that in a weak sense

$$\xi(t)-\xi(s)=\int_s^t \beta(\xi(\tau))\,d\tau + w(t)-w(s). \tag{3.59}$$

By Theorem 3.8 we have that $|w(t)-w(s)|'$ goes to zero as $t \to s$, and the conclusion of the theorem then follows by the triangle inequality for the norm. This proves the theorem.

*Remark.* A corresponding theorem for the finite dimensional case was given by Stroock and Varadhan in [35].

We shall close this section with a general result giving a sufficient condition for a measure to be quasi invariant.

Let us consider again the general situation of a rigged separable real Hilbert space $Q \subset K \subset Q'$, as in Section 2. We shall consider as before the splitting $Q' = R \oplus R^\perp$ induced on $Q'$ by the decomposition of the identity $I = P_R + (I - P_R)$, where $R$ is a one-dimensional subspace of $Q \subset K$ (identified with the real line) and $P_R$ is the continuous extension of the projection onto $R$ to all of $Q'$. The one-to-one bicontinous map $(x, \eta) \to x \oplus \eta$, $x \in R$, $\eta \in R^\perp$ permits to consider the measure $\mu$ on $Q'$ as a measure on $R \times R^\perp$. We shall assume as before that $Q'$ is such that the conditional probability measures with respect to $R^\perp$ exits (which is e.g. the case if $Q'$ is Suslin). Let $\mu(x|\eta)$ the conditional probability measure obtained on $R$ from $\mu$ by conditioning with respect to the $\sigma$-algebra of the measurable subsets of $R^\perp$. Then for any measurable set $B$ in $R^\perp$ and any Borel set $A$ in $R$ one has

$$\mu(A \times B) = \int_B \mu(A|\eta)\,d\nu(\eta),$$

where $\nu$ is the measure induced on $R^\perp$ by $\mu$. We shall say that a finite measure $\mu$ on $Q'$ is $q$-quasi-invariant, where $q$ is an element in $Q$, if $\mu(\xi + q)$ is equivalent $\mu(\xi)$, as measures on $Q'$.

**Theorem 3.10.** *Let $Q \subset K \subset Q'$ be a rigged Hilbert space as in Section 2, with $Q'$ such that for a probability measure $\mu$ on $Q'$ the conditional probability measure relative to $R^\perp$, in the splitting $Q' = R \oplus R^\perp$, exists, with $R \subset Q$ a one-dimensional subspace of $K$.*

*Suppose that for $q \in Q$ and all $g \in FC^\infty$ we have*

$$\left|\int_{Q'} [(q\cdot\nabla)^n g]\, d\mu\right| \leqq a^n n!\, \|g\|_\infty$$

*for all $n=0, 1, 2, \ldots$, where $\|\ \ \|_\infty$ is the supremum norm, and $a$ is independent of $n$ and $g$. Then $\mu$ is $q$-quasi-invariant.*

*Proof.* Let $q \in Q$ be given and let $R$ be the one-dimensional subspace of $K$ spanned by $q$ i.e. $R=\{\alpha q\}$, $\alpha$ real. Consider the splitting $Q'=R\oplus R^\perp$ as described before the Theorem and let $\mu(x|\eta)$ be the conditional probability measure on $R$, with respect to $R^\perp$ and $\nu(\eta)$ the projection of $\mu$ on $R^\perp$ so that for every $\eta\in Q'$, $\mu(\cdot|\eta)$ is a measure on $R$. Then for any $h\in FC^\infty(R^\perp)$

$$\sigma_h(\cdot)=\int_{R^\perp} \mu(\cdot|\eta)\, h(\eta)\, d\nu(\eta) \tag{3.60}$$

is a measure on $R$. Let now $f\in C_0^\infty(R)$, then $fh\in FC^\infty(Q')$. We have

$$\int [(q\cdot\nabla)^n fh]\, d\mu = \int_{R\times R^\perp} f^{(n)}(x)\, h(\eta)\, d\mu(x,\eta) = \int_R f^{(n)}(x)\, d\sigma_h(x). \tag{3.61}$$

By the assumption we then get

$$\left|\int_R f^{(n)}(x)\, d\sigma_h(x)\right| \leqq a^n n!\, \|f\|_\infty\, \|h\|_\infty \tag{3.62}$$

and thus, calling $\sigma_h^{(n)}$ the $n$-th distributional derivative of $\sigma_h$:

$$\left|\int f(x)\, d\sigma_h^{(n)}(x)\right| \leqq a^n n!\, \|f\|_\infty\, \|h\|_\infty. \tag{3.63}$$

This shows that $\sigma_h^{(n)}$ is a measure with bounded total variation, hence

$$\sigma_h^{(n)}(x)=\int_{-\infty}^{x} d\sigma_h^{(n+1)}(x') \tag{3.64}$$

with

$$|\sigma_h^{(n)}(x)| \leqq \int_{-\infty}^{x} d|\sigma_h^{(n+1)}(x')| \leqq a^n n!\, \|h\|_\infty. \tag{3.65}$$

This shows that $\sigma_h^{(n)}(x)$ is actually a real analytic function of $x$, hence it has an analytic continuation, denoted $\sigma_h(z)$, with $z=x+iy$, $y=\operatorname{Im} z$, to the strip $|\operatorname{Im} z|<1/a$. For fixed $x$ we have

$$\sigma_h(z)=\sum_{n=0}^{\infty} \frac{(iy)^n}{n!}\, \sigma_h^{(n)}(x) \tag{3.66}$$

and the series converges absolutely, uniformly in any closed strip $|\operatorname{Im} z| \leqq (1/a) - \varepsilon$, $\varepsilon>0$. But

$$\sigma_h^{(n)}(x)=\int \mu^{(n)}(x|\eta)\, h(\eta)\, d\nu(\eta), \tag{3.67}$$

with the bound (3.65). This bound then shows, $L_1(d\nu)$ being a closed space of continuous linear functionals, that $\|\mu^{(n)}(x|\cdot)\|_{L_1(d\nu)} \leqq a^n n!$. This then implies that the series

$$\sum_{n=0}^{\infty} \frac{(iy)^n}{n!} \mu^{(n)}(x|\cdot) \tag{3.68}$$

converges strongly in $L_1(d\nu)$, for all real $x$ and all real $y$ with $|y|<1/a$.

Define $\mu(z|\cdot) \equiv \mu(x+iy|\cdot)$ as the $L_1(d\nu)$-sum of this series. We shall now show that $\mu(z|\cdot)$ is the analytic continuation, as a $L_1(d\nu)$-valued function, of $\mu(x|\eta)$ to the strip $|\mathrm{Im}\, z|<1/a$. In fact $\sigma_h(z)$ is analytic in this strip by what we said above. Thus for any smooth closed curve $\Gamma$ in the strip $|\mathrm{Im}\, z|<1/a$ we have

$$\int_\Gamma \sigma_h(z)\, dz = 0.$$

Hence

$$\sum_{n=0}^{\infty} \int_\Gamma dz \int_{R^\perp} \frac{(iy)^n}{n!} \mu^{(n)}(x|\eta)\, h(\eta)\, d\nu(\eta) = 0$$

i.e.

$$\int_\Gamma dz \int_{R^\perp} \mu(z|\eta)\, h(\eta)\, d\nu(\eta) = 0.$$

By the Hahn-Banach theorem this implies $\int_{R^\perp} \mu(z|\eta)\, dz = 0$ in the strong $L_1(d\nu)$-sense. Thus we have that the function $\mu(z|\eta)$ from the strip $|\mathrm{Im}\, z|<1/a$ into $L_1(d\nu)$ is analytic.

In particular the strong $L_1(d\nu)$-derivative of $\mu(x|\eta)$, with respect to $x$ exists, call it $\mu^{(1)}(x|\eta)$, so that

$$s - L^1(d\nu) - \lim_{h\to 0} \frac{\mu(x+h|\cdot)-\mu(x|\cdot)}{h} = \mu^{(1)}(x|\cdot),$$

for all $x \in R$. For any given sequence $h_n \to 0$ we have that there exists a subsequence $h_{n_k}$, $h_k \to \infty$ such that $h_{n_k}^{-1}(\mu(x+h_{n_k}|\eta)-\mu(x|\eta)) \to \mu^{(1)}(x|\eta)$ as $n_k \to \infty$, $k \to \infty$, for all $\eta \in R^\perp - N(x, \{n_k\})$, with $\nu(N(x, \{n_k\}) = 0$.

This shows that $\mu(x|\eta)$ is differentiable with respect to $x$ for all $\eta \in R^\perp - N(x|\{\eta_k\})$, with derivative $\mu^{(1)}(x|\eta)$. Similarly we show that $\mu(x|\eta)$ is $n$-times differentiable with respect to $x$ for all $\eta \in R^\perp - N(x)$, with derivative $\mu^{(n)}(x|\eta)$, and with $\nu(N(x))=0$. Let now $\eta \in R^\perp - N(x)$ and $|\alpha|<1/a$. Then by the $L_1(d\nu)$-analyticity of $\mu(z|\cdot)$ in $|\mathrm{Im}\, z|<1/a$ we have $\mu(x+\alpha|\cdot) = \sum \frac{\alpha^n}{n!} \mu^{(n)}(x|\cdot)$ with the sum strongly convergent in $L_1(d\nu)$. Consider $\sum^N \frac{|\alpha|}{n!} |\mu^{(n)}(x|\cdot)|$. This is positive and as $N$ increases it increases monotonically. Moreover the limit as $N\to\infty$ exists in $L_1(d\nu)$ and is finite, hence $\sum^\infty \frac{|\alpha|^n}{n!} |\mu^{(n)}(x|\eta)| < \infty$ for $\nu$-a.e. $\eta$ and the exceptional set can be taken independent of $\alpha$. Thus $\sum \frac{\alpha^n}{n!} \mu^{(n)}(x|\eta)$ converges and is finite for $\nu$-a.e. $\eta$, so that $\mu(x+\alpha|\eta)=\sum \frac{\alpha^n}{n!} \mu^{(n)}(x|\eta)$, where the sum converges for $\nu$-a.e. $\eta$ and all $x \in R$, $|\alpha|<1/a$. Call $\hat{N}(x)$ the exceptional set. Let now $\{x_k\}$, be a sequence of points dense in $R$. Set $N=\bigcup_k \hat{N}(x_k)$. Then for $\eta \in R^\perp - N$ we have by the above that $\mu(x|\eta)$ is analytic at $x_k$ for any $k$ in any circle of radius smaller that $1/a$ around $x_k$. This implies however that $\mu(z|\eta)$ is actually analytic in $|\mathrm{Im}\, z|<1/a$, for all $\eta$ in $R^\perp - N$. Let now $\eta$ be fixed in $R^\perp - N$. Then by the analyticity of $\mu(z|\eta)$ in $|\mathrm{Im}\, z|<1/a$ we have $\mu(z|\eta)=0$ at most on a countable set $M_z$ or $\mu(z|\eta)\equiv 0$ for all $z$ in the strip. In both cases we have that $\mu(x+\alpha, \eta)$ is equivalent with $\mu(x|\eta)$, as a measure on the real line $R$. This implies that $\mu(x+\alpha|\eta)\, d\nu(\eta)$ is equivalent with $\mu(x|\eta)\, d\nu(\eta)$ as measures on $R \times R^\perp$ i.e. $d\mu^\alpha(x,\eta)$ is equivalent with $d\mu(x,\eta)$ i.e. $\mu$ is quasi invariant with respect to translations by elements in $R$, which proves the Theorem.

## 4. Weak Convergence of Measures and Dirichlet Operators

In this section we assume that $Q \subset K \subset Q'$ is a separable rigged Hilbert space as in the preceding section, and we also assume that $Q'$ is such that regular conditional probability measures exist for any probability measure on $Q'$ and any splitting $Q' = F \oplus F^\perp$, where $F$ is a finite dimensional subspace of $Q$.

If $\mu$ is a probability measure on $Q'$ we say that $\mu$ is *admissible* if the operator from a dense domain $V$ of $L_2(Q')$ into $K \hat{\otimes} L_2(Q')$ has a densely defined adjoint $V^*$. In which case it is closable and the Dirichlet operator $H = V^* \bar{V}$ exists and is a self adjoint operator in $L_2(Q', d\mu)$, where $\bar{V}$ is the closure of $V$. In particular we have that any measure in $\mathscr{P}_1(Q')$ is admissible. More generally we have that if the constants $q \otimes 1$ are in $D(V^*)$, the domain of $V^*$, then $FC^1 \subset D(V^*)$ and the measure is admissible.

**Proposition 4.1.** *Let $\mu$ be admissible. Then for any $q \in Q$ we have $\langle \xi, q\rangle \in D(\bar{V})$ and for any $f \in FC^1$ we have*

$$\bar{V}(\langle \xi, q\rangle f(\xi)) = q \otimes f(\xi) + \langle \xi, q\rangle \bar{V} f(\xi).$$

*Proof.* For any real $\alpha$ we have that $\frac{1}{\alpha}(e^{i\alpha\langle \xi, q\rangle} - 1) f(\xi)$ is in $D(V)$ and converges strongly in $L_2(Q', d\mu)$, as $\alpha \to 0$, towards $\langle \xi, q\rangle f(\xi)$, while

$$\begin{aligned} V \frac{1}{\alpha}(e^{i\alpha\langle \xi, q\rangle} - 1) f(\xi) &= q \otimes e^{i\alpha\langle \xi, q\rangle} f(\xi) + \frac{1}{\alpha}(e^{i\alpha\langle \xi, q\rangle} - 1) V f(\xi) \\ &\to q \otimes f(\xi) + \langle \xi, q\rangle V f(\xi), \end{aligned}$$

as $\alpha \to 0$. The closedness of $\bar{V}$ gives then the Proposition.

**Proposition 4.2.** *Let $q \otimes 1 \in D(V^*)$ for all $q \in Q$. Then $\mu$ is admissible and for any $q \in Q$ we have, with obvious notations, $\langle \xi, q\rangle FC^2 \subset D(H)$, $HFC^2 \subset D(\langle \xi, q\rangle)$ and,* *defining* $\pi(q)$ *as the symmetric operator* $\frac{1}{2i}((q\cdot\nabla)-(q\cdot\nabla)^*)$ *on* $FC^1$, *we have* $[H,\langle\xi,q\rangle]=\frac{2}{i}\pi(q)$, *as operators on* $FC^2$.

*Remark.* When $\mu$ is admissible, quasi invariant and in $\mathscr{P}_1(Q')$, then $\pi(q)$, as defined in Proposition 4.2, is the restriction to $FC^1$ of the infinitesimal generator of the one parameter unitary group

$$V(tq)f(\xi)=\sqrt{\frac{d\mu(\xi+tq)}{d\mu(\xi)}}\,f(\xi+tq),\qquad t\in R.$$

*Proof.* For any $f\in FC^1$ we have by Proposition 4.1 $\langle\xi,q\rangle f(\xi)\in D(\bar\nabla)$ and $\bar\nabla\langle\xi,q\rangle f(\xi)=q\otimes f(\xi)+\langle\xi,q\rangle\bar\nabla f(\xi)$, thus, for any $h\in FC^2$,

$$\begin{aligned}(Hh,\langle\cdot,q\rangle f)&=\int\bar\nabla h(\xi)\,\bar\nabla\langle\xi,q\rangle f(\xi)\,d\mu(\xi)\\&=\int\bar\nabla h(\xi)\,q\otimes f(\xi)\,d\mu(\xi)+\int\bar\nabla h(\xi)\langle\xi,q\rangle\,\bar\nabla f(\xi)\,d\mu(\xi),\end{aligned}$$

where $(\ ,\ )$ is the scalar product in $L_2(Q',d\mu)$. Using $\nabla^*(q\otimes f)=-\nabla f-\beta(q)$ and the fact that $\langle\xi,q\rangle\,\bar\nabla f(\xi)\in D(\nabla^*)$ we have that the right hand side is equal to

$$\int h(\xi)(-\nabla f(\xi)-\beta(q)(\xi))\,d\mu(\xi)+\int h(\xi)\nabla^*(\langle\xi,q\rangle\bar\nabla f(\xi))\,d\mu(\xi),$$

which then shows, being a bounded continuous function of $h$, that $\langle\xi,q\rangle FC^2\subset D(H)$. Since however, by the definition of $\pi(q)$, $D(\pi(q))\supset FC^2$ we get then that $\frac{2}{i}\pi(q)\,f(\xi)+H\langle\xi,q\rangle\,f(\xi)$ is well defined and equal to $\langle\xi,q\rangle Hf(\xi)$.

**Proposition 4.3.** *Under the same assumption as in Proposition* 4.2 *we have, for any* $f\in FC^2$ *and* $q,q'\in Q$

$$e^{-i\langle\xi,q\rangle}\pi(q')\,e^{i\langle\xi,q\rangle}f(\xi)=\frac{1}{i}\langle q',q\rangle f(\xi)+\pi(q')f(\xi)$$

*and*

$$e^{-i\langle\xi,q\rangle}H\,e^{i\langle\xi,q\rangle}f(\xi)=Hf(\xi)+2\pi(q)\,f(\xi)+\langle q,q\rangle\,f(\xi).$$

*Proof.* The first relation is proven using $2\pi(q')=q'\cdot\nabla+\frac{1}{2}\beta(q')$ on $FC^2$. To prove the second relation it suffices to compute, for $f_1,f_2$ in $FC^2$, $(e^{i\langle\xi,q\rangle}f_1, He^{i\langle\xi,q\rangle}f_2)$, using $H=\nabla^*\bar\nabla$ and

$$\bar\nabla e^{i\langle\xi,q\rangle}f_j(\xi)=q\otimes e^{i\langle\xi,q\rangle}f_j(\xi)+e^{i\langle\xi,q\rangle}\,\bar\nabla f_j(\xi),\qquad j=1,2.$$

**Corollary.** $\pm2\pi(q)\leqq H+\langle q,q\rangle$.

*Proof.* Use the second relation of Proposition 4.3 together with

$$e^{-i\langle\xi,q\rangle}H\,e^{i\langle\xi,q\rangle}\geqq0.$$

Let now $\mu_l$ be a sequence of probability measures on $Q'$ so that $\mu_l$ converge weakly to the probability measure $\mu$ and in addition let us assume that for any $q\in Q$ we have that there is a $C$ independent of $l$ such that

$$\left|\int((q\cdot\nabla)f)\,d\mu_l\right|\leqq C\left(\int|f|^2\,d\mu_l\right)^{\frac{1}{2}}\tag{4.1}$$

for any $f \in FC^\infty$. Then by the weak convergence, and note that we only need weak convergence on $FC^\infty$, we have that

$$(\int ((q \cdot \nabla) f) \, d\mu) \leqq C (\int |f|^2 \, d\mu)^{\frac{1}{2}}. \tag{4.2}$$

Since $FC^\infty$ is dense in $L_2(d\mu)$ it follows from (4.2) that the constants $q \otimes 1 \in D(\nabla^*) \subset L_2(d\mu)$. Since obviously for any $f \in FC^1$ we have $\nabla^* q \otimes f = (q \nabla) f + f \cdot \nabla^*(q \otimes 1)$ we see that also $FC^1 \subset D(\nabla^*)$ so that $\mu$ is admissible. Hence we have the following theorem

**Theorem 4.1.** *Let $\mu_l$ be probability measures on $Q'$ converging weakly on $FC^\infty$ to the probability measure $\mu$ such that the constants are uniformly in the domain of $\nabla^*$ i.e. there is a positive constant $C$ independent of* 1 *such that* (4.1) *holds. Then $\mu$ is admissible and $FC^1 \subset D(\nabla^*)$.*

We say that a probability measure $\mu$ on $Q'$ is analytic iff for any $q \in Q$ there is a positive real number $a_q$ such that

$$|\int ((q \cdot \nabla)^n g) \, d\mu| \leqq a_q^n n! \, \|g\|_\infty \tag{4.3}$$

for any $g \in FC^\infty$.

**Theorem 4.2.** *Let $\mu_l$ be a sequence of probability measures on $Q'$ which converge weakly on $FC^\infty$ to a probability measure $\mu$. If $\mu_l$ is an equi analytic sequence, i.e. for any $q \in Q$ there is a constant $a_q$ independent of $l$ such that*

$$|\int (q \cdot \nabla)^n g \, d\mu_l| \leqq a_q^n n! \, \|g\|_\infty,$$

*for any $g \in FC^\infty$, then $\mu$ is analytic and, by Theorem* 3.10, *$Q$-quasi invariant.*

*Proof.* By weak convergence we have (4.3) and hence $\mu$ is analytic.

Let now $\mu$ a $Q$-quasi invariant measure such that the constants are in the domain of $\nabla^*$ hence $FC^1 \subset D(\nabla^*)$. Let $\pi(q)$ be the self adjoint infinitesimal generator of the unitary translations in the direction $q \in Q$. It is easily seen that $\pi(q)$ is a self adjoint extension of $\frac{1}{2i}(q \cdot \nabla - (q \cdot \nabla)^*)$. It follows from $FC^1 \subset D(\nabla^*)$ that $FC^1$ is in the domain of $\frac{1}{2i}(q \cdot \nabla - (q \cdot \nabla)^*)$ so that $FC^1 \subset D(\pi(q))$. Especially $\pi(q) 1 = -\frac{1}{2i}(q \cdot \nabla)^* 1$, and therefore we have that $\mu \in \mathscr{P}_1(Q')$. From this together with Theorem 4.1 and Theorem 4.2 we have

**Theorem 4.3.** *Let $\mu_l$ be a sequence of probability measures on $Q'$ which converge weakly and the constants are uniformly in the domain of $\nabla^*$, then $\mu$ is in $\mathscr{P}_1(Q')$.*

Let now $\mu$ be an analytic probability measure on $Q'$. Let $F$ be a finite dimensional subspace $F \subset Q$ and let $F^\perp$ be its orthogonal complement in $Q'$. Then $Q' = F \oplus F^\perp$ and let $\mu(x|\eta)$ be the corresponding conditional probability measure, where $x \in F$ and $\eta \in F^\perp$, so that $d\mu = \mu(x|\eta) \, d\nu(\eta)$, as a probability measure on $F \times F^\perp$. In the proof of Theorem 3.10 we proved that $\mu(x|\eta) = \rho(x|\eta) \, dx$, where $\rho(x|\eta)$ was analytic in a uniform complex strip $|\mathrm{Im}\, z| < a$ for $\eta \in F^\perp - N_F$ with $\nu(N_F) = 0$, in the case of one dimensional $F$. It is easily seen from the proof of Theorem 3.10 that this result also holds for any finite dimensional $F \subset Q$. From the analyticity of $\mu$,

$$|\int (q \cdot \nabla)^n g \, d\mu| \leqq a_q^n n! \, \|g\|_\infty, \tag{4.4}$$

it follows by taking $q$ arbitrary in $F$ that there is a set $N_F \subset F^\perp$ of $\nu$-measure zero so that for any $\eta \in F^\perp - N_F$ we have that $\rho(x|\eta)$ is analytic in the strip $|\mathrm{Im}\, z| < \alpha$ for some $\alpha > 0$ and $z$ is in the complexification of $F$ and $|\ |$ is the norm in the real Hilbert space $K$.

Furthermore we also have as in the proof of Theorem 3.10 that $x \to \rho(x|\eta)$ is a strongly analytic function from $F$ into $L_1(F^\perp, d\nu)$ which is analytic in the strip $|\mathrm{Im}\, z| < \alpha$. From (4.4) we also have that for any $q \in F$

$$\iint |(q \cdot \nabla)^n \rho(x|\eta)| \, dx \, d\nu(\eta) \leqq a_q^n n!. \tag{4.5}$$

Let now $e_1, \ldots, e_k$ be an orthonormal base in $F$ relative to the inner product in $K$ and let $x_i = (e_i, x)$. From (4.5) we have that

$$\int |(q \cdot \nabla)^n \rho(x|\eta)| \, dx \tag{4.6}$$

is finite for $\nu$-almost all $\eta$. Hence we get, if necessary enlarging $N_F$, that there is a set $N_F$ of $\nu$-measure zero such that, for any $\eta \in F^\perp - N_F$, $\rho(x|\eta)$ is analytic in $x$ in the strip $|\mathrm{Im}\, z| < \alpha$ and

$$\int \left| \frac{\partial^{n_1}}{\partial x_1^{n_1}} \cdots \frac{\partial^{n_k}}{\partial x_k^{n_k}} \rho(x|\eta) \right| dx \tag{4.7}$$

is finite for any $n_1, \ldots, n_k$.

Especially we get that

$$\rho(x|\eta) = \int \cdots \int_{\Lambda_x} \frac{\partial}{\partial y_1} \cdots \frac{\partial}{\partial y_k} \rho(y|\eta) \, dy_1 \ldots dy_k \tag{4.8}$$

with

$$\Lambda_x \equiv \{y_i | y_j \leqq x_j, j = 1, \ldots, k\},$$

so that

$$|\rho(x|\eta)| \leqq \int \cdots \int_{\Lambda_x} \left| \frac{\partial}{\partial y_1} \cdots \frac{\partial}{\partial y_k} \rho(y|\eta) \right| dy_1 \ldots dy_k. \tag{4.9}$$

Hence, for any $\eta \in F^\perp - N_F$, $\rho(x|\eta)$ is an analytic function uniformly bounded on the real axis. Moreover from (4.9) we also get, by integrating with respect to $\eta$, that $x \to \rho(x|\eta)$ is uniformly bounded as a function from $F$ into $L_1(F^\perp, d\nu)$, and in fact we also have for any $q \in F$ that

$$\int |(q \nabla)^n \rho(x|\eta)| \, d\nu \leqq C \, a_q^n (n+k)!. \tag{4.10}$$

From this we get in the same way as in the proof of Theorem 3.10 that there is a set $N_F \subset F^\perp$ of $\nu$-measure zero so that for $\eta \in F^\perp - N_F$, $\rho(x|\eta)$ is analytic in a strip $|\mathrm{Im}\, z| < \alpha'$. By Theorem 3.10 we know that $\mu$ is $Q$-quasi invariant and for $y \in F$ it is easy to see that

$$\frac{d\mu(\xi+y)}{d\mu(\xi)}=\frac{\rho(x+y|\eta)}{\rho(x|\eta)} \tag{4.11}$$

with $\xi=x\oplus\eta$. Especially we get for any $q\in F$ that

$$q\cdot\beta=(q\cdot\nabla)\ln\rho(x|\eta) \tag{4.12}$$

where $\beta$ is the osmotic velocity associated with $\mu$. We summarize these results in the following theorem.

**Theorem 4.4.** *Let $\mu$ be an analytic probability measure. Then for any splitting $Q'=F\oplus F^{\perp}$ with $F$ finite dimensional in $Q$ we have that the corresponding conditional probability $\mu(x|\eta)$ given by $d\mu(x,\eta)=\mu(x|\eta)\,d\nu(\eta)$ has the form $\mu(x|\eta)=\rho(x|\eta)\,dx$ where $dx$ is the Lebesgue measure on $F$ and $x\to\rho(x|\eta)$ is a strongly analytic function from $F$ into $L_1(F^{\perp},d\nu)$ which is analytic and uniformly norm bounded in a strip $|\mathrm{Im}\,z|<\alpha$. Moreover there is a subset $N_F\subset F^{\perp}$ of $\nu$-measure zero so that, for any $\eta\in F^{\perp}-N_F$, $\rho(x|\eta)$ is analytic in the same strip $|\mathrm{Im}\,z|<\alpha$. Furthermore for any $y\in F$ we have that*

$$\frac{d\mu(\xi+y)}{d\mu(\xi)}=\frac{\rho(x+y|\eta)}{\rho(x|\eta)} \quad \textit{with} \quad \xi=x\oplus\eta,$$

*and for any $q\in F$ we have that*

$$q\cdot\beta=(q\cdot\nabla)\ln\rho(x|\eta),$$

*where $\beta$ is the osmotic velocity corresponding to the quasi invariant measure $\mu$.*

Let us now take a $Q$-quasi invariant probability measure $\mu\in\mathscr{P}_{\omega}$, i.e. $\mu\in\mathscr{P}_1$ and the cyclic vector $\Omega$ represented by the function $1\in L_2(d\mu)$ is an analytic vector for $\pi(q)$ for any $q\in Q$. Let $F$ be a finite dimensional subspace of $Q$ and let $\mu(x|\eta)$ be the conditional probability measure corresponding to the splitting $Q'=F\oplus F^{\perp}$. By the quasi invariance of $\mu$ it follows easily that $\mu(x|\eta)=\rho(x|\eta)\,dx$ where $\rho(x|\eta)>0$ for almost all $x$ with respect to the Lebesgue measure, for $\nu$-almost all $\eta$. Take now $q\in F$ and set $\varphi(x|\eta)=(\rho(x|\eta))^{\frac{1}{2}}$, then the condition that $\Omega$ is analytic for $\pi(q)$ is equivalent with the existence of $a_q>0$ such that

$$\iint|(q\cdot\nabla)^n\varphi(x|\eta)|^2\,dx\,d\nu(\eta)\leqq(a_q^n\,n!)^2. \tag{4.13}$$

Since

$$(q\cdot\nabla)^n\rho(x|\eta)=\sum_{s=0}^{n}\binom{n}{s}(q\cdot\nabla)^{n-s}\varphi(x|\eta)(q\cdot\nabla)^s\varphi(x|\eta) \tag{4.14}$$

we see that

$$\int|(q\cdot\nabla)^n\rho(x|\eta)|\,dx\,d\nu(\eta)\leqq\sum_{s=0}^{n}\binom{n}{s}\|(q\cdot\nabla)^{n-s}\varphi\|_2\,\|(q\cdot\nabla)^s\varphi\|_2 \tag{4.15}$$

so by (4.13) we have

$$\int|(q\cdot\nabla)^n\rho(x|\eta)|\,dx\,d\nu(\eta)\leqq(n+1)!\,a_q^n. \tag{4.16}$$

Since the right hand side is independent of $F$ we get, for any $g \in FC^{\infty}$, that

$$|\int (q \cdot \nabla)^n g \, d\mu| \leqq a_q^n (n+1)! \, \|g\|_{\infty}. \tag{4.17}$$

From this inequality we get the following theorem.

**Theorem 4.5.** *If $\mu \in \mathscr{P}_{\omega}$ then $\mu$ is an analytic measure. Moreover if $\mu_l$ is a sequence of probability measures on $Q'$ which are uniformly in $\mathscr{P}_{\omega}$ i.e. for any $q \in Q$ there are constants $a_q$ independent of $l$ such that*

$$\|\pi(q)^n \Omega_1\| \leqq a_q^n n!$$

*then $\mu_l$ is an equi analytic sequence of probability measures.*

We say that $f \in FC^{\omega}$ iff $f \in FC^{\infty}$ and for any $q \in Q$ there is a constant $\alpha_q > 0$ such that $\|(q \cdot \nabla)^n f\|_{\infty} \leqq \alpha_q^n$. Since, for any $q' \in Q$, $e^{i\langle q', \xi\rangle} \in FC^{\omega}$ we see that $FC^{\omega}$ is dense in $L_2(d\mu)$. Let us now assume that $\mu \in \mathscr{P}_{\omega}$ then for $f \in FC^{\omega}$ we have that for $q \in Q$

$$\begin{aligned}
\|\pi(q)^n f \Omega\| &\leqq \sum_{s=0}^{n} \binom{n}{s} \|(q \cdot \nabla)^s f \cdot \pi(q)^{n-s} \Omega\| \\
&\leqq \sum_{s=0}^{n} \binom{n}{s} \alpha_q^s \cdot a_q^{n-s} \cdot (n-s)! \\
&\leqq e \cdot n! (\max\{\alpha_q, a_q\})^n.
\end{aligned}$$

Hence we have seen that $FC^{\omega}\Omega$ is a dense set of analytic vectors for $\pi(q)$. This gives by Nelson's theorem on analytic vectors that $\pi(q)$ restricted to $FC^{\omega}\Omega$ is essentially self adjoint.

This gives us the following theorem.

**Theorem 4.6.** *If $\mu \in \mathscr{P}_{\omega}$ and $q \in Q$ then the restriction of $\pi(q)$ to $FC^{\omega}\Omega$ is essentially selfadjoint. Moreover any $f \in FC^{\omega}\Omega$ is an analytic vector for $\pi(q)$.*

## 5. Applications to Two-Dimensional Quantum Field Theoretical Models

In this section we shall apply the results and methods of the previous sections to quantum fields models with polynomial or exponential interactions. General references for the construction of the first class of models (the so called "$P(\varphi)_2$-models") are [70, 38–43, 71], and for the second class of models [44, 45, 41].[13] For our applications we use the rigged Hilbert space $Q \subset K \subset Q'$ with $Q = \mathscr{S}(R)$, $K = L_2(R)$, $Q' = \mathscr{S}'(R)$, where all spaces are here real, $R$ is the real line, $\mathscr{S}(R)$ is Schwartz space of test functions and $\mathscr{S}'(R)$ its dual space of tempered distributions. The measure $\mu$ on $Q' = \mathscr{S}'(R)$ (equipped with the $\sigma$-algebra generated by the cylinder sets) we shall consider and which will have the properties required for the application of the methods and results of the previous sections is the one

[13] Probabilistic aspects are particularly present since the systematic use of "Euclidean methods", starting from [72, 73], see e.g. [74, 78]

obtained, by a procedure given below, from the Euclidean measure $\mu^*$ on $\mathscr{S}'(R^2)$ of the Euclidean $P(\varphi)_2$-models or exponential models. We shall also describe below the measures $\mu^*$ and $\mu$. Let us first consider the simple case of the Gaussian measures belonging to the so called "free fields".

## 5.1. The Gaussian Measures

Let $\mu_0^*$ be the Gaussian measure on $\mathscr{S}'(R^2)$, with mean zero and covariance the Green's function of $-\Delta+m^2$ i.e. the kernel of $(-\Delta+m^2)^{-1}$, where $m^2$ is a positive constant and $\Delta$ is the Laplacian in $R^2$. So that $\mu_0^*$ is the Euclidean invariant Gaussian measure for the Euclidean Markov Gaussian generalized random field $\langle\xi^*,\psi\rangle$ over $R^2$, defined by

$$\int_{\mathscr{S}'(R^2)} e^{i\langle\xi^*,\psi\rangle}\, d\mu_0^*(\xi^*)=e^{-\frac{1}{2}(\psi,(-\Delta+m^2)^{-1}\psi)}, \tag{5.1}$$

where $\langle\ ,\ \rangle$ is the dualization between $\mathscr{S}(R^2)$ and $\mathscr{S}'(R^2)$, $(\ ,\ )$ is the scalar product in $L_2(R^2)$ and $\psi\in\mathscr{S}(R^2)$. $\langle\xi^*,\psi\rangle$ is the restriction to $\mathscr{S}(R^2)$ of a generalized Gaussian random field indexed by the real Sobolev space $\mathscr{H}_{-1}(R^2)$, the completion of $\mathscr{S}(R^2)$ in the norm $\|(-\Delta+m^2)^{-\frac{1}{2}}\psi\|^2$. Denote the extension again by $\langle\xi^*,\psi\rangle$.

For the introduction of the Gaussian measure $\mu_0^*$ into constructive field theory see [75].

Consider now another Gaussian measure $\mu_0$, defined on $\mathscr{S}'(R)$ and such that

$$\int_{\mathscr{S}'(R)} e^{i\langle\xi,\varphi\rangle}\, d\mu_0(\xi)=e^{-\frac{1}{4}(\varphi,(-\Delta+m^2)^{-\frac{1}{2}}\varphi)} \tag{5.2}$$

for all $\varphi\in\mathscr{S}(R)$, where $\langle\ ,\ \rangle$ denotes the dualization between $\mathscr{S}(R)$ and $\mathscr{S}'(R)$ and $(\ ,\ )$ the scalar product in $L_2(R)$, $\Delta\equiv\dfrac{d^2}{dx^2}$.

$\mu_0$ is the well known Gaussian measure giving the Fock space $L_2(\mathscr{S}'(R),d\mu_0)$ for the free quantum fields (in Schrödinger-Segal's realization).

Consider now the splitting $R^2=R\oplus R$ of the Euclidean space $R^2$ into a "time component" (first component) and a "space component" (second component), so that for an arbitrary point $y=(y^0,y^1)$ we have $y^0=t$ (time component) and $y^1=x$ (space component). The distributions of the form

$$\psi(t,x)=\delta_0(t)\,\varphi(x)\equiv(\delta_0\otimes\varphi)(t,x)$$

with $\varphi\in\mathscr{S}(R)$ and $\delta_0(\cdot)$ the $\delta$-function centered at the origin are in $\mathscr{H}_{-1}(R^2)$. The natural identification of $\langle\xi,\varphi\rangle$ with $\langle\xi^*,\delta_0\otimes\varphi\rangle$ induces an isometric injection $J_0$ i.e. an identification of

$$L_2(\mathscr{S}'(R),d\mu_0)\quad\text{with}\quad L_2(\mathscr{S}'(R^2),d\mu_0^*)\cap\Sigma_0,$$

where $\Sigma_0$ is the $\sigma$-subalgebra of subsets of $\mathscr{S}'(R^2)$ generated by the distributions $\langle\xi^*,\delta_0\otimes\varphi\rangle$. By this identification $\mu_0$ can be identified with the restriction of $\mu_0^*$ to $\Sigma_0$. As already discussed in [1], Section 4, Theorem 4.0, $\mu_0$ is $\mathscr{S}(R)$-quasi invariant, in fact the Radon-Nikodym derivative for the translated measure $d\mu_0(\xi+\varphi)$ with respect to the original one is, for any $\varphi\in\mathscr{S}(R)$, given by

$$\frac{d\mu_0(\xi+\varphi)}{d\mu_0(\xi)}=e^{-2\langle\xi,(-\Delta+m^2)^{\frac{1}{2}}\varphi\rangle}\,e^{-\langle\varphi,(-\Delta+m^2)^{\frac{1}{2}}\varphi\rangle} \tag{5.3}$$

and

$$\frac{d\mu_0(\cdot+\varphi)}{d\mu_0(\cdot)}\in L_p(\mathscr{S}'(R),d\mu_0) \quad \text{for all } 1\leqq p<\infty.$$

We also have, defining

$$(e^{i\pi(\varphi)}f)(\xi)=\sqrt{\frac{d\mu_0(\xi+\varphi)}{d\mu_0(\xi)}}\,f(\xi+\varphi) \tag{5.4}$$

for all $f\in L_2(\mathscr{S}'(R),d\mu_0)$ and with $\Omega_0(\xi)\equiv 1$ in $L_2(\mathscr{S}'(R),d\mu_0)$

$$(i\pi(\varphi)\Omega_0)(\xi)=-\langle\xi,(-\Delta+m^2)^{\frac{1}{2}}\varphi\rangle \tag{5.5}$$

hence $\pi(\varphi)\Omega_0\in L_2(\mathscr{S}'(R),d\mu_0)$. This shows that $\mu_0\in\mathscr{P}_1(\mathscr{S}'(R))$. Equivalently, the drift coefficient $\beta_0$ associated with $\mu_0$ defined by

$$\beta_0(\xi)(\varphi)\equiv 2i\pi(\varphi)\Omega_0(\xi)=-2\langle\xi,(-\Delta+m^2)^{\frac{1}{2}}\varphi\rangle$$

is in $L_2(\mathscr{S}'(R),d\mu_0)$. We also have easily from (5.4) and (5.5) that $\Omega_0\in D(\pi(\varphi)^n)$ for all positive integers $n$, moreover $\Omega_0$ is an analytic vector for $\pi(\varphi)$, thus $\mu_0$ is analytic. Moreover $\mu_0$ is strictly positive. Let $H_0$ be the Dirichlet operator given by $\mu_0$ i.e. such that. In this section Dirichlet operators are defined with an additional factor $\frac{1}{2}$

$$(f,H_0f)=\tfrac{1}{2}\int\nabla f\,\nabla f\,d\mu_0 \tag{5.6}$$

for all $f\in D(H_0^{\frac{1}{2}})$ i.e.

$$H_0f=-\tfrac{1}{2}\Delta f-\tfrac{1}{2}\beta_0\,\nabla f \tag{5.7}$$

for all $f\in FC^2$ in $L_2(\mathscr{S}'(R),d\mu_0)$. It was shown in Theorem 4.0 of Reference [1] that $H_0$ coincides with the usual Fock space free energy operator i.e. $H_0$ coincides, after identifying

$$L_2(\mathscr{S}'(R),d\mu_0) \quad \text{with} \quad L_2(\mathscr{S}'(R^2),d\mu_0^*)\cap\Sigma_0,$$

with the restriction of $E_0U_tE_0$ to $L_2(\mathscr{S}'(R^2),d\mu_0^*)\cap\Sigma_0$, $U_t$ being the unitary shift in $L_2(\mathscr{S}'(R^2),d\mu_0^*)$ generated by $U_t\langle\xi^*,\delta_0\otimes\varphi\rangle=\langle\xi^*,\delta_t\otimes\varphi\rangle$. The stochastic equation satisfied by the Markoff process $\xi_t$ with values in $\mathscr{S}'(R)$ generated by $H_0$ is

$$d\xi_t=\tfrac{1}{2}\beta_0(\xi_t)\,dt+dw_t, \tag{5.8}$$

where $dw_t$ is the standard Wiener process.[14]

[14] This equation is of the Orstein-Uhlenbeck type and for its probabilistic study see also e.g. [79]

All results of Sections 2 and 3 hold for the case of the measure $\mu_0$ and the rigging $\mathscr{S}(R)\subset L_2(R)\subset \mathscr{S}'(R)$, in particular, equivalence of $T$-ergodicity and $Q$ (i.e. $\mathscr{S}(R)$)-ergodicity. Note that $H_0$ has zero as a simple isolated eigenvalue at the bottom of its spectrum, thus in particular $\mu_0$ is $\mathscr{S}(R)$- and $T$-ergodic. We also observe for later use that one has, for any $f\in D(H_0)$

$$(e^{i\pi(\varphi)} H_0 e^{-i\pi(\varphi)} f)(\xi)=(H_0 f)(\xi)+\hat{L}_1^{\varphi} f(\xi)+\hat{L}_2^{\varphi}(\xi) f(\xi) \tag{5.9}$$

with $\hat{L}_1^{\varphi}\equiv\frac{1}{2}\langle\varphi,(-\Delta+m^2)\varphi\rangle$, $\hat{L}_2^{\varphi}(\xi)\equiv\langle\xi,(-\Delta+m^2)\varphi\rangle$.

### 5.2. *The Measures for the Space Time Cut-off Interactions*

Let now $P^{(\leqq n)}$ be the closed subspace of $L_2(\mathscr{S}'(R^2), d\mu_0^*)$ generated by all monomials $\prod_{i=1}^{k}\langle\xi^*,\varphi_i\rangle$ of degree $k\leqq n$ on $\mathscr{S}'(R^2)$, with $\xi^*\in\mathscr{S}'(R^2)$, $\varphi_i\in\mathscr{S}(R^2)$, $i=1,\dots,k$. Let $P^{(n)}\equiv P^{(\leqq n)}\ominus P^{\leqq(n-1)}$. The "$n$-th Wick power" of the field $\xi^*$ taken with test function $h\in L_{1+\varepsilon}(R^2)$, $0<\varepsilon\leqq 1$, is denoted by $:\xi^{*n}:(h)$ and is by definition the unique element in $P^{(n)}$ such that

$$\left(:\xi^{*n}:(h), \prod_{i=1}^{n}\langle\xi^*,\varphi_i\rangle\right)=n!\int_{R^2}\cdots\int_{R^2}\left(\prod_{i=1}^{n}(-\Delta+m^2)^{-1}(y_i-x)\,\varphi_i(y_i)\,dy_i\right)h(x)\,dx, \tag{5.10}$$

where $(\ ,\ )$ is the scalar product in $L_2(\mathscr{S}'(R^2), d\mu_0^*)$ and $\Delta$ is the Laplacian in $R^2$.

Note that the kernel $(-\Delta+m^2)^{-1}(\cdot)$ is translation invariant and in $L_1(R^2)$ as a function of the difference variables. $:\xi^{*n}:(h)$ is actually the strong $L_2(\mathscr{S}'(R^2), d\mu_0^*)$ limit as $\kappa\to\infty$ of the projection into $P^{(n)}$ of $\langle\xi^*,\chi_\kappa * h\rangle^n$, where $*$ is convolution and $\chi_\kappa$ is a $\delta$-sequence of functions in $\mathscr{S}(R^2)$. For discussions of the construction of Wick powers, related to Ito's multiple integrals, see e.g. [80].

Let now $u(s)$ be a real-valued function of the real variable $s$ of one of the forms

$$u(s)=\sum_{k=0}^{2m} a_s s^k \quad\text{with}\quad a_{2m}>0 \tag{5.11}$$

or

$$u(s)=\int e^{\alpha s}\,d\nu(\alpha) \tag{5.12}$$

with $d\nu(\alpha)$ any bounded positive measure with support on $(-2\sqrt{\pi}, 2\sqrt{\pi})$ and such that $d\nu(\alpha)=d\nu(-\alpha)$.

We shall call the case (5.11) "the $P(\varphi)_2$-case" and the case (5.12) "the exponential case".

Define $:u:(\xi^*)$ as the distribution in $\mathscr{S}'(R^2)$ obtained by replacing in the definition of $u(s)$, $s^k$ by $:\xi^{*k}:$, for all $k$ i.e.

$$:u:(\xi^*)=\sum_{k=0}^{2m} a_k :\xi^{*k}: \tag{5.13}$$

in the $P(\varphi)_2$-case and

$$:u:(\xi^*)=\int :e^{\alpha\xi^*}:d\nu(\alpha), \tag{5.14}$$

with

$$\int :e^{\alpha\xi^*}:d\nu(\alpha)=\sum_{n=0}^{\infty}\int \frac{\alpha^n}{n!}:\xi^{*n}:d\nu(\alpha), \tag{5.15}$$

in the exponential case.

In fact one shows that the sum (5.14) converges in the strong $L_2(\mathscr{S}'(R^2), d\mu_0^*)$-sense.

As test functions $h$ are allowed in (5.13) all $h\in L_1(R^2)\cap L_{1+\varepsilon}(R^2)$ and in (5.14) all $h\in L_1(R^2)\cap L_2(R^2)$. We shall use the notation

$$:u:(\xi^*)(h)$$

for the evaluation of $:u:(\xi^*)$ on the test function $h$, so that

$$:u:(\xi^*)(h)=\sum_{k=0}^{2m} a_k :\xi^{*k}:(h) \tag{5.16}$$

in the $P(\varphi)_2$ case and

$$:u:(\xi^*)(h)=\int :e^{\alpha\xi^*}:(h)\,d\nu(\alpha)=\sum_{n=0}^{\infty}\int \frac{\alpha^n}{n!}:\xi^{*n}:(h)\,d\nu(\alpha) \tag{5.17}$$

in the exponential case.

We shall also write shortly $U(h)$ for $:u:(\cdot)(h)$, considered as a function on $\mathscr{S}'(R^2)$. By well known estimates [72] (see also e.g. [41]) in the $P(\varphi)_2$ case $U(h)\in L_p(\mathscr{S}'(R^2), d\mu_0^*)$ for all $1\leqq p<\infty$ and, for $h\geqq 0$, $e^{-U(h)}\in L_p(\mathscr{S}'(R^2), d\mu_0^*)$, also for all $1\leqq p<\infty$. In the exponential case one has [45] $U(h)\in L_2(\mathscr{S}'(R^2), d\mu_0^*)$ and $U(h)\geqq 0$, the latter whenever $h\geqq 0$, thus, in this case, $e^{-U(h)}\in L_\infty(\mathscr{S}'(R^2), d\mu_0^*)$ for all $1\leqq p\leqq\infty$.

Thus in all cases we have that

$$d\mu_h^*\equiv\Big(\int_{\mathscr{S}'(R^2)} e^{-U(h)}\,d\mu_0^*\Big)^{-1} e^{-U(h)}\,d\mu_0^* \tag{5.18}$$

is a probability measure on $\mathscr{S}'(R^2)$, absolutely continuous with respect to the Gaussian measure $d\mu_0^*$. One calls $U(h)$ "the space-time cut off ($P(\varphi)_2$ resp. exponential) interaction", with "space-time cut off" $h$. $d\mu_h^*$ is the associated measure.

### *5.3. The Quasi Invariant Measures and the Associated Diffusion Processes for the Space Cut-off Interactions*

Define, for $\xi\in\mathscr{S}'(R)$, $\varphi\in\mathscr{S}(R)$, the Wick power $:\xi^n:(\varphi)$ in a corresponding way as we defined $:\xi^{*n}:(\psi)$, but with $\xi^*$ replaced by $\xi$, $\psi$ replaced by $\varphi$ and the covariance $(-\Delta+m^2)^{-1}$ replaced by $\left(-\frac{d^2}{dx^2}+m^2\right)^{-\frac{1}{2}}$. For notational convenience we shall write in this subsection $\Delta$ for $\frac{d^2}{dx^2}$. Thus $:\xi^n:(\varphi)$ is the space $P_0^{(n)} = P_0^{\leqq(n+1)} \ominus P_0^{\leqq(n)}$, with $P_0^{(\leqq n)}$ the closed linear span of the nonomials $\prod_{i=1}^{k} \langle \xi, \varphi_i \rangle$, $k \leqq n$ in $L_2(\mathscr{S}'(R), d\mu_0)$, such that

$$\left(:\xi^n:(\varphi), \prod_{i=1}^{n} \langle \xi, \varphi_i \rangle\right) = n! \int_R \cdots \int_R \prod_{i=1}^{n} ((-\Delta+m^2)^{-\frac{1}{2}}(y_i-x)\, \varphi_i(y_i)\, dy_i)\, \varphi(x)\, dx,$$

where $(\ ,\ )$ is the scalar product in $L_2(\mathscr{S}'(R), d\mu_0)$ and $\langle\ ,\ \rangle$ is the dualization between $\mathscr{S}(R)$ and $\mathscr{S}'(R)$. Define $:v:(\xi)$ in the same way as $:u:(\xi^*)$, formulae (5.13) resp. (5.14), but with $\xi$ replacing $\xi^*$ i.e.

$$:v:(\xi) = \sum_{k=0}^{2m} a_k :\xi^k: \tag{5.19}$$

resp.

$$:v:(\xi) = \int :e^{\alpha\xi}: d\mu(\alpha) = \sum_{n=0}^{\infty} \int \frac{\alpha^n}{n!} :\xi^n: d\mu(\alpha). \tag{5.20}$$

One shows that the evaluation $:v:(\xi)(g)$ of $:v:(\xi)$ on the test function $g$ in $L_1(R) \cap L_{1+\varepsilon}(R)$ resp. $L_1(R) \cap L_2(R)$ yields functions in $L_p(\mathscr{S}'(R), d\mu_0)$ for all $1 \leqq p < \infty$ resp. in $L_2(\mathscr{S}'(R), d\mu_0)$, and that the sum in (5.20) is convergent in the strong $L_2(\mathscr{S}'(R), d\mu_0)$-sense. We write also $V_g$ for $:v:(\xi)(g)$. One has that, for $g \geqq 0$, $e^{-V_g}$ has the same $L_p$-properties as $e^{-U(h)}$ i.e. $e^{-V_g}$ is in $L_p(\mathscr{S}'(R), d\mu_0)$ for all $1 \leqq p < \infty$ in "the $P(\varphi)_2$-case" (5.19) and for all $1 \leqq p \leqq \infty$ in the "exponential case" (5.20). One calls $V_g \equiv :v:(g)$ "the space cut-off ($P(\varphi)_2$- resp. exponential) interaction", with "space cut-off" $g$. For $g \geqq 0$ as above define $H_g$ on $D(H_0) \cap D(V_g)$ in $L_2(\mathscr{S}'(R), d\mu_0)$ by

$$H_g = H_0 + V_g. \tag{5.21}$$

From now on we shall always assume $g \geqq 0$, and $g \in L_1(R) \cap L_{1+\varepsilon}(R)$ with $0 < \varepsilon \leqq 1$ in the $P(\varphi)_2$ case resp. $\varepsilon = 1$ in the exponential case.

It is known that $H_g$ is essentially self-adjoint on $D(H_0) \cap D(V_g)$. Call also $H_g$ its unique self adjoint extension. $H_g$ is lower semibounded in the $P(\varphi)_2$ case resp. positive in the exponential case, with simple isolated lowest eigenvalue $E_g$. Call $\Omega_g$ the eigenvector in $L_2(\mathscr{S}'(R), d\mu_0)$ to the infimum of the spectrum of $H_g$, so that

$$H_g \Omega_g = E_g \Omega_g \tag{5.22a}$$

with $H_g \geqq E_g$. One has $\Omega_g(\xi) > 0$ $\mu_0$-a.e. and $\Omega_g \in L_1(\mathscr{S}'(R), d\mu_0)$. Define the measure $d\mu_g$ on $\mathscr{S}'(R)$, equivalent to $d\mu_0$, by

$$d\mu_g(\xi) \equiv \Omega_g^2(\xi)\, d\mu_0(\xi). \tag{5.22b}$$

**Lemma 5.2.** *$\mu_g$ is a $\mathscr{S}(R)$-quasi invariant measure i.e. $\mu_g \in \mathscr{P}_0(\mathscr{S}'(R))$.*

*Proof.* From the Definition (5.22) we have

$$d\mu_g^\varphi(\xi) \equiv d\mu_g(\xi+\varphi) = \Omega_g^2(\xi+\varphi)\, d\mu_0(\xi+\varphi),$$

for any $\varphi \in \mathscr{S}(R)$. Thus from (5.3)

$$\frac{d\mu_g^\varphi}{d\mu_g}(\xi) = \frac{\Omega_g^2(\xi+\varphi)}{\Omega_g^2(\xi)}\, e^{-2\langle \xi, (-\Delta+m^2)^{\frac{1}{2}}\varphi\rangle}\, e^{-\langle \varphi, (-\Delta+m^2)^{\frac{1}{2}}\varphi\rangle}$$

which is strictly positive $\mu_0$

**Lemma 5.3.** *Let $\pi(\varphi)$ be as in (5.4). Then on $D(H_0)\cap D(V_g)$ we have*

$$e^{i\pi(\varphi)} H_g e^{-i\pi(\varphi)} = H_0 + L_1^\varphi + L_2^\varphi + V_g^\varphi \tag{5.23}$$

*where*

$$V_g^\varphi \equiv e^{i\pi(\varphi)} V_g e^{-i\pi(\varphi)}. \tag{5.24}$$

*For any $f \in D(V_g)$ one has*

$$(V_g^\varphi f)(\xi) = :v:(\xi+\varphi)(g)\, f(\xi) = (V_g f)(\xi) + (R_{g,\varphi} f)(\xi) \tag{5.25}$$

*with*

$$(R_{g,\varphi} f)(\xi) \equiv \sum_{k=0}^{2m} a_k \sum_{j=1}^{k} \binom{k}{j} :\xi^{k-j}:(g\varphi^j) \tag{5.26}$$

*in the $P(\varphi)_2$-case and*

$$(R_{g,\varphi} f)(\xi) = \sum_{k=0}^{\infty} \int \frac{\alpha^{2k}}{(2k)!} \sum_{j=1}^{k} \binom{k}{j} :\xi^{2k-j}:(g\varphi^{2j})\, d\mu(\alpha) \tag{5.27}$$

*in the exponential case. $V_g^\varphi$, $R_{g,\varphi}$, $e^{-R_{g,\varphi}}$ are in $L_p(\mathscr{S}'(R), d\mu_0)$ for all $1 \leqq p < \infty$, in the $P(\varphi)_2$-case, and $V_g^\varphi, R_{g,\varphi} \in L_2(\mathscr{S}'(R), d\mu_0)$, $e^{-R_{g,\varphi}} \in L_p(\mathscr{S}'(R), d\mu_0)$ $1 \leqq p \leqq \infty$ in the exponential case.*

*Proof.* (5.23) follows from (5.9) and the fact that $V_g$ is a multiplication operator in $L_2(\mathscr{S}'(R), d\mu_0)$. (5.25), (5.26) come from the Definition (5.24) of $V_g^\varphi$ and the Definitions (5.19), (5.20) of $V_g$. The $L_p$ estimates are analogous to the ones for $V_g$, $e^{-V_g}$, q.e.d.

Set now, on $D(H_0)\cap D(V_g)$,

$$H_g^\varphi \equiv H_0^\varphi + V_g^\varphi \tag{5.28}$$

with

$$H_0^\varphi \equiv H_0 + \hat{L}_1^\varphi + \hat{L}_2^\varphi. \tag{5.29}$$

Then we have, by Lemma 5.3:

$$H_g^\varphi = e^{i\pi(\varphi)} H_g e^{-i\pi(\varphi)} \tag{5.30}$$

on $D(H_0)\cap D(V_g)$.

The following Lemma is an immediate consequence of Lemma 5.3.

**Lemma 5.4.** *For any real t, $H_g^{t\varphi}$ is n times strongly differentiable on $D(H_0)\cap D(V_g)$ with respect to t, and we have on $D(H_0)\cap D(V_g)$, for the strong derivatives at $t=0$:*

$$\left(\frac{d^n}{dt^n} H_g^{t\varphi}\right)_{t=0} = :v^{(n)}:(\cdot)(g\varphi^n) + L_n^\varphi, \quad L_1^\varphi \equiv \hat{L}_2^\varphi, \; L_2^\varphi \equiv 2\hat{L}_1^\varphi, \tag{5.30}$$

*and $L_n^\varphi \equiv 0$ for $n \geqq 3$ and $:v^{(n)}:(\xi)(f)$ is defined as $:v:(\xi)(f)$ but with the function v replaced by its n-th derivative, so that, in the $P(\varphi)_2$-case*

$$:v^{(n)}:(\cdot)(g\varphi^n) = \sum_{k=n}^{2m} a_k k(k-1)\ldots(k-n+1):\xi^{k-n}:(\cdot)(g\varphi^n) \tag{5.31}$$

*and*

$$:v^{(n)}:(\cdot)(g\varphi^n) = \sum_{k=n}^{\infty} \int \alpha^k k(k-1)\ldots(k-n+1):\xi^{k-n}:(\cdot)(g\varphi^n)\, d\mu(\alpha)$$

*in the exponential case.*

Define recursively $\operatorname{ad}^n A(B)$ as $[A,B]$ for $n=1$ and $\operatorname{ad}^n A(B) = [A, \operatorname{ad}^{n-1} A(B)]$.

**Lemma 5.5.** *On $D(H_g^{\frac{3}{2}})$ we have*

$$i^n \operatorname{ad}^n \pi(\varphi)(H_g) = :v^{(n)}:(\cdot)(g\varphi^n) + L_n^\varphi.$$

*Proof.* From Lemma 5.4, (5.30), we have for any $\psi \in D(H_0)\cap D(V_g)$ *and any* $\Phi \in D(\pi(\varphi))$

$$i((\pi(\varphi)\Phi, H_g\psi) - (H_g\Phi, \pi(\varphi)\psi)) = (\Phi, (:v^{(1)}:(g\varphi) + L_1^\varphi)\psi). \tag{5.32}$$

From known "local perturbations" bounds (":$\phi^j$:-bounds") in the $P(\varphi)_2$-case ([81, 82]) and an elementary consequence of $H_0 \leqq H_g$ for exponential interactions, we have

$$\pi(\varphi)^2 \leqq C(H_g + D), \tag{5.33}$$

where $C$ and $D$ are constants independent of g.

This then gives $D(\pi(\varphi)) \supset D(H_g^{\frac{1}{2}})$ hence, for $\psi \in D(H_g^{\frac{3}{2}})$, $H_g\psi \in D(H_g^{\frac{1}{2}}) \subset D(\pi(\varphi))$, thus the first term on the left hand side of (5.32) can be written $i(\Phi, \pi(\varphi)H_g\psi)$. Then from (5.32) we also get $\pi(\varphi)\psi \in D(H_g)$, and thus the statement of the lemma for $n=1$. The rest follows by applying repeatedly $\operatorname{ad}\pi(\varphi)$ to $:v^{(1)}:(\cdot)(g\varphi) + L_1^\varphi$ and observing that

$$[i\pi(\varphi), :v^{(n)}:(\cdot)(g\varphi^n)] = :v^{(n+1)}:(\cdot)(g\varphi^{n+1}) \quad \text{and} \quad [i\pi(\varphi), L_n^\varphi] = L_{n+1}^\varphi.$$

**Lemma 5.6.** *$\mu_g$ is in $\mathscr{P}_\omega$ i.e. 1 is an analytic vector in $L_2(\mathscr{S}'(R), d\mu_g)$ for $\pi_g(\varphi)$, where*

$$(e^{i\pi_g(\varphi)}f)(\xi) = \sqrt{\frac{d\mu_g(\xi+\varphi)}{d\mu_g(\xi)}}\, f(\xi+\varphi),$$

*for all $f \in L_2(\mathscr{S}'(R), d\mu_g)$. Equivalently $\Omega_g$ is an analytic vector for $\pi(\varphi)$ in $L_2(\mathscr{S}'(R), d\mu_0)$.*

*Proof.* From Lemma 5.5 we have, since $\Omega_g \in D(H_g^{\frac{3}{2}})$, $\Omega_g \in D(\pi(\varphi))$, hence $\mu \in \mathscr{P}_1(\mathscr{S}'(R))$. Now $H_g - E_g$ has zero as the infimum of its spectrum, separated from the rest of the spectrum by a positive distance $m^* > 0$, independent of $g$ (see [38, 39] resp. [45]). Now by the local perturbation estimates quoted above we have

$$\pm(:v^{(n)}:(g\varphi^n) + L_n^\varphi) \leqq C_\varphi^n (H_g - E_g + 1) \tag{5.34}$$

where $C_\varphi$ is independent of $g$.

Introducing this into the formula of Lemma 5.5 we have

$$\|(H_g - E_g + 1)^{-\frac{1}{2}} \operatorname{ad}^n \pi(\varphi)(H_g)(H_g - E_g + 1)^{-\frac{1}{2}}\| \leqq C_\varphi^n \tag{5.35}$$

for all $n = 1, 2, 3, \ldots$. Then by the proof of Theorem 3.7 we have

$$\|(H_g - E_g + 1)^{\frac{1}{2}} \pi(\varphi)^n \Omega_g\| \leqq n! \left(\frac{m^* + 1}{m^*} e\, C_\varphi\right)^n. \tag{5.36}$$

In particular we get that $\Omega_g$ is an analytic vector for $\pi(\varphi)$ in $L_2(\mathscr{S}'(R), d\mu_0)$. Using now the unitary equivalence of $L_2(\mathscr{S}'(R), d\mu_g)$ and $L_2(\mathscr{S}'(R), d\mu_0)$ given by $f \to \Omega_g f$ we finish easily the proof.

Let now $H_{\mu_g}$ be the Dirichlet operator given by the measure $\mu_g$. Then we have

**Lemma 5.7.**

$$(H_g - E_g)\psi = \Omega_g H_{\mu_g} \Omega_g^{-1} \psi$$

*for any* $\psi$ *in the linear hull of*

$$F C^2 \Omega_g \cup (\pi(\varphi) D(H_g^{\frac{3}{2}})) \cup (e^{i\pi(\varphi)} F C^2 \Omega_g).$$

*Proof.* If $\psi \in F C^2 \Omega_g$ then the statement follows by the fact that the assumptions of Theorem 3.4 in Ref. [1] are satisfied. Let now $\psi = \pi(\varphi) h$, with $h \in D(H_g^{\frac{3}{2}})$. Consider for $f \in F C^2$ the scalar product $(f \Omega_g, (H_g - E_g)\pi(\varphi) h)$ in $L_2(\mathscr{S}'(R), d\mu_0)$, well defined by Lemma 5.5, and equal to $((H_g - E_g) f \Omega_g, \pi(\varphi) h)$, hence, by what we have just proven, also to $(\Omega_g H_{\mu_g} f, \pi(\varphi) h)$. This then yields

$$(f, \Omega_g^{-1}(H_g - E_g)\pi(\varphi) h)_g = (H_{\mu_g} f, \Omega_g^{-1} \pi(\varphi) h)_g,$$

where $(\ ,\ )_g$ is the scalar product in $L_2(\mathscr{S}'(R), d\mu_g)$, which then proves the lemma for $\psi$ of the above form. Let finally $f_1 \in F C^2$, then by the fact that we have proven the lemma when $\psi \in F C^2 \Omega_g$ we have

$$(H_{\mu_g} f, e^{i\pi_g(\varphi)} f_1)_g = ((H_g - E_g) f \Omega_g, e^{i\pi(\varphi)} f_1 \Omega_g).$$

But $e^{i\pi(\varphi)} f_1 \Omega_g \in D(H_g)$ by Lemma 5.3, hence the right hand side is equal to

$$(f \Omega_g, (H_g - E_g) e^{i\pi(\varphi)} f_1 \Omega_g) = (f, \Omega_g^{-1}(H_g - E_g) e^{i\pi(\varphi)} f_1 \Omega_g)_g,$$

which proves the lemma.

*Remark.* In the case of the $(\varphi^4)_2$ interactions we have ([83]) $D(H_g) = D(H_0) \cap D(V_g)$, hence, by Theorem 3.5 in Reference [1], $H_g - E_g$ is equal to the isomorphic image of $H^F_{\mu_g}$ in $L_2(\mathscr{S}'(R), d\mu_0)$ under the isomorphism mapping $L_2(\mathscr{S}'(R), d\mu_g)$ onto $L_2(\mathscr{S}'(R), d\mu_0)$, where $H^F_{\mu_g}$ is the Friedrichs extension of $H_{\mu_g} \upharpoonright FC^2$.

**Lemma 5.8.** *On* $FC^2$

$$i^n \operatorname{ad}^n \pi(\varphi)(H_{\mu_g}) = :v^{(n)}:(g\varphi^n) + L^\varphi_n. \tag{5.37a}$$

*Moreover one has*

$$(H_g - E_g)\,\pi(\varphi)^n\,\Omega_g = \Omega_g\,H_{\mu_g}\,\Omega_g^{-1}\,\pi(\varphi)^n\,\Omega_g. \tag{5.37b}$$

*Proof.* By Lemma 5.5 we have, for any $f_1, f_2 \in FC^2$,

$$(-i)(((H_g - E_g)\,f_1, \pi(\varphi)\,f_2) - (\pi(\varphi)\,f_1, (H_g - E_g)\,f_2)) = (f_1, (:v^{(1)}:(g\varphi) + L^\varphi_1)\,f_2), \tag{5.38}$$

where the scalar product is the $L_2(\mathscr{S}'(R), d\mu_0)$ one. By Lemma 5.7,

$$((H_g - E_g)f_1, \pi(\varphi)f_2) = (f_1, \Omega_g\,H_{\mu_g}\,\Omega_g^{-1}\,\pi(\varphi)f_2). \tag{5.39}$$

On the other hand by Lemma 5.5 $(H_g - E_g)f_2 \in D(\pi(\varphi))$. Thus by Lemma 5.7, $(H_g - E_g)f_2 = H_{\mu_g} f_2 \in D(\pi(\varphi))$, hence

$$(\pi(\varphi)f_1, (H_g - E_g)f_2) = (f_1, \pi(\varphi)(H_g - E_g)f_2) = (f_1, \pi(\varphi)\,H_{\mu_g} f_2). \tag{5.40}$$

Introducing (5.39), (5.40) in (5.38) we have then

$$(-i)((f_1, \Omega_g H_{\mu_g} \Omega_g^{-1}\,\pi(\varphi)\,f_2) - (f_1, \pi(\varphi)\,H_{\mu_g} f_2)) = (f_1, (:v^{(1)}:(g\varphi) + L^\varphi_1)\,f_2). \tag{5.41}$$

This proves then

$$(-i)\,[H_{\mu_g}, \pi(\varphi)] = :v^{(1)}:(g\varphi) + L^\varphi_1 \tag{5.42}$$

on $FC^2$. Applying repeatedly ad $\pi(\varphi)$ to both sides of (5.42) we have then (5.37a). We shall now prove (5.37b) by induction with respect to $n$. By Lemma 5.7 we have

$$(H_g - E_g)\,\pi(\varphi)\,\Omega_g = \Omega_g\,H_{\mu_g}\,\Omega_g^{-1}\,\pi(\varphi)\,\Omega_g, \tag{5.43}$$

thus (5.37b) holds for $n = 1$.

Suppose now it holds for $n = 1, 2, \ldots, k$.

Using the algebraic relation (3.50) we have, since $(H_g - E_g)\,\Omega_g = 0$,

$$(H_g - E_g)\,\pi(\varphi)^{k+1}\,\Omega_g = -\sum_{j=1}^{k+1} \binom{k+1}{j} \operatorname{ad}^j \pi(\varphi)(H_g - E_g)\,\pi(\varphi)^{k-j}\,\Omega_g. \tag{5.44}$$

On the right hand side only terms of the form

$$\text{const.}\ \pi(\varphi)^i\,(H_g - E_g)\,\pi(\varphi)^l\,\Omega_g \quad \text{with} \quad i = 0, \ldots, k+1,\ l = 0, \ldots, k$$

enter, and using the induction hypothesis we rewrite them as const. $\pi(\varphi)^i\,\Omega_g H_{\mu_g} \Omega_g^{-1}\,\pi(\varphi)^l\,\Omega_g$. Using now ( .50) again we get the right hand side of (5.44) to be equal to $\Omega_g H_{\mu_g} \Omega_g^{-1}\,\pi(\varphi)^{k+1}\,\Omega_g$, which proves (5.37b).

**Lemma 5.9.** *$\mu_g$ is strictly positive.*

*Proof.* This follows easily by conditioning from the fact that $\Omega_g > 0\,\mu_0$-a.e. Another way of proving it is to observe that from (5.36) and (5.37) we have

$$\|(H_{\mu_g}+1)^{\frac{1}{2}}\,\Omega_g^{-1}\,\pi(\varphi)^n\,\Omega_g\|_g \leq n!\left(\frac{m^*+1}{m^*}\,e\,C_\varphi\right)^n, \tag{5.45}$$

where $\|\ \|_g$ is the norm in $L_2(\mathscr{S}'(R), d\mu_g)$.

But $2H_{\mu_g} = \nabla^* \nabla$ in $L_2(\mathscr{S}'(R), d\mu_g)$, hence (5.45) shows that $\Omega_g^{-1}\,\pi(\varphi)^n\,\Omega_g \in D(\nabla)$, thus $\pi_g(\varphi)^n 1 \in D(\nabla)$, where 1 is the function identically one in $L_2(\mathscr{S}'(R), d\mu_g)$, which, by the proof of Theorem 3.6 gives the strict positivity of $\mu_g$.

Let $\beta_g(\xi)(\varphi)$ be the drift coefficient (osmotic velocity) associated with the quasi invariant measure $\mu_g$ i.e.

$$\beta_g(\xi)(\varphi) \equiv (2i\pi_g(\varphi)\cdot 1)(\xi), \tag{5.46}$$

where 1 is the function identically one in $L_2(\mathscr{S}'(R), d\mu_g)$. We shall now summarize the main results on $\mu_g$ and the related quantities in the following

**Theorem 5.1.** *The measure $\mu_g$ for the space cut-off $P(\varphi)_2$ and exponential interactions is a $\mathscr{S}(R)$-quasi invariant measure on $\mathscr{S}'(R)$ which is in $\mathscr{P}_\omega$ (i.e. $\Omega_g$ is an analytic vector for $\pi(\varphi)$), hence in particular analytic, and strictly positive. Any $f \in FC^\infty\Omega_g$ is an analytic vector for $\pi(\varphi)$ and $\pi(\varphi)$ is essentially self-adjoint on $FC^\infty\Omega_g$. The associated Dirichlet operator $H_{\mu_g}$ coincides with $H_g - E_g$ on the domain $D$, where $D$ is the linear hull of $FC^2\Omega_g$, $\pi(\varphi)D(H_g^{\frac{3}{2}})$ and $\pi(\varphi)^n\Omega_g$, $n = 1, 2, \ldots$ (the coincidence is to be interpreted as the fact that the image of $H_g - E_g \restriction D$ by the natural isomorphism of $L_2(\mathscr{S}'(R), d\mu_0)$ and $L_2(\mathscr{S}'(R), d\mu_g)$ is $H_{\mu_g} \restriction \Omega_g^{-1} D$).*

*On $D(H_g^{\frac{3}{2}})$ resp. $FC^2$ we have for all $n = 1, 2, \ldots$*

$$i^n \operatorname{ad}^n \pi(\varphi)(\hat{H}_g) = :v^{(n)}:(g\varphi^n) + L_n^\varphi,$$

*where $\hat{H}_g$ stands for $H_g$ resp. $H_{\mu_g}\cdot H_{\mu_g}$ generates a Markoff process $\xi_t$ with values in $\mathscr{S}'(R)$ which solves the stochastic differential equation*

$$d\xi_t = \tfrac{1}{2}\beta_g(\xi)\,dt + d\hat{w}_t,$$

*in the sense of Section 3.*

*Remark.* In fact all results of Sections 2, 3, 4 hold.

### *5.4. Removal of the Space Cut-off*

**Lemma 5.11.** *For the exponential interactions* (5.20) *and for the "weak coupling $P(\varphi)_2$-interactions (i.e. those given by* (5.19) *with sufficiently small coefficients $a_k$) one has, for any $g \geq 0$, $\|g\|_\infty \leq 1$ with* $\operatorname{supp} g \subset [-l, l]$, $l > 0$:

$$\|\beta_g(\cdot)(\varphi)\|_g^2 \equiv \int_{\mathscr{S}'(R)} |\beta_g(\xi)(\varphi)|^2\,d\mu_g(\xi) \leq C_\varphi,$$

*where $C_\varphi$ is independent of g.*

*Proof.* By (5.45) it suffices to estimate $\|\pi(\varphi)\Omega_g\|_0$, where $\| \ \|_0$ is the norm in $L_2(\mathscr{S}'(R), d\mu_0)$. By Lemma 5.5 we have

$$(-i)[H_g - E_g, \pi(\varphi)]\,\Omega_g = :v^{(1)}:(g\varphi)\,\Omega_g + L_1^{\varphi}\,\Omega_g. \tag{5.47}$$

Noticing that $(\Omega_g, [\pi(\varphi), H_g - E_g]\,\Omega_g) = 0$ we have that $(H_g - E_g)^{-1}$ is well defined on $[\pi(\varphi), H_g - E_g]\,\Omega_g$, orthogonal to $\Omega_g$. We have thus, for any $C > 0$,

$$\pi(\varphi)\,\Omega_g = -(H_g - E_g)^{-1}(H_g - E_g + C)(H_g - E_g + C)^{-1}[\pi(\varphi), H_g]\,\Omega_g. \tag{5.48}$$

From the fact that $H_g - E_g$ has a strictly positive gap $m_g$ between zero and the rest of its spectrum ([38, 39] resp. [44], [45]) and that $m_g$ is bounded below by $m$ in the exponential case ([44, 45]) as well as by a positive constant independent of g for the weak $P(\varphi)_2$-case [38, 39], we obtain that $(H_g - E_g + C)^{-1}(H_g - E_g)^{-1}$ is bounded uniformly in g. Therefore

$$\|\pi(\varphi)\,\Omega_g\|_0 \leqq C_1\, \|(H_g - E_g + C)^{-1}[\pi(\varphi), H_g](H_g - E_g + C)^{-1}\,\Omega_g\|_0, \tag{5.49}$$

where $C_1$ is a constant that depends only on the function $v(s)$ giving the interaction and $C$. But by Lemma 5.5 we have

$$i\,[\pi(\varphi), H_g](H_g - E_g + C)^{-1}\,\Omega_g = (:v^{(1)}:(g\varphi) + L_1^{\varphi})(H_g - E_g + C)^{-1}\,\Omega_g, \tag{5.50}$$

hence by the local perturbations estimates (5.33) we have that the right hand side of (5.49) is bounded uniformly in g. This together with (5.45) completes the proof of the lemma.

**Lemma 5.12.** *For any $f_1 \in L_p(\mathscr{S}'(R), d\mu_0)$, $1 \leqq p \leqq \infty$, $f_2 \in FC^2$ we have that the scalar product in $L_2(\mathscr{S}'(R), d\mu_0)$*

$$(f_1\,\Omega_g, e^{-t(H_g - E_g)} f_2\,\Omega_g)$$

*is twice differentiable with respect to $t \geqq 0$ and one has*

$$\frac{d}{dt}(f_1\,\Omega_g, e^{-t(H_g - E_g)} f_2\,\Omega_g) = (f_1\,\Omega_g, e^{-t(H_g - E_g)}(H_g - E_g) f_2\,\Omega_g)$$

$$\frac{d^2}{dt^2}(f_1\,\Omega_g, e^{-t(H_g - E_g)} f_2\,\Omega_g) = (f_1\,\Omega_g, e^{-t(H_g - E_g)}(H_g - E_g)^2 f_2\,\Omega_g).$$

*Both the first and second derivatives are bounded uniformly in g and t.*

*Proof.* The existence of and the expressions for the derivatives follow from $FC^2\,\Omega_g \subset D(H_g)$, which holds by Lemma 5.7. The bounds follow from

$$\|e^{-t(H_g - E_g)}\| \leqq 1 \quad \text{and} \quad (H_g - E_g) FC^2\,\Omega_g = \Omega_g H_{\mu_g} FC^2$$

which yields, for $f \in F\,C^2$, $\|(H_g - E_g)\, f\,\Omega_g\| = \|H_{\mu_g} f\|_g$, which, by $2H_{\mu_g} = -\Delta - \beta_g \nabla$ on $F\,C^2$, $H_{\mu_g}$ being the Dirichlet operator given by $\mu_g$, is $\frac{1}{2}$ times

$$\|-\Delta f - \beta_g \cdot \nabla f\| \leqq \|-\Delta f\|_\infty + \sum_{j=1}^{n} \|\beta_g(\varphi_j)\|_g \sup_j \|\varphi_j \cdot \nabla f\|_\infty,$$

where $\{\varphi_j\}$ is an orthonormal basis for the base of the $FC^2$ function $f$. However the right hand side is bounded uniformly in $g$, since $f \in FC^2$ and the bound on $\|\beta_g(\varphi_j)\|_g$ given by Lemma 5.11 holds.

Here $\| \ \|_\infty$ is the $L_2(\mathcal{S}'(R), d\mu_0)$-norm, q.e.d.

Let now $\Sigma_0$ be the $\sigma$-subalgebra of $\mathcal{S}'(R^2)$ generated by the functions $\langle \xi^*, \delta_0 \otimes \varphi \rangle$, with $\varphi \in \mathcal{S}(R)$. We can identify $d\mu_g$ with a measure $d\mu_g^* \restriction \Sigma_0$, using the identification of

$$L_2(\mathcal{S}'(R^2), d\mu_0^*) \cap \Sigma_0 \quad \text{with} \quad L_2(\mathcal{S}'(R), d\mu_0)$$

given by $\langle \xi^*, \delta_0 \otimes \varphi \rangle \leftrightarrow \langle \xi, \varphi \rangle$. By the Feynman-Kac-Nelson formula one proves then, using the spectral Theorem and the fact that $H_g$ has a simple lowest eigenvalue, that $d\mu^*_{\chi_T \otimes g}$ converges weakly to $d\mu_g^*$ as $T \to \infty$, where $\chi_T$ is the characteristic function of the time interval $[-T, T]$ and $d\mu^*_{\chi_T \otimes g}$ is the measure (5.18) for the space-time cut-off interactions. It has also been shown for the exponential interactions (5.14) and for the "weakly coupled" $P(\varphi)_2$ interactions, i.e. the interactions (5.13) with sufficiently small coefficients $a_k$, that the measure $d\mu^*_{\chi_T \otimes g}$ converges weakly on $\mathcal{S}'(R^2)$ as $T \to \infty$ and $g \to 1$, where $g$ is the characteristic function of the space interval $[-L, L]$ so that $g \to 1$ means $L \to \infty$. This result was obtained by Glimm, Jaffe and Spencer [38–39] in the $P(\varphi)_2$-case and by ourselves in the exponential case [45]. Let $d\mu^*$ be the limit measure on $\mathcal{S}'(R^2)$. This measure is invariant under the transformation induced in $\mathcal{S}'(R^2)$ by the Euclidean transformations in $R^2$. $\mu^*$ is called the Euclidean measure for the models considered.

One has also that $d\mu^*$ is the weak limit of the measures $d\mu_g^*$ as $g \to 1$.

We shall now associate to the measure $\mu^*$ on $\mathcal{S}'(R^2)$ a measure $\mu$ on $\mathcal{S}(R)$. Consider first the $P(\varphi)_2$ interactions. Split again $R^2$ as $R \otimes R$, then $\mu$ will be a measure on $\mathcal{S}(R)$ of the space variables $x$ and is defined by

$$\int_{\mathcal{S}'(R)} e^{i\langle \xi, \varphi \rangle} d\mu(\xi) = \lim_{n \to \infty} \int_{\mathcal{S}'(R^2)} e^{i\langle \xi^*, \psi_n \rangle} d\mu^*(\xi^*), \tag{5.51}$$

where $\psi_n$ is a sequence of functions in $\mathcal{S}'(R^2)$ convergent in the $\mathcal{S}'(R) \otimes \mathcal{S}(R)$-sense to the element $\delta_0 \otimes \varphi$ in $\mathcal{H}_{-1}(R^2)$. That the limit on the right hand side exists and gives a continuous positive definite function on $\mathcal{S}(R)$ has been shown in [39, 77], so that $\mu$ is precisely, by Minlos Theorem, the measure associated with this function.

In the case of exponential interactions the existence of $\mu$ follows from [45].

We shall call $\mu$ the restriction of $\mu^*$ to the $\sigma$-algebra of time zero fields for the $P(\phi)_2$- resp. exponential interactions. It has also been proven, [38, 39] resp. [45], that $\mu$ is the weak limit of $\mu_g$ as $g \to 1$.

The moments

$$S^{(n)}(\psi_1, \ldots, \psi_n) = \int \prod_{i=1}^{n} \langle \xi^*, \psi_i \rangle \, d\mu^*(\xi^*)$$

of the measure $\mu^*$ ("Schwinger functions") yield, by an analytic continuation in the time variables, [38, 39] resp. [45], the Wightman functions of a relativistic quantum field theory [84, 85]. Let $H_{\text{ph}}$ be the infinitesimal generator of the unitary strongly continuous representation of time translations in the Hilbert space $\mathscr{H}_{\mathrm{ph}}$ of these Wightman models. We can consider $L_2(\mathscr{S}'(R), d\mu)$ as a subspace of $\mathscr{H}_{\mathrm{ph}}$, by identifying

$$\prod_{i=1}^{n} \langle \xi^*, \delta_0 \otimes \phi_i \rangle \, \Omega^*(\xi^*) \quad \text{with} \quad \prod_{i=1}^{n} \langle \xi, \phi_i \rangle \, \Omega(\xi)$$

and thus with $\prod_{i=1}^{n} \langle \Phi, \phi_i \rangle \, \Omega_{\mathrm{ph}}$, where $\Omega^*$ is the function 1 in $L_2(\mathscr{S}'(R), d\mu^*)$, $\Omega$ is the function 1 in $L_2(\mathscr{S}'(R), d\mu)$, $\Omega_{\mathrm{ph}}$ is the eigenvector to the eigenvalue zero of $H_{\mathrm{ph}}$ in $\mathscr{H}_{\mathrm{ph}}$, $\langle \Phi, \phi \rangle$ is the time zero Wightman field.

We shall now give some consequences of the results of the preceding sections to the case of the measure $\mu$. Combining the above results with Propositions 4.1–4.3 and the results of Sections 2 and 3 (observing that $\mathscr{S}'$, being the dual of the nuclear space $\mathscr{S}(R)$, satisfies all assumptions) we have the following Theorems 5.2, 5.3 and 5.4:

**Theorem 5.2.** *$\mu$ is admissible and $\mathscr{S}(R) \otimes FC^1 \subset D(\nabla^*)$. Moreover $\mu$ is $\mathscr{S}(R)$-quasi invariant and analytic, hence has the properties given in Theorem 4.4. Furthermore $\mu$ is in $\mathscr{P}_1(\mathscr{S}'(R))$ i.e. the function identically one in $L_2(\mathscr{S}'(R), d\mu)$ is in the domain of $\pi(\varphi)$.*

*Proof.* The admissibility and the fact that $\mathscr{S}(R) \otimes FC^1 \subset D(\nabla^*)$ follow from Theorem 4.1, its assumption being satisfied by Lemma 5.11 and the weak convergence of $\mu_g$. By (5.36) and Theorem 4.5 we have that the $\mu_g$ are equi analytic, hence by Theorem 4.2 $\mu$ is analytic and by Theorem 3.10 $\mu$ is $\mathscr{S}(R)$-quasi invariant. Furthermore from Theorem 4.3 we have, by the equi analyticity of $\mu_g$ and Lemma 5.11, that $\mu$ is in $\mathscr{P}_1(\mathscr{S}'(R))$. Since $\mu$ is an $\mathscr{S}(R)$-quasi invariant measure the correspondent Dirichlet operator is well defined, by the results of Section 2, and we set

$$H = \tfrac{1}{2} \nabla^* \bar{\nabla}.$$

We have

$$H = -\tfrac{1}{2} \Delta - \tfrac{1}{2} \beta \cdot \nabla$$

on $FC^2$.

**Theorem 5.3.** *For any $f_1$, $f_2$ in $FC^2$ we have*

$$(f_1, H_{\mu_g} f_2)_g \to (f_1, H f_2),$$

*where $(\ ,\ )_g$ is the scalar product in $L_2(\mathscr{S}'(R), d\mu_g)$ and $(\ ,\ )$ is the scalar product in $L_2(\mathscr{S}'(R), d\mu)$. Moreover*

$$(f_1, H\, f_2) = (f_1\, \Omega_{\mathrm{ph}}, H_{\mathrm{ph}}\, f_2\, \Omega_{\mathrm{ph}})_{\mathrm{ph}},$$

*where $(\ ,\ )_{\mathrm{ph}}$ is the scalar product in $\mathscr{H}_{\mathrm{ph}}$.*

*Proof.* By the weak convergence of the Schwinger functions we have that $(f_1\, \Omega_g, e^{-t(H_g - E_g)} f_2\, \Omega_g)_0$ converges as $g \to 1$ and from the construction of

$(\mathscr{H}_{\rm ph}, \Omega_{\rm ph}, H_{\rm ph})$ we have that the limit is $(f_1 \Omega_{\rm ph},\ e^{-tH_{\rm ph}} f_2 \Omega_{\rm ph})$. The uniform bounds on the derivatives (Lemma 5.12) gives the convergence of the first derivatives at $t=0$ to $(\hat{f}_1 \Omega_{\rm ph}, H_{\rm ph} \hat{f}_2 \Omega_{\rm ph})_{\rm ph}$. On the other hand, by Lemma 5.7,

$$(f_1 \Omega_g, (H_g - E_g) f_2 \Omega_g)_0 = (f_1, H_{\mu_g} f_2)_g = \tfrac{1}{2} \int \nabla f_1 \cdot \nabla f_2 \, d\mu_g .$$

By the weak convergence of $\mu_g$ to $\mu$ we have

$$\lim_{g \to 1} (f_1, H_{\mu_g} f_2)_g = \tfrac{1}{2} \int \nabla f_1 \cdot \nabla f_2 \, d\mu ,$$

and the Theorem is proven. Again by the results of Sections 2, 3, 4 we have

**Theorem 5.4.** *For the weak coupling $P(\varphi)_2$ models and the exponential models the Euclidean measure $\mu^*$ restricted to the $\sigma$-algebra generated by the time zero fields defines a measure $\mu$ on $\mathscr{S}'(R)$ which is a $\mathscr{S}(R)$-quasi invariant admissible analytic measure in $\mathscr{P}_1(\mathscr{S}'(R))$, with respect to the nuclear rigging $\mathscr{S}(R) \subset L_2(R) \subset \mathscr{S}'(R)$. $L_2(\mathscr{S}'(R), d\mu)$ carries an irreducible representation of the Weyl canonical commutation relations, with the function 1 as a cyclic vector for the fields $\langle \xi, \varphi \rangle$, $\xi \in \mathscr{S}'(R)$, $\varphi \in \mathscr{S}(R)$ and in the domain of the canonical conjugate momentum $\pi(\varphi)$. The self-adjoint positive Dirichlet operator $H = \frac{1}{2} \nabla^* \overline{\nabla}$ in $L_2(d\mu)$ given by $\mu$ coincides as a form on the dense domain $FC^2 \times FC^2$ in $L_2(d\mu)$ with the restriction of the infinitesimal generator of time translations of the corresponding Wightman models. Moreover $H$ is the limit, as a form on $FC^2 \times FC^2$, of the Dirichlet operators $H_{\mu_g}$ for the space cut-off interactions. On $FC^2$ one has the operator equalities $i[\bar{H}, \langle \xi, \varphi \rangle] = \pi(\varphi)$ and $e^{-i\langle \xi, \varphi \rangle} H e^{i\langle \xi, \varphi \rangle} = H + \pi(\varphi) + \frac{1}{2} \langle \varphi, \varphi \rangle$. $\mu$ is the invariant measure of a Markov diffusion process $\xi(t)$ with state space $\mathscr{S}'(R)$ solving, in the weak sense, the stochastic differential equation*

$$d\xi(t) = \tfrac{1}{2} \beta(\xi(t)) \, dt + dw(t),$$

*with initial distribution $d\mu$, where $\beta$ is the osmotic velocity to $\mu$ and $w$ is the standard Wiener process associated with the rigged Hilbert space.*

*Remark.* It is clear from the method of proof and the fact that the $:\Phi^j:$ bounds are available also in these cases that corresponding results hold for the $P(\varphi)_2$ models with Dirichlet boundary conditions and zero as an isolated eigenvalue of the Hamiltonian $H_{\rm ph}$ (the simplicity of this eigenvalue is not needed, although it is known [42, 87]). By the local perturbation bounds coming from the positivity of the interaction one sees that also the exponential interactions with Dirichlet boundary conditions, considered in [45, 41], can be covered. Further results on the circle of problems discussed in Section 5 are given in Reference [88].

## Appendix

We first prove that the following formula

$$q \cdot \nabla E_R f = E_R q \cdot \nabla f + E_R((\beta \cdot q - E_R \beta \cdot q) f) \tag{1}$$

holds for all $f \in D(q\cdot\nabla)\cap L_\infty(Q', d\mu)$, all $q\in Q$, where $E_R$ is the conditional expectation with respect to the $\sigma$-algebra $\Sigma_R$ and $\beta\cdot q$ is defined by (2.7). Let $\mu_{\Sigma_R}$ be the restriction of the measure $\mu$ to the $\sigma$-algebra $\Sigma_R$. Then we have, for any $f$ as above and any $h$ which is $\Sigma_R$-measurable and $d\mu_{\Sigma_R}$-integrable:

$$\int_{Q'} h(\xi)(E_R q\cdot\nabla f)(\xi)\,d\mu_{\Sigma_R}(\xi) = \int_{Q'} h(\xi)(q\cdot\nabla f)(\xi)\,d\mu(\xi). \tag{2}$$

Recall now that for any $g\in D(q\cdot\nabla)\cap L_\infty(Q', d\mu)$ we have

$$((q\cdot\nabla)^* g)(\xi) = -((q\cdot\nabla)g(\xi) - \beta\cdot q(\xi)\,g(\xi)). \tag{3}$$

This implies if $h\in D(q\cdot\nabla)\cap L_\infty(Q', d\mu)$ that the right hand side of (2) is equal to

$$-\int_{Q'} ((q\cdot\nabla)h)(\xi)f(\xi)\,d\mu(\xi) - \int_{Q'} \beta\cdot q(\xi)\,h(\xi)f(\xi)\,d\mu(\xi). \tag{4}$$

But consider now the first term in this formula. Since $q\cdot\nabla h$ is $\Sigma_R$-measurable and $E_R$ is self-adjoint on $L_2(Q', d\mu)$, since it is a projection, we have

$$\int_{Q'} (q\cdot\nabla h)(\xi)f(\xi)\,d\mu(\xi) - \int_{Q'} E_R(q\cdot\nabla h)(\xi)f(\xi)\,d\mu_{\Sigma_R}(\xi)$$
$$= \int_{Q'} (q\cdot\nabla h)(\xi)(E_R f)(\xi)\,d\mu_{\Sigma_R}(\xi). \tag{5}$$

Suppose now $f$ is such that $E_R f\in D(q\cdot\nabla)\cap L_\infty(Q', d\mu)$. Then we can again use formula (3) to obtain that the right hand side of (5) is equal to

$$-\int_{Q'} h(\xi)(q\cdot\nabla)(E_R f)(\xi)\,d\mu_{\Sigma_R}(\xi) - \int_{Q'} h(\xi)(\beta\cdot q)(E_R f)(\xi)\,d\mu_{\Sigma_R}(\xi). \tag{6}$$

Again using that $h$ is $\Sigma_R$-measurable and $E_R$ is self-adjoint we have that the second term in (6) can be written in the form

$$\int_{Q'} h(\xi)(E_R(\beta\cdot q))f(\xi)\,d\mu_{\Sigma_R}(\xi) = \int_{Q'} h(\xi)\,E_R(E_R(\beta\cdot q)f)(\xi)\,d\mu_{\Sigma_R}(\xi). \tag{7}$$

Thus the first term of (4) is equal by (5), (6) and (7) to

$$\int_{Q'} h(\xi)(q\cdot\nabla)(E_R f)(\xi)\,d\mu_{\Sigma_R}(\xi) - \int_{Q'} h(\xi)\,E_R(E_R(\beta\cdot q)f)(\xi)\,d\mu_{\Sigma_R}(\xi). \tag{8}$$

The second term of (4) can be written, by the fact that $h$ is $\Sigma_R$-measurable and $E_R$ is self-adjoint as

$$-\int_{Q'} h(\xi)\,E_R(\beta\cdot q f)(\xi)\,d\mu(\xi). \tag{9}$$

Thus from (3), (8) and (9) we have that the right hand side of (2) is equal to the sum of (8) and (9), which then proves the formula (1) integrated with respect to $h(\xi)\,d\mu_{\Sigma_R}(\xi)$. Since however the set of $h$ for which this integrated formula has been shown to hold are the $d\mu_{\Sigma_R}$-integrable functions we have that, in particular, (1) holds in the sense of $L_2(Q', d\mu_{\Sigma_R})$-functions, whenever $f$ is the domain

$$D_0 \equiv \{f\in D(q\cdot\nabla)\cap L_\infty(Q', d\mu)\,|\,E_R f\in D(q\cdot\nabla)\}.$$

However the right hand side of (1) exists for all $f \in D(q \cdot \nabla) \cap L_\infty(Q', d\mu)$. The operators $q \cdot \nabla$ and $\beta \cdot q$ being closed, $E_R$ being a projection in $L_2(Q', d\mu_{\Sigma_R})$, and $D(q \cdot \nabla) \cap L_\infty(Q', d\mu)$ being dense in $L_2(Q', d\mu)$, we have that for any $f \in L_2(Q', d\mu_{\Sigma_R})$, $(q \cdot \nabla) E_R f_n$ converges strongly in $L_2(Q', d\mu_{\Sigma_R})$ whenever $f_n \in D_0$ and $f_n \to f$ strongly in $L_2(Q', d\mu_{\Sigma_R})$. Since $E_R f_n \to E_R f$ and $q \cdot \nabla$ is closed, this then implies that $E_R f \in D(q \cdot \nabla)$ and (1) holds for all $f \in D(q \cdot \nabla) \cap L_\infty(Q', d\mu)$, which is what we wanted to prove. To prove Lemma 2.1 it suffices to observe that (1) implies now that $E_R f$ converges to $f$ strongly in the graph norm of $q \cdot \nabla$. Moreover we see easily that $D(q \cdot \nabla) \cap L_\infty(Q', d\mu)$ is dense in $D(q \cdot \nabla)$ in the graph norm, which then concludes the proof of Lemma 2.1.

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