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# Random Processes of Manifolds in $R^n$

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Dedicated to Professor Leopold Schmetterer
on the occasion of his 60th Birthday

**Summary.** We compute the expected values of certain random variables associated with a random process of manifolds in $R^n$ by inserting certain general formulas of integral geometry into the definition of the moment measures of a point process.

## 1. Introduction

Increasing interest in stochastic processes of geometric figures has arisen recently in view of their applications and their connections with geometry. Berman [1], Coleman [2], Fava and Santaló [3], and Parker and Cowan [8] are some of the authors who have dealt with problems closely connected with the subject of this paper.

In this article we show how some previous results concerning processes of geometric figures in the plane and in three dimensional space can be studied in a unified manner in the more general context of $R^n$, by taking into account certain invariants linked to each manifold, namely, the integrals of mean curvature and the Euler-Poincaré characteristic. This is done by inserting some general formulas of integral geometry into the definition of the moment measures of a point process.

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## 2. Geometric Preliminaries

All manifolds appearing in the sequel are assumed to have a finite simplicial decomposition and the symbol $\chi(\Sigma)$ is used to denote the Euler-Poincaré

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characteristic of $\Sigma$. Recall that the Euler-Poincaré characteristic is a topological invariant which has the value zero for the empty set, and for a compact manifold $\Sigma^n$ of dimension $n$ which has a finite simplicial decomposition with $\alpha_i$ simplexes of dimension $i$, equals

$$\chi(\Sigma^n)=\alpha_0-\alpha_1+\ldots+(-1)^n\alpha_n.$$

That is, $\chi(\Sigma^n)$ can be computed simply by counting the simplexes in any simplicial decomposition of $\Sigma^n$. For instance, for a simplex of dimension $n$ or equivalently, for a topological ball $D^n$ in $R^n$, we have $\chi(D^n)=1$ and for a topological sphere $S^{n-1}=\partial D^n$ in $R^n$ we have $\chi(S^{n-1})=1-(-1)^n$. For a 2-dimensional torus $T$ in $R^3$ we have $\chi(T)=0$. In general, for a closed surface $\Sigma_g^2$ of genus $g$ (i.e. a compact manifold of dimension 2 homeomorphic to a 2-dimensional sphere with $g$ handles) we have $\chi(\Sigma_g^2)=2(1-g)$. If $\Sigma^n$ is composed of $m$ disjoint compact manifolds $\Sigma^{ni}$ of dimension $n$, then $\chi(\Sigma^n)=\chi(\Sigma^{n1})+\chi(\Sigma^{n2})+\ldots+\chi(\Sigma^{nm})$.

Concerning the Euler-Poincaré characteristic from the point of view which is of interest in Integral geometry, see Hadwiger [5], Lefschetz [7] and Groemer [4].

With reference to the integrals of mean curvature $M_i(\Sigma)$ $(i=0,1,\ldots,n-1)$ they are well defined for hypersurfaces of class $C^2$ in $R^n$ in terms of their principal curvatures, see [10] or [5]. For compact manifolds of dimension $n$ in $R^n$ (which will be called "bodies") the integrals of mean curvature refer to the boundary $\partial\Sigma^n$. For manifolds of dimension less than $n-1$ and for non smooth manifolds, the integrals of mean curvature have to be computed by the familiar device of considering the integrals for the parallel set to a distance $\varepsilon$ and letting $\varepsilon$ tend to zero [10].

For the ordinary space, $n=3$, the following cases are of interest: (a) $\Sigma^3$ = convex polyhedron; then $M_0(\Sigma^3)=M_0(\partial\Sigma^3)$ = surface area of $\Sigma^3$, $M_1(\Sigma^3)=M_1(\partial\Sigma^3)=(1/2)\,\Sigma(\pi-\alpha_i)\,a_i$, where $a_i$ are the lengths of the edges and $\alpha_i$ the corresponding dihedral angles, the sum being extended over all edges of $\Sigma^3$. (b) $\Sigma^2$ = convex plate of area $f$ and perimeter $u$; then $M_0=2f$, $M_1=(\pi/2)\,u$, $M_2=4\pi$. (c) $\Sigma^1$ = linear segment of length $s$; then $M_0=0$, $M_1=\pi s$, $M_2=4\pi$.

Throughout the paper, we shall denote by

$$O_k=\frac{2\pi^{(k+1)/2}}{\Gamma((k+1)/2)}$$

the surface area of the $k$-dimensional unit sphere $S^k$; thus $O_0=2$, $O_1=2\pi$, $O_2=4\pi$, $O_3=2\pi^2$, etc.

By $R^n$ we mean the $n$-dimensional euclidean space.

## 3. Processes of Manifolds in $R^n$. Processes of Bodies

Let us consider in $R^n$ a random process of compact manifolds, the location of each manifold being given by a point $H$ lying in it and an orthonormal $n$-frame composed of the origin $H$ and a set of orientation vectors $u_{n-1},\ldots,u_1$, where $u_k$

is a point of the $k$-dimensional unit sphere $S^k$, while the shape of the manifold is given by an element $\rho$ of a certain probability space $(\mathfrak{S}, Q)$, where $\mathfrak{S}$ is a locally compact Hausdorff space with a countable base which we call the shape space, in such a way that two manifolds of the family which have assigned the same value of $\rho$ are congruent.

We assume that the manifolds of a given realization can be enumerated according to the distance of the corresponding point $H$ to the origin and that this enumeration is measurable (for two or more points at an equal distance a systematic method for continuing the enumeration is chosen).

We shall also assume that (for an arbitrary realization) our process can be decomposed into $n+1$ mutually independent processes, namely,

(i) the points $H$ corresponding to the different manifolds of the realization, which form a point process in $R^n$ with the property that the expected value of the random variable $N(A)$ = number of points $H$ within the Borel set $A \subset R^n$ is invariant under translations in $R^n$ and finite if $A$ is bounded.

(ii) for each fixed $k$, the unit vectors $u_k$ corresponding to the different elements of the realization, which we assume to be mutually independent and uniformly distributed on the unit sphere $S^k$, with density $du_k/O_k$, where $du_k$ stands for the area element of $S^k (k=1,2,\ldots,n-1)$.

(iii) the sequence of values of the "shape parameter" $\rho$ corresponding to the different elements of the realization, which we assume to be mutually independent and distributed in $\mathfrak{S}$ according to the same probability law $Q$.

Finally, we assume that if $V(\rho)$ is the volume, $M_i(\rho)$ the $i$-th integral of mean curvature and $\chi(\rho)$ the Euler-Poincaré characteristic of a manifold which has assigned the value $\rho$ of the shape parameter, then the mean values

$$E(V)=\int_{\mathfrak{S}} V(\rho)\,Q(d\rho),\quad E(M_i)=\int_{\mathfrak{S}} M_i(\rho)\,Q(d\rho) \qquad (i=1,2,\ldots,n-1),$$

$$E(\chi)=\int_{\mathfrak{S}} \chi(\rho)\,Q(d\rho)$$

are all finite.

## 4. The Mathematical Model

We recall that a point process (more precisely its law $P$) in a locally compact second countable Hausdorff space $Z$ is a probability measure in the space $\mathcal{M}$ of all point measures in $Z$, i.e., locally finite sums of $\delta$-measures. Denoting by $\mu^{\otimes h} = \mu \otimes \ldots \otimes \mu$ the $h$-fold product of $\mu \in \mathcal{M}$, the $h$-th moment measure of $P$ is the mixture

$$\nu^{(h)} = \int_{\mathcal{M}} \mu^{\otimes h}\, P(d\mu)$$

provided that this exists as a locally finite measure in $Z^h$, in which case $P$ is called an $h$-th order process. Explicitly, this means that the functional defined on $\mathcal{M}$ by

$$\zeta_f^{(h)}(\mu) = \mu^{\otimes h}(f) \qquad (\mu \in \mathcal{M}), \tag{1}$$

where $f$ is a Borel function on $Z^h$ such that (1) exists for every $\mu \in \mathscr{M}$ (for example, $f$ bounded with compact support or $f \geqq 0$), is a random variable on the probability space $(\mathscr{M}, P)$ and

$$\nu^{(h)}(f) = \int_{\mathscr{M}} \mu^{\otimes h}(f)\, P(d\mu) = E(\zeta_f^{(h)}). \tag{2}$$

A point $z \in Z$ such that $\mu(\{z\}) \geqq 2$ is called a multiple point of the realization $\mu$. In this paper we shall be dealing exclusively with simple processes, that is, processes having almost surely no multiple points. Accordingly, every realization $\mu$, except for a set of $P$-measure zero, can be identified with its support (a countable set without accumulation points), each point of which carries a mass equal to one.

If $f$ is a function on $Z^h$ of the tipe described above, then $\zeta_f^{(h)}(\mu) = \mu^{\otimes h}(f)$ is the sum of the values $f(z_1, z_2, \ldots, z_h)$ over the set of all $h$-tuples $(z_1, z_2, \ldots, z_h)$ formed from the "points" of the realization $\mu$. We consider two particular cases:

(a) $h=1$; then $\zeta_f(\mu) = \zeta_f^{(1)}(\mu)$ is the sum of the values $f(z)$ over the set of all points $z$ of the realization $\mu$;

(b) $h=1$ and $f = 1_A$ = the indicator function of a bounded Borel set $A \subset Z$; then $\zeta_A(\mu) = \zeta_{1_A}(\mu) = \mu(A)$ is the number of points of $\mu$ falling into $A$, and

$$\nu^{(1)}(A) = E(\zeta_A) = \int_{\mathscr{M}} \mu(A)\, P(d\mu)$$

is the average number of such points, i.e., $\nu^{(1)}$ is the intensity measure of $P$.

Returning to the random process described in the preceding section, since each manifold of the family under consideration can be identified through a one to one correspondence with the ordered set $z = (H, u_{n-1}, \ldots, u_1, \rho)$, mathematically we are dealing with a point process in the locally compact space $Z = R^n \times S^{n-1} \times \ldots \times S^1 \times \mathfrak{S}$ and the assumptions we made in that section imply that the first moment measure $\nu^{(1)}$ of this point process is given by

$$\nu^{(1)}(dz) = \frac{\lambda}{O_{n-1} \ldots O_1}\, dH\, du_{n-1} \ldots du_1\, Q(d\rho) \tag{3}$$

where $\lambda$ is a positive constant and $dH = dx_1 \ldots dx_n$ is the volume element of $R^n$. Let us prove this fact.

Recalling that $N(A)$ denotes the number of points $H$ within the Borel set $A \subset R^n$ and taking into account that $EN(.)$ is a measure in $R^n$, our hypothesis (i) of Sect. 3 entails the equation

$$EN(A) = \lambda V(A),$$

where $\lambda$ is a positive constant and $V(A)$ the volume of $A$.

Let us consider a rectangular set $A \times B \subset Z$, where $A \subset R^n$ is bounded, $B = B_{n-1} \times \ldots \times B_1 \times B_0$ with $B_k \subset S^k (k = 1, 2, \ldots, n-1)$, $B_0 \subset \mathfrak{S}$ and all sets involved are Borel sets. To compute the conditional expectation of the random variable $\zeta_{A \times B}$ (see notation above) given that $N(A) = m$, let us write $z_1, \ldots, z_m$ to denote those points of the realization whose projections on $R^n$ fall into $A$. Setting $z_j$

$=(H_j, \omega_j)$, where $H_j \in A$ and $\omega_j \in S^{n-1} \times \ldots \times S^1 \times \mathfrak{S}$, we have

$$\zeta_{A \times B} = 1_B(\omega_1) + \ldots + 1_B(\omega_m).$$

Hence, taking expected values

$$E(\zeta_{A \times B} | N(A) = m) = m \Pr\{\omega_j \in B\},$$

that is

$$E(\zeta_{A \times B} | N(A)) = N(A) \int_B \frac{du_{n-1}}{O_{n-1}} \ldots \frac{du_1}{O_1} Q(d\rho)$$

and taking expectations again in the last equation, we get

$$\begin{aligned} E\zeta_{A \times B} = \nu^{(1)}(A \times B) &= \lambda V(A) \int_B \frac{du_{n-1} \ldots du_1 \, Q(d\rho)}{O_{n-1} \ldots O_1} \\ &= \frac{\lambda}{O_{n-1} \ldots O_1} \int_{A \times B} dH \, du_{n-1} \ldots du_1 \, Q(d\rho) \end{aligned}$$

which proves our assertion.

*Remark.* For brevity of notation, we shall occassionally use the symbol $dz$ instead of the strictly correct ones $\nu^{(1)}(dz)$ or $d\nu^{(1)}(z)$.

For our next computations it is of the utmost importance to observe that the measure (3) can be written in the form

$$d\nu^{(1)} = \frac{\lambda}{O_{n-1} \ldots O_1} dK Q(d\rho) \tag{4}$$

where $dK = dH \, du_{n-1} \ldots du_1$ is the so called kinematic density of integral geometry [10, Chap. 15] which, as is well known, is the unique density, up to a constant factor, which is invariant with respect to the euclidean motions of $R^n$.

## 5. Mean Values of Intersections

Assuming first that all the manifolds $z$ of our family as well as the set $A \subset R^n$ are bodies, we regard the function $f$ defined on $Z$ by the formula $f(z) = \chi(A \cap z)$. Then, the random variable $\zeta_f$ which we call $X(A)$ is the sum

$$X(A) = \sum \chi(A \cap z) \tag{5}$$

where summation extends to all bodies $z$ of the realization.

To compute the expected value of (5) we have the formula

$$\begin{aligned} EX(A) = \nu^{(1)}(f) = \int f \, d\nu^{(1)} &= \int \chi(A \cap z) \, dz \\ &= \frac{\lambda}{O_{n-1} \ldots O_1} \int_{\mathfrak{S}} Q(d\rho) \int \chi(A \cap z) \, dK. \end{aligned}$$

On the other hand, according to the kinematic fundamental formula of integral geometry [10, formula (15.36)], the value of the last integral on the right hand is

$$O_1 \ldots O_{n-2}\Big[O_{n-1}\,\chi(A)\,V(\rho)+O_{n-1}\,\chi(\rho)\,V(A)$$
$$+(1/n)\sum_{i=0}^{n-2}\binom{n}{i+1}M_i(A)\,M_{n-i-2}(\rho)\Big].$$

Hence

$$EX(A)=\lambda\Big[\chi(A)\,E(V)+E(\chi)\,V(A)$$
$$+(1/nO_{n-1})\sum_{i=0}^{n-2}\binom{n}{i+1}M_i(A)\,E(M_{n-i-2})\Big] \tag{6}$$

Since the Euler-Poincaré characteristic of a convex body equals one, if we assume that all the bodies involved in the present context are convex, then $X(A)$ represents the number of bodies of the realization which intersect with $A$ and formula (6) becomes

$$EX(A)=\lambda\Big[V(A)+E(V)+(1/nO_{n-1})\sum_{i=0}^{n-2}\binom{n}{i+1}M_i(A)\,E(M_{n-i-2})\Big] \tag{7}$$

Let us consider some particular cases of (7):

(a) For $n=3$, the integrals of mean curvature of an arbitrary convex body $K$ are $M_0(K)=F(K)=$surface area of $K$ and $M_1(K)=M(K)=$integral of mean curvature of the boundary of $K$, so that in this case we have

$$EX(A)=\lambda[V(A)+E(V)+(1/4\pi)\,F(A)\,E(M)+(1/4\pi)\,M(A)\,E(F)] \tag{8}$$

where $F=F(\rho)$ is the surface area of any body of the family which corresponds to the value $\rho$ of the shape parameter and $E(F)=\int F(\rho)\,Q(d\rho)$.

(b) For $n=3$, if each manifold $z$ is a convex plate we may consider it as a flattened convex body, and we only have to insert the values $F=2f=$twice the are of the plate and $M=(\pi/2)\,u$, where $u=u(\rho)$ is the perimeter of $z$, to get the formula

$$EX(A)=\lambda[V(A)+(1/8)\,F(A)\,E(u)+(1/2\pi)\,M(A)\,E(f)]. \tag{9}$$

(c) For $n=3$, suppose that each manifold $z$ is a linear segment of length $s=s(\rho)$. If we think of each segment as the limit of a narrowing convex plate, we may insert the values $u=2s$ and $f=0$ in the above formula and we obtain

$$EX(A)=\lambda\,V(A)+(\lambda/4)\,F(A)\,E(s). \tag{10}$$

(d) For $n=2$, the integrals of mean curvature of a plane convex set $K$ are $M_0(K)=u(K)=$the perimeter of $K$ and $M_1(K)=2\pi$, thus

$$EX(A)=\lambda[f(A)+E(f)+(1/2\pi)\,u(A)\,E(u)] \tag{11}$$

where $f=f(\rho)$ is the area and $u=u(\rho)$ the perimeter of $z$, while $f(A)$ denotes the area of $A$.

(e) For $n=2$ and each $z$ a linear segment of length $s=s(\rho)$, by inserting in (11) the values $f=0$ and $u=2s$, we obtain

$$EX(A)=\lambda f(A)+(\lambda/\pi)\,u(A)\,E(s) \tag{12}$$

in agreement with [8].

Once more we emphasize that for the validity of the last formulas (7), ..., (12) all the manifolds involved must be convex.

Turning back our attention to a process of (not necessarily convex) bodies in $R^n$, we fix a number $q$ in the set $\{1,2,\dots,n-1\}$ and consider the function $f$ defined on $Z$ by $f(z)=M_{q-1}(A\cap z)$, where $A$ is some fixed body. Then $\zeta_f$ is the random variable $Y_{q-1}(A)$ defined by

$$Y_{q-1}(A)=\sum M_{q-1}(A\cap z) \tag{13}$$

with summation extended over all bodies $z$ of the realization (we complete the definition by writing $M_{q-1}(\emptyset)=0$).

To compute the expected value of (13), we have

$$\begin{aligned}EY_{q-1}(A)&=\int f\,dv^{(1)}=\int M_{q-1}(A\cap z)\,dz\\&=\frac{\lambda}{O_{n-1}\cdots O_1}\int Q(d\rho)\int M_{q-1}(A\cap z)\,dK.\end{aligned}$$

On the other hand, the formula (15.72) of [10] gives for the last integral the value

$$O_{n-2}\cdots O_1\Bigg[O_{n-1}M_{q-1}(A)\,V(\rho)+O_{n-1}\,V(A)\,M_{q-1}(\rho)$$
$$+\frac{(n-q)\,O_{q-1}}{O_{n-q-1}}\sum_{h=n-q}^{n-2}\frac{\binom{q-1}{q+h-n}O_{2n-h-q}O_h}{(h+1)\,O_{n-h}O_{h+q-n}}M_{h+q-n}(A)\,M_{n-2-h}(\rho)\Bigg],$$

where, for $q=1$, the last sum must be deleted. It follows from here that

$$EY_{q-1}(A)=\lambda\Bigg[M_{q-1}(A)\,E(V)+V(A)\,E(M_{q-1})$$
$$+\frac{(n-q)O_{q-1}}{O_{n-1}\,O_{n-q-1}}\sum_{h=n-q}^{n-2}\frac{\binom{q-1}{q+h-n}O_{2n-h-q}O_h}{(h+1)\,O_{n-h}O_{h+q-n}}M_{h+q-n}(A)\,E(M_{n-2-h})\Bigg]. \tag{14}$$

We consider two particular cases of the preceding relation for $n=3$:

1. $q=1$. Then $Y_0=\sum M_0(A\cap z)=$ total surface area of all sets $A\cap z$; $M_0(A)=F(A)=$ surface area of $A$; $M_0(\rho)=F(\rho)=F=$ surface area of $z$, and formula (14) yields

$$EY_0(A)=\lambda[F(A)\,E(V)+V(A)\,E(F)].$$

In particular, if the manifolds are plates with area $f=f(\rho)$, then $Y_0(A) = 2V_2(A)$ = twice the total area within $A$ of all plates $z$ which intersect with $A$, and we only have to write $V=0$, $F=2f$ in the last relation to get $EV_2(A) = \lambda V(A)E(f)$.

2. $q=2$. Then $Y_1(A)=\sum M_1(A\cap z)$ and we obtain

$$EY_1(A)=\lambda[M_1(A)E(V)+V(A)E(M_1)+(\pi^2/16)F(A)E(F)].$$

## 6. Manifolds of Dimension Less Than $n$

Let $A=A^q$ be a $q$-dimensional compact manifold in $R^n$ and let us suppose that all manifolds $z$ are compact manifolds of dimension $r$, where $r\leqq n$ and $r+q-n\geqq 0$. We shall denote by $\sigma_r(M^r)$ the volume of the $r$-dimensional manifold $M^r$. For a manifold $M$ of dimension zero, $\sigma_0(M)$ denotes the number of points of $M$.

If $f$ is the function defined on $Z$ by the formula $f(z)=\sigma_{r+q-n}(A\cap z)$, then $\zeta_f$ (see the definition in section 3) is the random variable

$$V_{r+q-n}(A)=\sum \sigma_{r+q-n}(A\cap z)$$

where the summation extends over all manifolds $z$ of the realization. Its expected value is

$$EV_{r+q-n}(A)=E(\zeta_f)=v^{(1)}(f)=\int \sigma_{r+q-n}(A\cap z)\,dz$$
$$=\frac{\lambda}{O_{n-1}\dots O_1}\int_{\mathfrak{S}} Q(d\rho)\int \sigma_{r+q-n}(A\cap z)\,dK.$$

Now, the formula (15.20) of [10] gives for the last integral on the right hand the value $(O_n\dots O_1)O_{r+q-n}(O_q O_r)^{-1}\sigma_q(A)\sigma_r(\rho)$, where $\sigma_r(\rho)=\sigma_r(z)$. Hence

$$EV_{r+q-n}(A)=\lambda\frac{O_n O_{r+q-n}}{O_q O_r}\sigma_q(A)E(\sigma_r). \tag{15}$$

If $r+q-n=0$, then $V_0$ denotes the number of intersection points of $A$ with the manifolds of the realization.

Formula (15) contains many particular cases which may be useful in practical situations of the type encountered in stereology. Let us consider some examples.

(a) If $q=n$, we get $EV_r(A)=\lambda V(A)E(\sigma_r)$, where $V_r(A)$ denotes the total $r$-dimensional volume within $A$ of all manifolds $z$ which intersect with $A$. Recall that the manifolds $z$ have dimension $r$.

(b) If $n=3$, $q=2$, $r=1$ ($A$ is a surface of area $F(A)$ and $z$ are curves of length $L(\rho)$), then $V_0(A)$ denotes the number of intersection points of $A$ with the curves of the realization, and (15) gives $EV_0(A)=(\lambda/2)F(A)E(L)$.

(c) If $n=3$, $q=2$, $r=2$, (15) gives $EV_1(A)=(\lambda/4)\pi F(A)E(F)$, where $V_1(A)$ is the total length of the intersections of the surface $A$ with the surfaces of the realization (of area $F=F(\rho)$).

## 7. Mean Values of Multiple Intersections

Suppose that all manifolds $z$ of our process have the same dimension $r$. We shall make an additional assumption concerning the structure of the $h$-th moment measure of $P$, namely, we assume that if $A_1$ $A_2, \dots, A_h$ are disjoint Borel sets in $R^n$, then we have

$$E[N(A_1)\dots N(A_h)]=EN(A_1)\dots EN(A_h). \tag{16}$$

For example, (16) is satisfied in the case of the Poisson process as defined in the next section.

Recalling that $\sigma_i(M^i)$ denotes the volume of the $i$-th dimensional manifold $M^i$ and assuming that $q+rh-nh\geqq 0$, we wish to compute the expected value of the random variable

$$W(A)=\sum \sigma_{q+rh-nh}(A\cap z_1\cap\dots\cap z_h)$$

where $A$ is a compact $q$-dimensional manifold and summation extends over all sets $\{z_1,\dots,z_h\}$ formed by $h$ different elements of the realization.

To this end, we start by considering the mappings $\pi_1$ and $\pi_2$ defined on $Z^h$ by $(H, u_{n-1},\dots,u_1,\rho)\to H$ and $(H,u_{n-1},\dots,u_1,\rho)\to(u_{n-1},\dots,u_1)$ respectively.

Let $D_1$ be the closed subset of $Z^h$ formed by all points $(z_1,z_2,\dots,z_h)$ such that $\pi_1(z_i)=\pi_1(z_j)$ for some pair of indexes $i$ and $j$ such that $1\leqq i<j\leqq h$ and similarly, let $D_2$ be the closed subset of $Z^h$ formed by all points $(z_1,\dots,z_h)$ such that $\pi_2(z_i)=\pi_2(z_j)$ for some pair of different indexes $i$ and $j$. We write $D=D_1\cup D_2$ and define the function $f$ on $Z^h$ by

$$f(z_1,\dots,z_h)=\begin{cases}\sigma_{q+rh-nh}(A\cap z_1\cap\dots\cap z_h) & \text{outside } D\\ 0 \quad \text{on } D.\end{cases}$$

It is clear that almost surely

$$W(A)=\frac{1}{h!}\zeta_f^{(h)},$$

the factorial in the denominator accounting for all permutations.

We postpone the proof of the following facts:

(a) The restriction of $\nu^{(h)}$ to the open set $Z^h-D$ equals the restriction of the $h$-fold product $\nu^{(1)}\otimes\dots\otimes\nu^{(1)}$ to the same set;

(b) $\nu^{(1)}\otimes\dots\otimes\nu^{(1)}(D)=0$.

Taking them for granted, the computation runs as follows:

$$EW(A)=\frac{1}{h!}E(\zeta_f^{(h)})=\frac{1}{h!}\int f\,d\nu^{(h)}=\frac{1}{h!}\int_{Z^h-D} f\,d\nu^{(h)}$$
$$=\frac{1}{h!}\int_{Z^h-D} f\,d(\nu^{(1)}\otimes\dots\otimes\nu^{(1)})=\frac{1}{h!}\int_{Z^h} f(z_1,\dots,z_h)\,dz_1\dots dz_h$$

where $dz_i = \nu^{(1)}(dz_i) = \lambda(O_{n-1} \dots O_1)^{-1} dK_i Q(d\rho_i)$, $i = 1, 2, \dots, h$. Hence

$$EW(A) = \frac{\lambda^h}{h!(O_1 \dots O_{n-1})^h} \int_{\mathfrak{S}} Q(d\rho_1) \dots \int_{\mathfrak{S}} Q(d\rho_h)$$
$$\times \int \sigma_{q+rh-nh}(A \cap z_1 \cap \dots \cap z_h) dK_1 \dots dK_h.$$

Now, the formula (15.22) of [10] gives for the last integral the value $(O_1 \dots O_n)^h O_{q+rh-nh}(O_q O_r^h)^{-1} \sigma_q(A) \sigma_r(\rho_1) \dots \sigma_r(\rho_h)$, where $\sigma_r(\rho_i) = \sigma_r(z_i)$. Therefore

$$EW(A) = \frac{\lambda^h}{h!} \frac{O_n^h O_{q+rh-nh}}{O_q O_r^h} \sigma_q(A) [E(\sigma_r)]^h. \tag{17}$$

Next, we give the proof of (a) and (b) in the case $h = 2$.

Let $U_1 = A_1 \times B_1$ and $U_2 = A_2 \times B_2$, where $A_1$ and $A_2$ are disjoint bounded open sets in $R^n$, while $B_1$ and $B_2$ are rectangular subsets of $S^{n-1} \times \dots \times S^1 \times \mathfrak{S}$. To compute the conditional expectation of $\zeta^{(2)}_{U_1 \times U_2}$ given that $N(A_1) = m$ and $N(A_2) = p$, let us write $z_j = (H_j, \omega_j)$ and $z'_k = (H'_k, \omega'_k)$ to denote the points of the realization $\mu$ such that $H_j \in A_1$ $(1 \leqq j \leqq m)$ and $H'_k \in A_2 (1 \leqq k \leqq p)$. Then

$$\zeta^{(2)}_{U_1 \times U_2}(\mu) = \zeta_{U_1}(\mu) \zeta_{U_2}(\mu) = \{1_{B_1}(\omega_1) + \dots + 1_{B_1}(\omega_m)\} \{1_{B_2}(\omega'_1)$$
$$+ \dots + 1_{B_2}(\omega'_p)\} = \sum_{j=1}^{m} \sum_{k=1}^{p} 1_{B_1}(\omega_j) 1_{B_2}(\omega'_k).$$

Hence

$$E(\zeta^{(2)}_{U_1 \times U_2} | N(A_1) = m, N(A_2) = p) = mp \Pr(B_1) \Pr(B_2)$$

where $\Pr(B) = \int_B (O_{n-1} \dots O_1)^{-1} du_{n-1} \dots du_1 Q(d\rho)$. That is

$$E(\zeta^{(2)}_{U_1 \times U_2} | N(A_1), N(A_2)) = N(A_1) N(A_2) \Pr(B_1) \Pr(B_2).$$

Taking expected values, we get

$$\nu^{(2)}(U_1 \times U_2) = E(\zeta^{(2)}_{U_1 \times U_2}) = E(N(A_1) N(A_2)) \Pr(B_1) \Pr(B_2)$$
$$= \{EN(A_1) \Pr(B_1)\} \{EN(A_2) \Pr(B_2)\}$$
$$= \nu^{(1)}(U_1) \nu^{(1)}(U_2) = \nu^{(1)} \otimes \nu^{(1)}(U_1 \times U_2)$$

which proves (a).

As for (b), we may assume without loss of generality that each manifold $z$ is a point in $R^n$, so that $Z = R^n$ and $D_1$ becomes the diagonal of $Z^2$.

If $E$ is a bounded subset of $D_1$, let $E' = \{z : (z, z) \in E\}$. Then, from the fact that $\nu^{(1)}$ assigns the value zero to each set consisting of a single point,

$$\nu^{(1)} \otimes \nu^{(1)}(E) = \int \nu^{(1)}(E_z) \nu^{(1)}(dz) = \int_{E'} \nu^{(1)}(\{z\}) \nu^{(1)}(dz) = 0.$$

Hence $\nu^{(1)} \otimes \nu^{(1)}(\mathrm{D}_1) = 0$ and similarly for $D_2$. Thus $\nu^{(1)} \otimes \nu^{(1)}(D) = 0$ and the proof of (17) is complete.

*Examples.* 1. Assuming that $n=3$ and that each $z$ is a surface, we may consider two particular cases:

(a) $q=3$, $h=2$. Then $W(A)$ represents the total length within $A$ of all intersections of two different surfaces of the realization and its expected value is

$$EW(A)=(\pi\lambda^2/8)\,V(A)\,[E(f)]^2$$

where $f=f(\rho)$ denotes the surface area of $z$.

(b) $q=3$, $h=3$. Then $W(A)$ becomes the number of intersection points within $A$ of every three different surfaces of the realization and its expected value is

$$EW(A)=(\pi\lambda^3/48)\,V(A)\,[E(f)]^3.$$

2. Assuming that $n=2$ and that each $z$ is a linear segment of length $s(\rho)$, we consider the case $q=2$, $h=2$; then $W(A)$ is the number of segment-segment crossings within $A$, and we get the result of Parker and Cowan [8]:

$$EW(A)=(\lambda^2/\pi)\,F(A)\,[E(s)]^2,$$

where $F(A)$ denotes the area of $A$.

## 8. Processes of Poisson of Convex Manifolds

If in addition to the previously stated hypothesis, the point process (i) of Sect. 3 satisfies the following two conditions:

(a) For every finite set of disjoint Borel subsets of $R^n$, say $\{A_1, A_2, \ldots, A_k\}$, the random variables $N(A_1), \ldots, N(A_k)$ are mutually independent,

(b) For every bounded Borel set $A\subset R^n$, $P\{N(A)=m\}=[\lambda V(A)]^m(m!)^{-1}\exp(-\lambda V(A))$ $(m=0,1,2,\ldots)$ then we say that our process of manifolds $P$ is a Poisson process of intensity $\lambda$.

In this section we assume that both (a) and (b) hold and that the arbitrary set $A$ as well as the manifolds $z$ are convex.

Under these assumptions, it is easy to see that the random variable $X(A)$ = the number of manifolds which intersect with $A$ has a Poisson distribution with the expectation given by (7). Calling $D$ the distance from the origin to the nearest manifold of the process, if we take for $A$ the ball of radius $r$ centered at the origin, then $P\{D>r\}=P\{X(A)=0\}$. But for this particular $A$, we have

$$V(A)=(O_{n-1}/n)\,r^n, \qquad M_i(A)=O_{n-1}\,r^{n-i-1}\,(i=0,1,\ldots,n-1)$$

so that the expected value of $X(A)$ is the number

$$\psi=\lambda\left[\frac{O_{n-1}}{n}r^n+E(V)+\frac{1}{n}\sum_{i=0}^{n-2}\binom{n}{i+1}r^{n-i-1}E(M_{n-i-2})\right]. \tag{18}$$

Hence, under the Poisson assumption,

$$P\{D>r\}=\exp(-\psi)$$

with $\psi$ given by (18). It follows from here that the probability density function of the random variable $D$ is

$$\lambda \exp(-\psi)\left[O_{n-1} r^{n-1} + \sum_{i=0}^{n-2} \binom{n-1}{i+1} E(M_{n-i-2}) r^{n-i-2}\right], \qquad r>0.$$

For the ordinary space ($n=3$) we have three possibilities:

(a) Each manifold $z$ is a convex body. Then $M_0(z)=F(\rho)=$surface area of $z$, $M_1(z)=M_1(\rho)=$first integral of mean curvature and $\psi=\lambda[4\pi r^3/3+E(V)+E(M_1) r^2+E(F) r]$.

(b) Each manifold $z$ is a convex plate. Then $M_0(z)=2f$, where $f=f(\rho)=$area of $z$; $M_1(z)=(\pi/2) u$, where $u=u(\rho)=$perimeter of $z$ and $\psi=\lambda[4\pi r^3/3+(\pi r^2/2) E(u)+2r E(f)]$.

(c) Each set $z$ is a linear segment of length $s=s(\rho)$. Then $M_0(z)=M_0(\rho)=0$, $M_1(z)=M_1(\rho)=\pi s$ and $\psi=\lambda[4\pi r^3/3+\pi r^2 E(s)]$.

## References


1. Berman, M.: Distance distributions associated with Poisson processes of geometric figures. J. Appl Probability **14**, 195–199 (1977)
2. Coleman, R.: The distance from a given point to the nearest end of one member of a random process of linear segments. In Stochastic Geometry. Ed. Harding and Kendall pp. 192–201. New York: Wiley 1974
3. Fava, N., Santaló, L.A.: Plate and line segment processes, J. Appl. Probability **15**, 494–501 (1978)
4. Groemer, H.: Eulersche Characteristic, Projectionen und Quermassintegrale. Math. Ann. **198**, 23–56 (1972)
5. Hadwiger, H.: Vorlesungen über Inhalt, Oberfläche und Isoperimetrie. Berlin-Heidelberg-New York: Springer 1957
6. Krickeberg, K.: Moments of Point-processes. Stochastic Geometry. Ed. Harding and Kendall, pp. 89–113. New York: Wiley 1974
7. Lefschetz, S.: Introduction to Topology. Princeton University Press 1949
8. Parker, P., Cowan, R.: Some properties of line segment processes. J. Appl. Probability **13**, 96–107 (1976)
9. Santaló, L.A.: Integralgeometrie 5, Exposés de Géométrie. Paris: Hermann 1936
10. Santaló, L.A.: Integral Geometry and Geometric Probability, Encyclopedia of Mathematics and its Applications. Reading, Mass.: Addison-Wesley 1976


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