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A New Proof of Kesten's Theorem on the Growth of the Sum of Independent and Identically Distributed Random Variables

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0. In this paper, a more direct and probabilistically intuitive proof of Kesten's theorem on the growth of the sum of independent and identically distributed random variables ([2]) is presented. The technique used is a modified version of "centering" employed by Wolfowitz ([4]) and Kesten ([3]).

1. Theorem (Kesten [2], Corollary 3). *Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of independent and identically distributed random variables with $EX_i^+ = EX_i^- = +\infty$. Then either*

(i) $\lim_{n\to\infty} \frac{S_n}{n} = +\infty$ *w.p.l, or*

(ii) $\lim_{n\to\infty} \frac{S_n}{n} = -\infty$ *w.p.l, or*

(iii) $\limsup_{n\to\infty} \frac{S_n}{n} = +\infty$ *and* $\liminf_{n\to\infty} \frac{S_n}{n} = -\infty$ *w.p.l,*

where $S_n = \sum_{i=1}^{n} X_i$, $n = 1, 2, \ldots$.

Proof. Let $D = \liminf_{n\to\infty} S_n/n$ where $-\infty \leqq D \leqq +\infty$. By the Kolmogorov Zero-One Law, D is constant w.p.l. To prove the theorem, it suffices to show that $D = +\infty$ or $D = -\infty$. Suppose $-\infty < D < +\infty$. Without loss of generality we may assume $D > 0$; for otherwise we may replace X_i by $Y_i = X_i - D + \frac{1}{2}$ and $H_n = \sum_{i=1}^{n} Y_i$ by noticing that $\liminf_{n\to\infty} H_n/n = \frac{1}{2} > 0$ and $\liminf_{n\to\infty} H_n/n$ is finite iff $\liminf_{n\to\infty} S_n/n$ is finite.

Consider the two-sided sequence $\{X_i\}_{i=-\infty}^{+\infty}$ which is the extension of $\{X_i\}_{i=1}^{\infty}$ into "the past" ([1] Ch. 6). Define $\tilde{S}_k = \sum_{i=-k+1}^{0} X_i$ and $\tilde{S} = 0 = S_0$. It follows that

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$\liminf_{n\to\infty} \tilde{S}_n/n = D$ w.p.l. Define stopping times $\gamma_0 \equiv 0$,

$$\begin{aligned} \gamma_1 &= \min\{n \geqq 1 : \tilde{S}_n > 0\} \\ &= +\infty \text{ if no such } n \text{ exists.} \end{aligned} \tag{1}$$

Since $D > 0$, $\tilde{S}_n \to +\infty$ as $n \to \infty$ so $\gamma_1 < \infty$ w.p.l. In the same manner, define

$$\begin{aligned} \gamma_k &= \min\{n > \gamma_{k-1} : \tilde{S}_n - \tilde{S}_{\gamma_{k-1}} > 0\} \\ &= +\infty \text{ if no such } n \text{ exists for } k = 2, 3, \ldots. \end{aligned} \tag{2}$$

Now $\{\tilde{S}_{\gamma_k} - \tilde{S}_{\gamma_{k-1}}\}_{k=1}^{\infty}$ are i.i.d. random variables and are positive, so the Birkhoff Ergodic Theorem implies that

$$\begin{aligned} \lim_{n\to\infty} \tilde{S}_{\gamma_n}/n &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (\tilde{S}_{\gamma_k} - \tilde{S}_{\gamma_{k-1}}) \\ &= E(\tilde{S}_{\gamma_1} - \tilde{S}_{\gamma_0}) \quad \text{w.p.l.} \end{aligned}$$

To conclude the proof of the theorem, we need the following facts:

$$\liminf_{n\to\infty} X_{-\gamma_n - 1}/n = -\infty, \tag{3}$$

$$0 < E(\gamma_1 - \gamma_0) < +\infty, \quad \text{and} \tag{4}$$

$$0 < E(\tilde{S}_{\gamma_1} - \tilde{S}_{\gamma_0}) < \infty. \tag{5}$$

For (4) implies that $\lim_{n\to\infty} \tilde{S}_{\gamma_n}/\gamma_n$ exists and then (3) and (5) imply that $\lim_{n\to\infty} \tilde{S}_{\gamma_n+1}/(\gamma_n + 1) = -\infty$ which would contradict the fact that D is finite.

Now $X_{-\gamma_k - 1} = \tilde{S}_{\gamma_k+1} - \tilde{S}_{\gamma_k}$ so that $\{X_{-\gamma_k-1}\}_{k=1}^{\infty}$ is an i.i.d sequence of random variables with the distribution of $X_{-\gamma_1-1}$ the same as the distribution of X_1 ([1]). Hence $EX^+_{-\gamma_k-1} = EX^-_{-\gamma_k-1} = +\infty$ and so (3) holds.

To prove (4) and (5) we adopt the notation of Kesten ([3]). In particular

$$\begin{aligned} v_0 &= \min\{j \geqq 0 : \inf_{n>j} S_n - S_j > 0\} \quad \text{and} \\ v_{k+1} &= \min\{j > v_k : \inf_{n>j} S_n - S_j > 0\}. \end{aligned} \tag{6}$$

Since $D > 0$, $\lim_{n\to\infty} S_n = +\infty$ w.p.l so that $v_0 < \infty$ and $v_{k+1} < \infty$. Furthermore, as was shown in ([3]), $q \equiv P(v_0 = 0) > 0$. This allows us to define a new probability measure Q on the Borel subsets of $\mathbb{R}$ by

$$Q(A) = \frac{1}{q} P(A, v_0 = 0). \tag{7}$$

Let

$$v^* = \min\{m \geqq 1 : \sum_{-m+1 \leqq i \leqq v_0} X_i > 0\}. \tag{8}$$

Then since $\{X_0, X_{-1}, \ldots\}$ and $\{X_1, X_2, \ldots\}$ generate independent σ-fields

$$P(\gamma_1=k)=\frac{P(\gamma_1=k, v_0=0)}{P(v_0=0)}$$
$$=\frac{1}{q} P\left(\sum_{-j+1\leqq i\leqq 0} X_i \begin{matrix}\leqq 0 \text{ for } 1\leqq j<k \\ >0 \text{ for } j=k\end{matrix}, v_0=0\right)$$
$$=\frac{1}{q} P(v^*=k, v_0=0)$$
$$=\frac{1}{q} P(v_0=0, v_1=k)=Q(v_1-v_0=k). \tag{9}$$

Thus

$$0<\int \gamma_1 \, dP=\int (v_1-v_0)\, dQ=\frac{1}{q}<\infty. \tag{10}$$

where the last equality is a simple result which may be found in [3].
To prove (5) we use ince again the independence of the σ-fields generated by X_0, X_{-1}, ... and X_1, X_2, ... respectively to obtain

$$P(\tilde{S}_{\gamma_1}-\tilde{S}_{\gamma_0}\in A)=\frac{1}{q} P(\tilde{S}_{\gamma_1}-\tilde{S}_{\gamma_0}\in A, v_0=0)$$
$$=\frac{1}{q}\sum_{l=1}^{\infty} P(v^*=l, v_0=0, \tilde{S}_l\in A), \tag{11}$$

where A is a Borel set.

Now increasing the indices by l on the set

$$\{v^*=l, v_0=0, \tilde{S}_l\in A)$$

yields the set

$$\{v_1=_l v_0=0, S_l\in A\}.$$

From this and (10) it follows that

$$P(\tilde{S}_{\gamma_1}-\tilde{S}_{\gamma_0}\in A)=Q(S_{v_1}-S_{v_0}\in A) \tag{12}$$

Therefore

$$0\leqq\int(\tilde{S}_{\gamma_1}-\tilde{S}_{\gamma_0})\, dP=\int(S_{v_1}-S_{v_0})\, dQ\leqq+\infty. \tag{13}$$

Now by [3], $\lim_{n\to\infty} S_{v_k}/k\equiv\beta$ exists w.p.l, $\beta>0$, and $E(\beta)=\int(S_{v_1}-S_{v_0})\, dQ$. Suppose $E(\beta)=+\infty$. Then for any integer $K>0$ $Q(\beta/\alpha>K)>0$, where $\alpha\equiv\lim_{n\to\infty} v_k/k$, since if $\beta/\alpha<K$ w.p.1 Q, then

$$EQ(\beta)<KE_Q(\alpha)=K\int(v_1-v_0)\, dQ<\infty. \tag{14}$$

Since we assumed $D<\infty$, we may choose $K_0>D$. Then since $P(\liminf_{n\to\infty} S_n/n>K_0)=0$ and $Q\ll P$, it follows that

$$\begin{aligned} 0 &= Q(\liminf_{n\to\infty} S_n/n>K_0)\\ &\geqq Q(\liminf_{n\to\infty} S_n/n>K_0,\ \beta/\alpha>K_0)\\ &= Q(\beta/\alpha>K_0)\geqq 0. \end{aligned} \tag{15}$$

Hence $E(\beta)<\infty$ and so (5) is proved. As stated earlier, this, in conjunction with (3) and (4), proves the theorem.

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