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On Vector-Valued Amarts and Dimension of Banach Spaces

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If $(X_n)_{n\in\mathbb{N}}$ is an amart of class (B) taking values in a Banach space with the Radon-Nikodym property, then X_n converges weakly a.s., as proved in [4]. Examples exist in [4] and [7] which show that *strong* convergence may fail, but recently Alexandra Bellow [2] proved the following result: A Banach space **E** is finite-dimensional if (and only if) every **E**-valued amart of class (B) converges *strongly* a.s. We prove here that if p is fixed, $1\leqq p<\infty$, then a Banach space **E** is finite-dimensional if (and only if) every L^p-bounded **E**-valued amart converges *weakly* a.s. The point of this is that in the amart convergence theorem for an infinite-dimensional Banach space, the assumption (B) cannot be weakened any more than the conclusion that weak a.s. convergence holds can be strengthened.

Let $(\Omega, \mathscr{F}, P)$ be a probability space, $\mathbb{N}=\{1,2,\ldots\}$, and let $(\mathscr{F}_n)_{n\in\mathbb{N}}$ be an increasing sequence of σ-algebras contained in $\mathscr{F}$. A *stopping time* is a mapping τ: $\Omega\to\mathbb{N}\cup\{\infty\}$, such that $\{\tau=n\}\in\mathscr{F}_n$ for all $n\in\mathbb{N}$. The collection of bounded stopping times is denoted by T; under the natural ordering T is a directed set. (The notation and the terminology of the present note are close to those of our longer article [7].)

Let **E** be a Banach space and consider a sequence $(X_n)_{n\in\mathbb{N}}$ of **E**-valued random variables adapted to $(\mathscr{F}_n)_{n\in\mathbb{N}}$, i.e. such that X_n: $\Omega\to E$ is $\mathscr{F}_n$-strongly measurable. We will write EX (expectation of X) for the Pettis integral [9] of the random variable X. The sequence (X_n) is called an *amart* iff each X_n is Pettis integrable and $\lim_{\tau\in T} EX_\tau$ exists in the strong topology of **E**. An adapted sequence (X_n) is said to be of *class* (B) iff

$$\sup_{\tau\in T} E\|X_\tau\|<\infty. \tag{B}$$

Let $1\leqq p<\infty$. (X_n) is said to be *L^p-bounded* iff

$$\sup\nolimits_{n\in\mathbb{N}} E\|X_n\|^p<\infty. \tag{1}$$

Theorem 1. *Let a number p with $1\leqq p<\infty$ be fixed. Given a Banach space* **E**, *the following conditions are equivalent.*

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(i) **E** *is finite-dimensional.*

(ii) *Every* **E**-*valued* L^p-*bounded amart converges weakly a.s.*

(iii) *Every* **E**-*valued* L^p-*bounded amart adapted to a sequence of* σ-*algebras* $\mathscr{F}_n$ *with* $\mathscr{F}_n = \mathscr{F}$ *for all n, converges weakly a.s.*

We will prove a slightly stronger result. The condition (2) includes not only L^p-boundedness, but boundedness in vector-valued Orlicz spaces as well.

Theorem 2. *Let* $\Phi\colon [0,\infty) \to [0,\infty)$ *be a continuous increasing function with* $\Phi(0)=0$ *and* $\liminf_{t\to\infty} \Phi(t)/t > 0$. *Let* **E** *be a Banach space. The following are equivalent.*

(i) **E** *is finite-dimensional.*

(ii) *Every* **E**-*valued amart* (X_n) *such that*

$$\sup_n E\Phi(\|X_n\|) < \infty \tag{2}$$

converges weakly a.s.

(iii) *Every* **E**-*valued amart* (X_n) *satisfying* (2) *and adapted to a sequence of* σ-*algebras* $(\mathscr{F}_n)$ *with* $\mathscr{F}_n = \mathscr{F}$ *for all n, converges weakly a.s.*

Proof. (i)⇒(ii). Since $\liminf_{t\to\infty} \Phi(t)/t > 0$, an amart satisfying (2) is L^1-bounded. As observed in [2], the convergence is an immediate consequence of the (ascending) amart convergence theorem [1] which asserts that an L^1-bounded scalar amart converges a.s. It suffices to point out that projections on different coordinates of an L^1-bounded amart are L^1-bounded amarts.

(ii)⇒(iii) is obvious.

(iii)⇒(i). This is the main part of the theorem. We may note the non-probabilistic nature of condition (iii). Indeed, to say that $(X_n, \mathscr{F})_{n\in\mathbb{N}}$ is an amart, is to say that for every sequence (ρ_n) of finite measurable partitions of Ω, $\rho_n = \{A_n^i,\ i = 1, 2, \ldots, r_n\}$, one has

$$\lim_n \sum_{i=1}^{r_n} E(X_{n+1}\colon A_n^i) = z \in \mathbf{E}$$

exists, the same for all $(\rho_n)_{n\in\mathbb{N}}$. Thus the implication (iii)⇒(i) is a result in the geometry of Banach spaces, and our proof is based on a fundamental result of that theory, the theorem of Dvoretzky, reducing the situation to Hilbert space. A. Bellow used the more elementary lemma of Dvoretzky-Rogers, which could also be applied, but the theorem of Dvoretzky seems to shed more light on the construction.

Given two Banach spaces **F** and **E**, **F** is said to be *finitely representable* in **E** iff, given any finite-dimensional subspace $\mathbf{F}_1$ of **F** and any number $\varepsilon > 0$, there is an isomorphism V of $\mathbf{F}_1$ into **E** such that

$$(1-\varepsilon)\|x\| \leqq \|Vx\| \leqq (1+\varepsilon)\|x\|$$

for all $x \in \mathbf{F}_1$. The theorem of Dvoretzky [6] asserts that ℓ^2 is finitely representable in any infinite-dimensional Banach space.

Suppose now that $\mathbf{E}$ is an infinite-dimensional Banach space and let $\mathbf{F}=\ell^2$. We will define an amart on $\Omega=[0,1]$ for $\mathscr{F}_n$: the Lebesgue measurable sets for all n. For each n, let r_n be the least integer larger than $2^n \Phi(2n)$.

Let $\{\bar{e}_n^i\colon n\in\mathbb{N},\ 1\leqq i\leqq r_n\}$ be a collection of orthonormal vectors in $\mathbf{F}$. By the theorem of Dvoretzky, for each n there is an isomorphism V_n of the Hilbert space $\mathbf{F}_n$ spanned by $\{\bar{e}_n^i\colon 1\leqq i\leqq r_n\}$ into $\mathbf{E}$, such that

$$\|V_n x\|\leqq 2\|x\| \tag{3}$$

for all $x\in\mathbf{F}_n$. Write $e_n^i=V_n\bar{e}_n^i$. Let $\rho_n=\{A_n^i\colon i=1,2,\ldots,r_n\}$ be a partition of Ω such that $P(A_n^i)=1/r_n$. For $n\in\mathbb{N}$ and $i=1,\ldots,r_n$, set

$$Y_n^i=n\,e_n^i\,1_{A_n^i}.$$

Let the sequence $(X_m)_{m\in\mathbb{N}}$ be $\{Y_n^i\colon n\in\mathbb{N},\ 1\leqq i\leqq r_n\}$ ordered so that Y_n^i is before $Y_{n'}^{i'}$ if $n<n'$, or $n=n'$ and $i<i'$. Thus if $X_m=Y_n^i$, then $m=R_{n-1}+i$, where $R_{n-1}=r_1+r_2+\cdots+r_{n-1}$. For $\tau\in T$, let $B_n^i=A_n^i\cap\{\tau=R_{n-1}+i\}$. Then

$$X_\tau=\sum_n\sum_{i=1}^{r_n} n\,e_n^i\,1_{B_n^i}.$$

Now the sequence $X_m(\omega)$ is unbounded for each $\omega\in\Omega$, hence does not converge weakly.

In order to show that (X_n) is an amart, note that $r_n\geqq 2^n\Phi(2n)\geqq 2^n\Phi(2)$, so

$$\sum n r_n^{-1/2}\leqq\sum n 2^{-n/2}\,\Phi(2)^{-1/2}<\infty. \tag{4}$$

Now if $\tau\in T$, $\tau>R_{N-1}$, then

$$\begin{aligned}\|EX_\tau\|&\leqq\sum_{n>N}\left\|\sum_{i=1}^{r_n}E(n e_n^i\,1_{B_n^i})\right\|\\&\leqq\sum_{n>N}2\left\|\sum_{i=1}^{r_n}E(n\bar{e}_n^i\,1_{B_n^i})\right\|.\end{aligned}$$

But the $\bar{e}_n^i$ are orthonormal, so

$$\begin{aligned}\left\|\sum_{i=1}^{r_n}E(n\bar{e}_n^i\,1_{B_n^i})\right\|^2&=\sum_{i=1}^{r_n}n^2P(B_n^i)^2\\&\leqq\frac{n^2}{r_n}\sum_{i=1}^{r_n}P(B_n^i)\leqq\frac{n^2}{r_n}.\end{aligned}$$

Thus

$$\|EX_\tau\|\leqq 2\sum_{n>N}\frac{n}{r_n^{1/2}},$$

which goes to 0 as N goes to ∞ by (4).

We next verify that (X_n) satisfies (2). Set

$$\bar{Y}_n^i=n\bar{e}_n^i\,1_{A_n^i}.$$

Then for each ω,

$$\|Y_n^i(\omega)\| \leqq 2\,\|\bar{Y}_n^i(\omega)\|.$$

Hence

$$E\Phi(\|Y_n^k\|) \leqq E\Phi(2\,\|\bar{Y}_n^k\|) = E\Phi(2n\,1_{A_n^k}) = \frac{\Phi(2n)}{r_n} \leqq \frac{\Phi(2n)}{2^n\,\Phi(2n)} = \frac{1}{2^n},$$

so $E\Phi(\|X_m\|) \leqq 1$ for all m.

We finally observe that we have only used Dvoretzky's theorem with $\varepsilon = 1$. A simple proof for this weaker form of the theorem was given by L. Tzafriri [10].

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