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Characterization of Palm Distributions and Infinitely Divisible Random Measures

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The purpose of the present paper is to characterize Campbell measures and Palm distributions of random measures and to apply these results in a new approach to the characterizations of infinitely divisible random measures by their Laplace functionals and their Palm distributions. The results on infinitely divisible random measures are well known. They can be found together with a detailed list of references in Kallenberg's monograph [2], which also contains proofs of almost all statements in Section 1 of this paper (see his note on page 9 concerning the Polish space setting).

1. Introduction

Let A be a Polish space, let $\mathfrak{A}$ be its Borel algebra and $\mathscr{B}$ the ring of bounded Borel sets (bounded with respect to some complete metric which generates the topology). The space M of Radon measures on $(A, \mathfrak{A})$ with the vague topology is also a Polish space. Its Borel algebra is denoted by $\mathfrak{M}$. A probability measure P on the measurable space $(M, \mathfrak{M})$ is called a random measure on A. We will deal not only with probability measures on $(M, \mathfrak{M})$ but also with measures P which may be infinite. In any case it is assumed that the inequalities

(1) $\lambda_P(B) = \int \mu B\, P(d\mu) < \infty. \quad B \in \mathscr{B},$

hold.

Let $\mathscr{F}$ be the set of measurable and bounded functions $f\colon A \to \mathbb{R}$ with bounded support and let $\mathscr{F}^+(\mathscr{F}_c^+)$ be the subset of positive (and continuous) functions.

The Campbell measure C_P of a measure P on $(M, \mathfrak{M})$ is the σ-finite measure on $(M \times A, \mathfrak{M} \otimes \mathfrak{A})$ defined by the equations

(2) $\int_{M \times A} g\, dC_P = \iint_{M \times A} g(\mu, x)\, \mu(dx)\, P(d\mu), \quad g \geqq 0$ measurable.

There exists (uniquely up to λ_P-null sets) a Markov kernel $P(x,\cdot)$ from $(A,\mathfrak{A})$ to $(M,\mathfrak{M})$ such that for measurable functions $g \geqq 0$ the equations

$$(3) \qquad \int_{M\times A} g(\mu,x)\,\mu(dx)\,P(d\mu) = \iint_{A\times M} g(\mu,x)\,P(x,d\mu)\,\lambda_P(dx)$$

hold. The probability measures $P(x,\cdot)$ are called the Palm distributions, the kernel $(P(x,\cdot))_{x\in A}$ the Palm kernel of P.

An easy consequence of (3) is the following Lemma 1, which can be used to calculate Palm distributions explicitely. Let P be a random measure, let

$$L_P(f) = \int_M \exp(-\mu f)\,P(d\mu), \qquad f\in\mathscr{F}^+$$

and

$$L_P(x,f) = \int_M \exp(-\mu f)\,P(x,d\mu), \qquad f\in\mathscr{F}^+,\ x\in A,$$

be the Laplace functionals of P and its Palm distributions.

Lemma 1. *The Laplace functionals of a random measure P and its Palm distributions satisfy the equations*

$$(4) \quad \frac{d}{ds} L_P(sf+g) = -\int_A L_P(x,sf+g)\,f(x)\,\lambda_P(dx), \qquad f,g\in\mathscr{F}^+,\ s\geqq 0.$$

If K is a Markov kernel from $(A,\mathfrak{A})$ to $(M,\mathfrak{M})$ with Laplace functionals $L_K(x,\cdot)$, $x\in A$, and if the equations

$$\frac{d}{ds} L_P(sf+g)\bigg|_{s=0} = -\int L_K(x,g)\,f(x)\,\lambda_P(dx), \qquad f,g\in\mathscr{F}^+$$

hold, then K is a Palm kernel of P.

Remark. The method of differentiation of the Laplace or characteristic functional of P in order to obtain the Palm distributions has been used frequently in earlier papers (see for instance Mecke [4], Jagers [1], Krickeberg [3], and Kallenberg [2]).

The following results will be needed in the sequel:

Lemma 2 (Mecke [4]). *There exists a measurable function $h: M\times A\to\mathbb{R}^+$ satisfying the equalities*

$$\int h(\mu,x)\,\mu(dx) = 1 \qquad \textit{for all } \mu\in M\setminus\{0\}.$$

Lemma 3. *If the sequence of Laplace functionals L_n of random measures P_n converges pointwise on $\mathscr{F}_c^+$ and the limit is continuous from above at 0, the limit is the Laplace functional of some random measure P, and the sequence (P_n) converges weakly to P.*

2. Characterization of Campbell Measures and Palm Kernels

Theorem 1. *A measure C on the measurable space $(M\times A, \mathfrak{M}\otimes\mathfrak{A})$ is the Campbell measure of some measure P with the property* (1) *if and only if* (5) *and* (6) *hold.*

(5) $C(M\times B)<\infty, \quad B\in\mathscr{B}.$

(6) $\int g\,dC=0$ *whenever* $\int g(\mu,x)\,\mu(dx)=0$ *for each* $\mu\in M$.

The measure C is the Campbell measure of some random measure P if in addition to (5) *and* (6) *the inequalities*

(7) $\iint\limits_{(\mu B>0)\times B} (\mu B)^{-1}\,dC\leqq 1, \quad B\in\mathscr{B},$

are valid.

Proof. If C is a Campbell measure, the statements (5), (6), and (7) are immediate consequences of (2). Assume now the validity of (5) and (6) and define the measure P on $(M,\mathfrak{M})$ by

$$\int\limits_M f(\mu)\,P(d\mu)=\int\limits_{M\times A} f\,h\,dC, \qquad f \text{ measurable, bounded,}$$

where h is the function introduced in Lemma 2. Then we have because of (6) for measurable $g\colon M\times A\to\mathbb{R}^+$

$$\int g\,dC_P=\iiint g(\mu,x)\,\mu(dx)\,h(\mu,y)\,C(d(\mu,y))=\int g\,dC$$
$$(\text{for } \int(\int g(\mu,x)\,\mu(dx)\,h(\mu,y)-g(\mu,y))\,\mu(dy)=0,\ \mu\in M).$$

Thus $C=C_P$, and P satisfies (1) because of (5). In case we also have (7) the measure P' which equals P on $M\setminus\{0\}$ and has weight $1-\sup\limits_{B\in\mathscr{B}}\iint\limits_{(\mu B>0)\times B}(\mu B)^{-1}\,dC$ on the null measure 0 is a random measure with $C_{P'}=C$.

Theorem 1 implies the following characterization of Palm kernels:

Theorem 2. *A Markov kernel K from $(A,\mathfrak{A})$ to $(M,\mathfrak{M})$ is a Palm kernel of some measure P with property* (1) *if and only if there exists a measure $\lambda\in M$ such that*

(8) $\int\limits_A K(x,\{0\})\,\lambda(dx)=0$

and

(9) $\int\limits_C\int\limits_D \mu B\,K(x,d\mu)\,\lambda(dx)=\int\limits_B\int\limits_D \mu C\,K(x,d\mu)\,\lambda(dx), \qquad B,C\in\mathscr{B},\ D\in\mathfrak{M}$

hold. It is a Palm kernel of some random measure P if and only if in addition to (8) *and* (9)

(10) $m=\sup\limits_{B\in\mathscr{B}}\int\limits_B\int\limits_{\mu B>0}(\mu B)^{-1}\,K(x,d\mu)\,\lambda(dx)<\infty$

is valid.

Proof. The necessity of (8), (9), and (10) for Palm kernels K is an obvious consequence of formula (3). To prove the sufficiency we observe that (9) can be written in the form

$$(11)\quad \iiint g(x,y,\mu)\,\mu(dx)\,K(y,d\mu)\,\lambda(dy)=\iiint g(y,x,\mu)\,\mu(dx)\,K(y,d\mu)\,\lambda(dy)$$

with g being a product of indicator functions. By standard methods it can be shown that (11) is also valid for arbitrary integrable functions g. We use Theorem 1 to prove that $C=K\otimes\lambda$ is a Campbell measure. Because of $C(M\times B)=\lambda B$ for $B\in\mathscr{B}$ condition (5) is satisfied. Let $\int f(\mu,x)\,\mu(dx)=0$ for each $\mu\in M$ and define the function g by $g(x,y,\mu)=f(\mu,x)\,h(\mu,y)$, $x,y\in A$, $\mu\in M$ (h being the function introduced in Lemma 2). Then the left hand side of (11) is equal to 0, and the right hand side equals because of (8)

$$\iint_{M\setminus\{0\}\times A} f\,dC=\int f\,dC.$$

This proves the existence of a measure P on $(M,\mathfrak{M})$ with Palm kernel K. Under the assumption (10) the measure $\frac{1}{m}P$ is a random measure with the desired properties.

Condition (9) can be formulated in terms of Laplace functionals, which is the appropriate form for the application in the next section.

Corollary. *A Markov kernel with the Laplace functionals $L_K(x,\cdot)$, $x\in A$, is a Palm kernel if and only if there exists a measure $\lambda\in M$ such that* (8) *and*

$$(12)\quad \frac{d}{ds}\int L_K(x,sf_1+g)\,f_2(x)\,\lambda(dx)=\frac{d}{ds}\int L_K(x,sf_2+g)\,f_1(x)\,\lambda(dx)$$

for $f_1,f_2,g\in\mathscr{F}_c^+$ *and* $s=0$

hold.

3. Infinitely Divisible Random Measures

Theorem 3. *Let P be a random measure on A. Then the following three statements are equivalent:*

(13) *P is infinitely divisible.*

(14) *There exists a Markov kernel K' from $(A,\mathfrak{A})$ to $(M,\mathfrak{M})$ such that $P*K'(x,\cdot)$, $x\in A$, is a Palm kernel of P.*

(15) *There exists a measure $\alpha\in M$ and a measure K on $(M,\mathfrak{M})$ with property* (1) *such that*

$$L_P(f)=\exp\left(-\alpha f-\int(1-\exp(-\mu f))\,K(d\mu)\right),\qquad f\in\mathscr{F}^+.$$

Proof. Assume (13). Let P_n be the random measure on A satisfying $P=P_n^n$ ($n=1,2,\ldots$). Let L, L_n, $L(x,\cdot)$. and $L_n(x,\cdot)$ be the Laplace functionals of P, P_n, $P(x,\cdot)$, and $P_n(x,\cdot)$ respectively. Because of

$$L(f)=L_n(f)^n, \quad f\in\mathscr{F}^+,$$

we obtain from (4) for $s=0$

$$\begin{aligned}\frac{d}{ds}L(sf+g)&=n\,L_n(g)^{n-1}\frac{d}{ds}L_n(sf+g)\\&=-\int L_n(g)^{n-1}L_n(x,g)\,f(x)\,\lambda(dx)\end{aligned}$$

and by the second part of Lemma 1

$$L(x,f)=L(f)^{1-1/n}L_n(x,f), \qquad x\in A,\ f\in\mathscr{F}^+.$$

For $x\in A$ the limit

$$\lim_{n\to\infty}L_n(x,f)=L(x,f)/L(f)=L_{K'}(x,f)$$

exists pointwise on $\mathscr{F}_c^+$ and is continuous from above at 0. It is therefore (Lemma 3) a Laplace functional of some random measure $K'(x,\cdot)$. This proves (14).

Now assume the validity of (14). Define the function d and the Markov kernel $K(x,\cdot)$, $x\in A$, by the equations

$$(16)\quad d(x)=K'(x,\{0\}), \qquad K'(x,\cdot)=d(x)\,\varepsilon_0+(1-d(x))\,K(x,\cdot), \qquad x\in A.$$

Applying the corollary to Theorem 2 we show that $K(x,\cdot)$ is a Palm kernel. Condition (8) is satisfied by definition. Condition (12) holds for $L(x,\cdot)$ instead of $L_K(x,\cdot)$. The Laplace functional $L(x,\cdot)$ is given by

$$(17)\quad L(x,f)=L(f)\,L_{K'}(x,f)=L(f)(d(x)+(1-d(x))\,L_K(x,f)).$$

Therefore we obtain for $s=0$

$$\begin{aligned}&\frac{d}{ds}\int L(sf_1+g)\,f_2(x)\,\lambda(dx)\\&\quad=\int\left[\frac{d}{ds}L(sf_1+g)\,L_{K'}(x,g)+L(g)\,(1-d(x))\,\frac{d}{ds}L_K(x,sf_1+g)\right]f_2(x)\,\lambda(dx).\end{aligned}$$

Applying (4) to the term $\frac{d}{ds}L(sf_1+g)$ and using (14) this equals

$$L(g)\left[-\prod_{i=1}^{2}\int L_{K'}(x,g)\,f_i(x)\,\lambda(dx)+\frac{d}{ds}\int L_K(x,sf_1+g)\,f_2(x)(1-d(x))\,\lambda(dx)\right]$$

and equating this expression to $\frac{d}{ds} \int L(x, sf_2+g) f_1(x) \lambda(dx)\Big|_{s=0}$ proves (12) with $(1 - d(x)) \lambda(dx)$ instead of $\lambda(dx)$.

Thus $K(x, \cdot)$, $x \in A$, is a Palm kernel of some measure K with $\lambda_K(dx) = (1 - d(x)) \lambda(dx)$.

From (4) and (17) we obtain

$$\frac{d}{ds} \log L(sf) = -\int f(x) d(x) \lambda(dx) - \int L_K(x, sf) f(x) \lambda_K(dx)$$

and with $\alpha(dx) = d(x) \lambda(dx)$

$$\begin{aligned} \log L(f) &= -\alpha f - \int_0^1 \int \exp(-s \mu f) f(x) K(x, d\mu) \lambda_K(dx) ds \\ &= -\alpha f - \int (1 - \exp(-\mu f)) K(d\mu). \end{aligned}$$

This proves (15).

The proof of (13) under the assumption (15) is trivial.

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