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Asymptotically Distribution-Free Aligned Rank Order Tests for Composite Hypotheses for General Multivariate Linear Models*

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For general multivariate linear models, a composite hypothesis does not usually induce invariance of the joint distribution under appropriate groups of transformations, so that genuinely distribution-free tests do not usually exist. For this purpose, some aligned rank order statistics are incorporated in the proposal and study of a class of asymptotically distribution-free tests. Tests for the parallelism of several multiple regression surfaces are also considered. Finally the optimal properties of these tests are discussed.

1. Introduction

In the context of a general form of the multivariate linear model, we consider a sequence $\{\mathbf{X}_i=(X_{1i},\dots,X_{pi})', i\geqq 1\}$ of independent random vectors (i.rv) with continuous cumulative distribution functions (cdf)

$$F_i(\mathbf{x})=P\{\mathbf{X}_i\leqq\mathbf{x}\}=F(\mathbf{x}-\boldsymbol{\alpha}-\boldsymbol{\beta}\mathbf{c}_i),\ i\geqq 1,\ \mathbf{x}\in R^p,\ p\geqq 1, \tag{1.1}$$

where $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_p)'$ and $\boldsymbol{\beta}=((\beta_{jk}))_{j=1,\dots,p,k=1,\dots,q}$ $(q\geqq 1)$ are unknown parameters, and $\mathbf{c}_i=(c_{1i},\dots,c_{qi})'$, $i\geqq 1$ are vectors of known regression constants. The form of the cdf F is not assumed to be specified. By reference to the usual canonical reduction of the multivariate linear hypothesis [viz., Anderson (1958), Ch. 8], we consider the following. We partition

$$\underset{p\times q}{\boldsymbol{\beta}}=(\underset{p\times q_1}{\boldsymbol{\beta}_1},\ \underset{p\times q_2}{\boldsymbol{\beta}_2}),\quad q_1+q_2=q,\ q_1\geqq 0 \text{ and } q_2\geqq 0, \tag{1.2}$$

and consider the null hypothesis

$$H_0\colon \boldsymbol{\beta}_2=\mathbf{0} \quad \text{against the alternative} \quad H_1\colon \boldsymbol{\beta}_2\neq\mathbf{0}. \tag{1.3}$$

Note that whenever q_i is $\geqq 1$, both H_0 and H_1 are composite hypotheses.

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For the particular case of $q_1=0$ i.e., for H_0: $\boldsymbol{\beta}=\mathbf{0}$, the $\mathbf{X}_i$ are identically distributed (i.d.) under H_0 and a class of conditionally as well as asymptotically distribution-free rank order tests has already been studied by Puri and Sen (1969). For $q_1 \geqq 1$, under H_0: $\boldsymbol{\beta}_2=\mathbf{0}$, the $\mathbf{X}_i$ are no longer i.i.d., and this invalidates the approach of the above mentioned paper. In fact, in such a case, genuinely distribution-free rank order tests may not generally exist. In the univariate case (i.e., for $p=1$), this difficulty has been circumvented in some specific problems by Sen (1969) and Puri and Sen (1973) by using suitable aligned rank order tests where the alignment is based on rank order estimates of the nuisance parameters. This approach is systematically explored and developed here for the general multivariate linear model, and, basically, the theory of rank order estimators of regression parameters, developed in Sen and Puri (1969) and Jurečková (1971), is employed here for the estimation of the nuisance parameters (i.e., $\boldsymbol{\beta}_1$) and incorporated in the construction of suitable aligned rank order statistics on which tests for H_0: $\boldsymbol{\beta}_2=\mathbf{0}$ are based. In the univariate parametric case, the classical likelihood ratio test possesses some (asymptotic) optimality properties. The picture is somewhat different in the multivariate general linear models where the likelihood ratio tests may not perform uniformly better than others, even asymptotically. However, under quite general regularity conditions, the Wald-optimality (viz., best average power and the most stringency) applies to the likelihood ratio types tests. In the nonparametric multivariate problems too, a variety of tests is available in the literature [viz., Puri and Sen (1971)], and some of these can be adapted for the general linear model under consideration. Among other possibilities, we consider here a class of rank order tests having some analogy with the likelihood ratio type tests. For such rank tests, it is possible to develop distribution theory in some closed forms comparable to those of the likelihood ratio tests and permitting an asymptotic comparison of these tests for local alternatives.

The proposed rank order tests for H_0 are considered in Section 3 following the preliminary notions and basic assumptions in Section 2. Section 4 deals with asymptotic comparisons of parametric (mostly, normal theory and likelihood ratio) and rank order tests. Asymptotic optimality of the proposed aligned rank tests is also considered in this section. The last section deals with a special case of (1.3), namely, testing the hypothesis of parallelism of several (multiple) regression surfaces, which turns out to be the multivariate multiparameter analogue of Sen (1969).

2. Preliminary Notions

Let $R_{ji}=\sum_{s=1}^{n} u(X_{ji}-X_{js})$ (where $u(t)=1$ or 0 according as t is $\geqq$ or <0) be the rank of X_{ji} among the X_{js}, $s=1,\dots,n$, for $i=1,\dots,n$, $j=1,\dots,p$. Since F is continuous, ties among the observations may be neglected, in probability. For each $j(=1,\dots,p)$ and $n(\geqq 1)$, we consider a set of real valued *scores*

$a_n^{(j)}(1), \ldots, a_n^{(j)}(n)$, generated by a *score function* $\phi_j(u)$, $0<u<1$, in either of the following ways:

$$a_n^{(j)}(i)=\phi_j(i/(n+1)) \quad \text{or} \quad E\phi_j(U_{ni}), \qquad i=1, \ldots, n;\ j=1, \ldots, p, \tag{2.1}$$

where $\phi_j(u)$ is assumed to be square integrable inside $(0, 1)$, and $U_{n1}<\cdots<U_{nn}$ are the ordered random variables of a sample of size n from the rectangular distribution over $(0, 1)$. Our proposed testing procedure is based on the following type of rank order statistics:

$$\mathbf{S}_n=((S_{n,jk})); \qquad S_{n,jk}=\sum_{i=1}^{n}(c_{ki}-\bar{c}_{kn})\, a_n^{(j)}(R_{ji}) \tag{2.2}$$

where $\bar{c}_{kn}=n^{-1}\sum_{i=1}^{n} c_{ki}$, for $k=1, \ldots, q$; $j=1, \ldots, p$. We denote by $\bar{\mathbf{c}}=n^{-1}\sum_{i=1}^{n}\mathbf{c}_i$.

Following Hájek (1968) and Hoeffding (1973), we assume that for every $j(=1, \ldots, p)$,

$$\varphi_j(u)=\varphi_j^{(1)}(u)-\varphi_j^{(2)}(u) \tag{2.3}$$

where $\varphi_j^{(s)}(u)$, $s=1,2$ is absolutely continuous and non-decreasing in $u\in(0, 1)$ and

$$\int_0^1 |\varphi_j^{(s)}(u)|\,\{u(1-u)\}^{-\frac{1}{2}}\,du<\infty; \qquad s=1,2;\ j=1, \ldots, p. \tag{2.4}$$

Denote

$$\bar{\varphi}_j=\int_0^1 \varphi_j(u)\,du, \qquad j=1, \ldots, p, \tag{2.5}$$

and

$$\lambda_{jj'}(F)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi_j(F_{[j]}(x))\,\varphi_{j'}(F_{[j']}(y))\,dF_{[jj']}(x, y)-\bar{\varphi}_j\bar{\varphi}_{j'} \tag{2.6}$$

where $F_{[j]}(x)$ and $F_{[jj']}(x, y)$ are the marginal cdfs of the j-th and the (j, j')-th components respectively. Assume

$$\Lambda(F)=((\lambda_{jj'}(F))) \quad \text{is positive definite (p.d.) and finite.} \tag{2.7}$$

We also denote by

$$\begin{aligned}\mathbf{C}_n=((C_{n,kk'}))&=\sum_{i=1}^{n}(\mathbf{c}_i-\bar{\mathbf{c}}_n)(\mathbf{c}_i-\bar{\mathbf{c}}_n)'\\ &=\left(\left(\sum_{i=1}^{n}(c_{ki}-\bar{c}_{kn})(c_{k'n}-\bar{c}_{k'n})\right)\right)\end{aligned} \tag{2.8}$$

and assume that

$$\mathbf{C}_n \quad \text{is p.d. for every } n\geqq n_0, \tag{2.9}$$

and there exists a p.d. and finite matrix $\mathbf{C}$, such that

$$n^{-1}\mathbf{C}_n \to \mathbf{C} \quad \text{as } n\to\infty. \tag{2.10}$$

Further, we assume that

$$\mathbf{c}_i = \mathbf{c}_i^{(1)} - \mathbf{c}_i^{(2)}, \quad \text{for } i=1,\dots,n, \tag{2.11}$$

where for each $k(=1,\dots,q)$ and $s(=1,2)$, $c_{ki}^{(s)}$ is non-decreasing in i. This assumption is a slightly simplified version of a parallel assumption made by Jurečková (1971). For $q=1$, (2.11) is not necessary.

Finally, we assume that for every $\varepsilon>0$, there exists an integer $n_o=n_o(\varepsilon)$, such that for $n\geqq n_o$.

$$n^{-1}C_{n,kk} > \varepsilon\{\max_{1\leqq i\leqq n} |c_{ki}-\bar{c}_{kn}|^2\}, \quad k=1,\dots,n. \tag{2.12}$$

Regarding the c.d.f. F, we assume that for each $j(=1,\dots,p)$, the marginal c.d.f. $F_{[j]}$ has an absolutely continuous probability density function (p.d.f.) $f_{[j]}$ with a finite Fisher information

$$I_j = I(f_{[j]}) = \int_{-\infty}^{\infty} \{(d/dx)\log f_{[j]}(x)\}^2\, dF_{[j]}(x), \quad j=1,\dots,p. \tag{2.13}$$

As has been mentioned in Section 1, our testing procedure rests on some aligned rank order statistics. To explain the alignment procedure, we need the following notations.

Let $\mathbf{B}=((b_{jk}))_{j=1,\dots,p;\,k=1,\dots,q}$ be a real matrix, and we write $\mathbf{B}'=((\mathbf{b}_1',\dots,\mathbf{b}_p'))$ where $\mathbf{b}_j'=(b_{j1},\dots,b_{jq})$, for $j=1,\dots,p$. Let then

$$\mathbf{X}_i(\mathbf{B}) = \mathbf{X}_i - \mathbf{B}\mathbf{c}_i = (X_{1i}(\mathbf{b}_1),\dots,X_{pi}(\mathbf{b}_p))' \quad \text{for } i=1,\dots,n; \tag{2.14}$$

$$R_{ji}(\mathbf{B}) = R_{ji}(\mathbf{b}_j) = \sum_{s=1}^{n} u(X_{ji}(\mathbf{b}_j) - X_{js}(\mathbf{b}_j)), \quad i=1,\dots,n;\ j=1,\dots,p, \tag{2.15}$$

so that $R_{ji}(\mathbf{B})$ is the rank of $X_{ji}(\mathbf{b}_j)$ among $X_{js}(\mathbf{b}_j)$, $s=1,\dots,n$, for $i=1,\dots,n$.

Now, replace the R_{ji} in (2.2) by $R_{ji}(\mathbf{b}_j)$ for $i=1,\dots,n$, $j=1,\dots,p$ and denote the corresponding matrix of rank order statistics by

$$\mathbf{S}_n(\mathbf{B}) = ((S_{n,jk}(\mathbf{b}_j)))_{j=1,\dots,p;\,k=1,\dots,q}. \tag{2.16}$$

Note that $\mathbf{S}_n(\mathbf{B})$ in (2.16), viewed as a function of the pq elements in $\mathbf{B}$, generates a pq-dimensional stochastic process (on R^{pq}). We shall make use of the same in the next section to introduce the proposed aligned rank order statistics.

3. The Proposed Aligned Rank Order Tests

As in (1.2), we introduce the following partitionments:

$$\underset{p\times q}{\mathbf{B}} = (\underset{p\times q_1}{\mathbf{B}_1}, \underset{p\times q_2}{\mathbf{B}_2}); \quad \underset{q\times 1}{\mathbf{c}_i'} = (\underset{q_1\times 1}{\mathbf{c}_{i(1)}'}, \underset{q_2\times 1}{\mathbf{c}_{i(2)}'}), \quad i=1,\dots,n. \tag{3.1}$$

Then, under H_0 in (1.3), we have

$$F_i(\mathbf{x})=F(\mathbf{x}-\boldsymbol{\alpha}-\boldsymbol{\beta}_1\mathbf{c}_{i(1)}), \quad i=1,\ldots,n. \tag{3.2}$$

First, we proceed to estimate the nuisance parameter (matrix) $\boldsymbol{\beta}_1$ for the model (3.2). For this, we consider the $p\times q_1$ matrix

$$\mathbf{S}_{n(1)}(\mathbf{B}_1,\mathbf{0})=((S_{n,jk}(\mathbf{b}_j^{(1)})))_{j=1,\ldots,p;\,k=1,\ldots,q_1}, \tag{3.3}$$

where

$$(\mathbf{b}_j^{(1)})'=(b_{j1},\ldots,b_{jq_1},0,\ldots,0), \quad j=1,\ldots,p. \tag{3.4}$$

Now, under (3.2), $\mathbf{S}_{n(1)}(\boldsymbol{\beta}_1,\mathbf{0})$ has the same distribution as of $\mathbf{S}_{n(1)}(\mathbf{0},\mathbf{0})$ under the hypothesis that $\boldsymbol{\beta}=\mathbf{0}$; for the latter case, we may use the results of Puri and Sen (1969) and obtain the following: (a) under (3.2), $\mathbf{S}_{n(1)}(\boldsymbol{\beta}_1,\mathbf{0})$ has expectation $\mathbf{0}$ and dispersion matrix

$$A(F)\otimes\mathbf{C}_{n(11)} \quad \text{where } \mathbf{C}_n=\begin{pmatrix}\mathbf{C}_{n(11)}, & \mathbf{C}_{n(12)}\\ \mathbf{C}_{n(21)} & \mathbf{C}_{n(22)}\end{pmatrix}. \tag{3.5}$$

$\mathbf{C}_{n(ij)}$ is of order $q_i\times q_j$, $i,j=1,2$ and $\otimes$ stands for the Kronecker product, and (b) as $n\to\infty$,

$$\mathscr{L}(n^{-\frac{1}{2}}\mathbf{S}_{n(1)}(\boldsymbol{\beta}_1,\mathbf{0}))\to\mathscr{N}_{p\times q_1}(\mathbf{0},A(F)\otimes\mathbf{C}_{(11)}), \tag{3.6}$$

where $\mathbf{C}_{(11)}$ is the upper $q_1\times q_1$ principal minor of $\mathbf{C}$, defined by (2.10). To estimate $\boldsymbol{\beta}_1$, we adopt the alignment procedure studied in detail by Sen and Puri (1969) and Jurečková (1971), and define

$$\mathbf{D}_n=\left\{\mathbf{B}_1:\sum_{j=1}^{p}\sum_{k=1}^{q_1}|S_{n,jk}(\mathbf{b}_j^{(1)})|=\text{minimum}\right\}. \tag{3.7}$$

Then, our estimator of $\boldsymbol{\beta}_1$, under the model (3.2), is given by

$$\hat{\boldsymbol{\beta}}_{1,n}=\text{centre of gravity of } \mathbf{D}_n. \tag{3.8}$$

By arguments parallel to those of Jurečková (1971), it follows that

$$\sup_{\mathbf{B}_1\in\mathbf{D}_n}\|\boldsymbol{\beta}_1-\hat{\boldsymbol{\beta}}_{1,n}\|\xrightarrow{P}0, \quad \text{as } n\to\infty \tag{3.9}$$

$$\mathscr{L}(n^{\frac{1}{2}}[\hat{\boldsymbol{\beta}}_{1,n}-\boldsymbol{\beta}_1])\to\mathscr{N}_{p\times q}(0,\boldsymbol{\Gamma}(F)\otimes\mathbf{C}_{(11)}) \tag{3.10}$$

where

$$\mathbf{T}(F)=((\tau_{jj'}(F)))=((\lambda_{jj'}(F)/A_jA_{j'})) \tag{3.11}$$

and

$$A_j=\int_{-\infty}^{\infty}(d/dx)\,\varphi_j(F_{[j]}(x))\,dF_{[j]}(x), \quad j=1,\ldots,p. \tag{3.12}$$

Under the model (3.2), $\hat{\boldsymbol{\beta}}_{1,n}$ is a translation-invariant, robust, consistent and asymptotically normally distributed estimator of $\boldsymbol{\beta}_1$. We use the same for our alignment process and consider the following aligned rank order statistics:

$$\hat{\mathbf{S}}_{n(2)}=((\hat{S}_{n,jk}))_{j=1,\dots,p;\,k=q_1+1,\dots,q} \tag{3.13}$$

where

$$\hat{S}_{n,jk}=\sum_{i=1}^{n}(c_{ki}-\bar{c}_{kn})\,a_n^{(j)}(\hat{R}_{ji}),\qquad j=1,\dots,p;\,k=q_1+1,\dots,q, \tag{3.14}$$

and the aligned-ranks are defined by

$$\hat{R}_{ji}=R_{ji}(\hat{\boldsymbol{\beta}}_{1,n},\mathbf{0})\quad\text{ for } i=1,\dots,n;\ j=1,\dots,p. \tag{3.15}$$

To introduce the proposed test statistics, we first define

$$\begin{aligned} M_n&=((m_{jj',n}))_{j,j'=1,\dots,p}\\ &=\left(\left((n-1)^{-1}\sum_{i=1}^{n}a_n^{(j)}(R_{ji})\,a_n^{(j')}(R_{j'i})-\bar{a}_n^{(j)}\bar{a}_n^{(j')}\right)\right), \end{aligned} \tag{3.16}$$

where

$$\bar{a}_n^{(j)}=n^{-1}\sum_{i=1}^{n}a_n^{(j)}(i),\qquad\text{for } j=1,\dots,p. \tag{3.17}$$

Also, replacing R_{ji} by $\hat{R}_{ji}$ and $R_{j'i}$ by $\hat{R}_{j'i}$ for $i=1,\dots,n$ and $j,j'=1,\dots,p$ in (3.16), we denote the corresponding matrix by

$$\hat{M}_n=((\hat{m}_{jj',n}))_{j,j'=1,\dots,p}. \tag{3.18}$$

Let then

$$\mathbf{C}_n^*=\mathbf{C}_{n(22)}-\mathbf{C}_{n(21)}\mathbf{C}_{n(11)}^{-1}\mathbf{C}_{n(12)}, \tag{3.19}$$

$$\hat{\mathbf{G}}_n=\hat{\mathbf{M}}_n\otimes\mathbf{C}_n^*\qquad(\text{of order } pq_2\times pq_2), \tag{3.20}$$

$$\mathbf{H}_n=((\hat{S}_{n,jk}\hat{S}_{n,j'k'}))_{j,j'=1,\dots,p;\,k,k'=q_1+1,\dots,q} \tag{3.21}$$

where $\mathbf{H}_n$ is also of the order $pq_2\times pq_2$. Then, our proposed test statistic is

$$\begin{aligned}\mathscr{L}_n&=\operatorname{Tr}(\mathbf{H}_n\hat{\mathbf{G}}_n^{-1})\\ &=\sum_{j=1}^{p}\sum_{j'=1}^{p}\sum_{k=q_1+1}^{q}\sum_{k'=q_1+1}^{q}\hat{S}_{n,jk}\hat{S}_{n,j'k'}\,\hat{m}_n^{jj'}\,c_n^{*kk'}\end{aligned} \tag{3.22}$$

where

$$\hat{\mathbf{M}}_n^{-1}=((\hat{m}_n^{jj'}))_{j,j'=1,\dots,p}\quad\text{ and }\quad\mathbf{C}_n^{*-1}=((c_n^{*kk'}))_{k,k'=q_1+1,\dots,q}. \tag{3.23}$$

The analogy of $\mathscr{L}_n$ to the classical Lawley-Hotelling trace criterion for the multivariate analysis of variance (MANOVA) problem can readily be identified. Whereas the latter is based on the least squares estimators of the parameters involved in the model, our $\mathscr{L}_n$ is based on the corresponding rank order estimators of the nuisance parameters. Further, the asymptotic equivalence of the Lawley-Hotelling trace and the likelihood-ratio criteria (under the normal theory model) is well-known [viz., Anderson (1959), Ch. 8]. In the same spirit, we could have proposed an alternative test-statistic

$$\mathscr{L}_n^0 = \|\hat{\mathbf{G}}_n\| / \|\mathbf{H}_n + \hat{\mathbf{G}}_n\|, \tag{3.24}$$

where $\|\mathbf{A}\|$ stands for the determinant of the matrix $\mathbf{A}$. By using Theorem 3.1 of Jurečková (1971) and proceeding as in the proof of Theorem 3.3 of Puri, Sen and Gokhale (1970), it can be shown that $\mathscr{L}_n$ and $-2\log \mathscr{L}_n^0$ are asymptotically equivalent under the null hypothesis and for local alternatives too. As such, in the sequel, we shall be mainly concerned with the statistic $\mathscr{L}_n$.

In the remainder of this section, we show that Under H_0: $\boldsymbol{\beta}_2 = \mathbf{0}$, when the assumptions of Section 2 are met, $\mathscr{L}_n$ has asymptotically a chi-square distribution with pq_2 degrees of freedom. This provides an ADF (asymptotically distribution-free) test for H_0.

Lemma 3.1. *Under H_0: $\boldsymbol{\beta}_2 = \mathbf{0}$ and the assumptions of Section 2,*

$$n\hat{\mathbf{G}}_n^{-1} \xrightarrow{p} \Lambda^{-1}(F) \otimes \mathbf{C}^{*-1}, \qquad \text{as } n \to \infty, \tag{3.25}$$

where

$$\mathbf{C}^* = \mathbf{C}_{(22)} - \mathbf{C}_{(21)} \mathbf{C}_{(11)}^{-1} \mathbf{C}_{(12)}; \qquad \mathbf{C} = \begin{pmatrix} \mathbf{C}_{(11)} & \mathbf{C}_{(12)} \\ \mathbf{C}_{(21)} & \mathbf{C}_{(22)} \end{pmatrix}. \tag{3.26}$$

Proof. By virtue of (2.10), to prove (3.25), it suffices to show that

$$\hat{\mathbf{M}}_n \xrightarrow{p} \Lambda(F), \qquad \text{as } n \to \infty. \tag{3.27}$$

Also since $\hat{m}_{jj',n} = m_{jj,n} = (n-1)^{-1} \left\{ \sum_{i=1}^{n} [a_n^{(j)}(i) - \bar{a}_n^{(j)}]^2 \right\} \to \lambda_{jj}(F) = \lambda_{jj}$ by (2.1) and some routine computations, we need only to show that for every $j \neq j'$,

$$\hat{m}_{jj',n} \xrightarrow{p} \lambda_{jj'}(F) \qquad \text{when } H_0 \text{ holds.} \tag{3.28}$$

By assumption (2.3), (see also Hájek (1968), section 5) for every $\varepsilon > 0$, there exists a decomposition

$$\varphi_j(u) = \varphi_j^{(1)}(u) + \varphi_j^{(2)}(u) - \varphi_j^{(3)}(u), \qquad 0 < u < 1, \ j = 1, \ldots, p \tag{3.29}$$

where $\varphi_j^{(1)}$ is a polynomial, $\varphi_j^{(2)}$ and $\varphi_j^{(3)}$ are non-decreasing, and

$$\sum_{k=2}^{3} \int_0^1 [\varphi_j^{(k)}(u)]^2 \, du < \varepsilon \lambda_{jj}, \qquad 1 < j < p. \tag{3.30}$$

Using (3.29) we decompose $\hat{m}_{jj',n}$ into 9 terms. Using the Cauchy-Schwarz inequality for the eight terms for which at least one factor is non polynomial along with (3.30), it follows that to prove (3.28), it suffices to take $\varphi_j = \varphi_j^{(1)}$, $1 \leqq j \leqq p$. Since the $\varphi_j^{(1)}$ are absolutely continuous and are polynomials, for them, the corresponding $\hat{m}_{jj',n}$ can be written as

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi_j^{(1)}(\hat{H}_{nj}(x))\,\varphi_{j'}^{(1)}(\hat{H}_{nj'}(y))\,d\hat{H}^*_{njj'}(x, y)+o(1)$$

where $\hat{H}_{nj}$ is the sample c.d.f. for all the aligned observations on the j-th variate, $j=1, \ldots, p$ and $\hat{H}^*_{njj'}$ is the bivariate sample c.d.f. (for the j-th and j'-th variates) for these aligned observations. For the aligned vectors $\mathbf{X}_i(\boldsymbol{\beta}_1, \mathbf{0})$, $i=1, \ldots, n$, the sample c.d.f.'s are denoted by H_{nj} and $H^*_{njj'}$, for $j(\neq j')=1, \ldots, p$. Then, by (2.14), (3.10) and the continuity of the parent c.d.f., it follows that as $n\to\infty$,

$$\sup\{|\hat{H}^*_{njj'}(x, y)-H^*_{njj'}(x, y)|:\ x, y\in R^2\}\to 0\ \text{ a.s.},$$

for every $j\neq j'=1, \ldots, p$. Also, note that the $\phi_j^{(1)}$ are bounded, continuous functions. So, first replacing $\hat{H}_n$ by H_n, $\hat{H}^*_n$ by H^*_n and then using Theorem 4.1 of Puri and Sen (1969), the desired result follows. In fact, it can be shown that (3.27) holds a.s.

Lemma 3.2. *Under* H_0: $\boldsymbol{\beta}_2=\mathbf{0}$ *and the assumptions of Section* 2, *as* $n\to\infty$,

$$n^{-\frac{1}{2}}\{\hat{\mathbf{S}}_{n(2)}-\mathbf{S}_{n(1)}(\boldsymbol{\beta}_1, \mathbf{0})+\mathbf{A}(\hat{\boldsymbol{\beta}}_{1,n}-\boldsymbol{\beta}_1)\,\mathbf{C}_{n(12)}\}\xrightarrow{p} 0, \tag{3.31}$$

where defining the A_j *by* (3.12),

$$\mathbf{A}=\operatorname{Diag}(A_1, \ldots, A_p). \tag{3.32}$$

The proof follows as a direct multivariate extension of Theorem 3.1 of Jurečková (1971), and hence, the details are omitted.

By noting that cf. Jurečková (1971) $\mathbf{S}_{n(1)}(\hat{\boldsymbol{\beta}}_1, \mathbf{0})=\mathbf{o}_p(1)$, the following lemma also follows from Theorem 3.1 of Jurečková (1971).

Lemma 3.3. *Under the hypothesis of Lemma* 3.2, *as* $n\to\infty$,

$$n^{-\frac{1}{2}}\{\mathbf{S}_{n(1)}(\boldsymbol{\beta}_1, \mathbf{0})-\mathbf{A}(\hat{\boldsymbol{\beta}}_{1,n}-\boldsymbol{\beta}_1)\,\mathbf{C}_{n(11)}\}\xrightarrow{p}\mathbf{0}. \tag{3.33}$$

From (3.31) and (3.33), we arrive at the following

Lemma 3.4. *Under the hypothesis of Lemma* 3.2 *as* $n\to\infty$,

$$n^{-\frac{1}{2}}\{\hat{\mathbf{S}}_{n(2)}-\mathbf{S}_{n(2)}(\boldsymbol{\beta}_1, \mathbf{0})+\mathbf{S}_{n(1)}(\boldsymbol{\beta}_1, \mathbf{0})\,\mathbf{C}^{-1}_{n(11)}\,\mathbf{C}_{n(12)}\}\xrightarrow{p}\mathbf{0}, \tag{3.34}$$

where $\mathbf{S}_{n(2)}(\boldsymbol{\beta}_1, \mathbf{0})$ *is defined as in* (3.13)–(3.14) *with the* $\hat{R}_{ji}$ *being replaced by* $R_{ji}(\boldsymbol{\beta}_1, \mathbf{0})$.

Consider now H_0^*: $\boldsymbol{\beta}=\mathbf{0}$. Then, under H_0: $\boldsymbol{\beta}_2=\mathbf{0}$, the pair $(\mathbf{S}_{n(1)}(\boldsymbol{\beta}_1, \mathbf{0}),\ \mathbf{S}_{n(2)}(\boldsymbol{\beta}_1, \mathbf{0}))$ have the same joint distribution as that of $\mathbf{S}_n$ under H_0^*, and since, [viz., Puri and Sen (1969)], the latter is asymptotically multinormal with mean $\mathbf{0}$ and dispersion matrix

$$\Lambda(F)\otimes\mathbf{C}_n, \tag{3.35}$$

it follows that under H_0 in (1.3),

$$\mathscr{L}(n^{-\frac{1}{2}}\{\mathbf{S}_{n(2)}(\boldsymbol{\beta}_1,\mathbf{0})-\mathbf{S}_{n(1)}(\boldsymbol{\beta}_1,\mathbf{0})\,\mathbf{C}_{n(11)}^{-1}\,\mathbf{C}_{n(12)})$$
$$\to\mathscr{N}_{pxq_2}(\mathbf{0},\Lambda(F)\otimes\{\mathbf{C}_{(22)}-\mathbf{C}_{(21)}\,\mathbf{C}_{(11)}^{-1}\,\mathbf{C}_{(12)}\}). \tag{3.36}$$

Hence, by (3.34) and (3.36), we obtain that under H_0 in (1.3), as $n\to\infty$,

$$\mathscr{L}(n^{-\frac{1}{2}}\,\hat{\mathbf{S}}_{n(2)}\to\mathscr{N}_{pxq_2}(0,\Lambda(F)\otimes\mathbf{C}^*). \tag{3.37}$$

From (3.37), Lemma 3.1 and the asymptotic distribution theory of quadratic forms associated with asymptotically multinormal vectors, it follows that under the hypothesis of Lemma 3.3, $\mathscr{L}_n$ has asymptotically a chi square distribution with pq_2 degrees of freedom.

Thus, the proposed ADF test of size $\alpha(0<\alpha<1)$ is as follows:

Let $\chi^2_{pq_2,\alpha}$ be the upper $100\alpha\,\%$ point of the chi square c.d.f. with pq_2 degrees of freedom. Then, the null hypothesis H_0 in (1.3) is accepted or rejected according as $\mathscr{L}_n$ is $<$ or $\geqq\ \chi^2_{pq_2,\alpha}$.

4. Asymptotic Comparison with Parametric Counterparts

We confine ourselves to local alternatives for which the power of the proposed tests are away from 0 and 1. We consider a sequence $\{K_n\}$ of Pitman-type alternative hypotheses, viz., for some (fixed) non-null γ_2,

$$K_n\colon\ \boldsymbol{\beta}_2=\boldsymbol{\beta}_2^{(n)}=n^{-\frac{1}{2}}\gamma_2,\qquad \gamma_2=((\gamma_{jk}))_{j=1,\dots,p;\,k=q_1+1,\dots,q} \tag{4.1}$$

for which an asymptotic power function can be traced and compared with the parallel function for some parametric tests for the same problem.

For the normal theory model (where the underlying c.d.f. F is assumed to be multivariate normal), classical parametric (likelihood ratio, Lawley-Hotelling trace or the largest characteristic root criterion of S.N. Roy – see Chapter 8 of Anderson (1959)) tests are all based on the least squares estimators. Sen and Puri (1970) have studied the asymptotic properties of the likelihood ratio (as well as the Lawley-Hotelling Trace) statistic when the underlying F is not necessarily normal. It follows that if F possesses a finite and positive definite dispersion matrix

$$\Sigma(F)=((\mathrm{cov}(X_{ji},X_{j'i})))=((\sigma_{jj'}(\mathrm{F}))),$$

then, under H_0 in (1.3), L_n ($=-2\log$-likelihood ratio criterion when F is assumed to be normal) has asymptotically a chi-square distribution with pq_2 degrees of freedom. Also, under $\{K_n\}$ in (4.1), L_n has asymptotically a non-central chi square distribution with pq_2 degrees of freedom and non-centrality parameter

$$\Delta_L = \mathrm{Trace}(\bar{\boldsymbol{\Gamma}}(\boldsymbol{\Sigma}(F) \otimes \mathbf{C}^*)^{-1}), \tag{4.2}$$

where

$$\bar{\boldsymbol{\Gamma}} = ((\gamma_{jk}\, \gamma_{j'k'}))_{j, j' = 1, \ldots, p;\, k, k' = q_1 + 1, \ldots, q}, \tag{4.3}$$

and the γ_{jk} are defined in (4.1).

Consider now a sequence of alternatives $\{K_n^*\}$, specified by

$$K_n^*: \ \boldsymbol{\beta} = (\mathbf{0}, n^{-\frac{1}{2}} \boldsymbol{\gamma}_2), \qquad \boldsymbol{\gamma}_2 \text{ defined in (4.1).} \tag{4.4}$$

Then, $(\mathbf{S}_{n(1)}(\boldsymbol{\beta}_1, \mathbf{0}), \mathbf{S}_{n(2)}(\boldsymbol{\beta}_1, \mathbf{0}))$, under K_n, has the same joint distribution as that of $\mathbf{S}_n$ under K_n^*. Noting this fact, using the results of Puri and Sen (1969) and our lemmas in Section 3, it follows by some routine computations that under $\{K_n\}$, as $n \to \infty$,

$$\mathscr{L}(n^{-\frac{1}{2}} \hat{\mathbf{S}}_{n(2)}) \to \mathscr{N}_{pq_2}(\mathbf{A}\boldsymbol{\gamma}_2 \mathbf{C}^*, \boldsymbol{A}(F) \otimes \mathbf{C}^*). \tag{4.5}$$

From Lemma 3.1 and (4.5), we conclude that under $\{K_n\}$, $\mathscr{L}_n$ has asymptotically a non-central chi square distribution with pq_2 degrees of freedom and non-centrality parameter

$$\Delta_{\mathscr{L}} = \mathrm{Trace}(\bar{\boldsymbol{\Gamma}}(\mathbf{T}(F) \otimes \mathbf{C}^*)^{-1}), \tag{4.6}$$

where $\mathbf{T}(F)$ is defined by (3.11).

From (4.2) and (4.6), we conclude that the Pitman asymptotic relative efficiency (ARE) of $\mathscr{L}_n$ with respect to L_n is given by

$$e_{\mathscr{L}, L} = \Delta_{\mathscr{L}} / \Delta_L = \mathrm{tr}(\bar{\boldsymbol{\Gamma}}(\mathbf{T}(F) \otimes \mathbf{C}^*)^{-1}) / \mathrm{tr}(\bar{\boldsymbol{\Gamma}}(\boldsymbol{\Sigma}(F) \otimes \mathbf{C}^*)^{-1}) \tag{4.7}$$

which depends on $\bar{\boldsymbol{\Gamma}}$, F and $\mathbf{C}^*$. If F is multinormal and for $\mathscr{L}_n$ we use the normal scores (i.e., all the ϕ_j being the inverse of a standard normal c.d.f.), then, it can easily be checked that $\mathbf{T}(F) = \boldsymbol{\Sigma}(F)$, and hence, (4.7) reduces to 1, i.e., the aligned rank order normal scores test and the likelihood-ratio test are asymptotically power-equivalent for normal F and local alternatives in (4.1). However, in general, for arbitrary F, $e_{\mathscr{L}, L}$ is bounded as follows:

$$\mathrm{ch}_p(\boldsymbol{\Sigma}(F)\mathbf{T}^{-1}(F)) \leqq e_{\mathscr{L}, L} \leqq \mathrm{ch}_1(\boldsymbol{\Sigma}(F)\mathbf{T}^{-1}(F)), \tag{4.8}$$

where $\mathrm{ch}_j(A)$ stands for the j-th (largest) root of A for $j \geqq 1$. The bounds in (4.8) may be studied as in Sen and Puri (1967) or Puri and Sen (1969), and hence, the details are omitted. For testing simple hypotheses in multivariate linear models, Puri and Sen (1969) have studied (in their Theorem 6.2) the optimality of rank order tests for local alternatives. In passing, we may remark that under the same set of regularity conditions as in Theorem 6.2 of Puri and Sen (1969), $\mathscr{L}_n$ has asymptotically the best average power with respect to suitable surfaces in the parameter space (of $\boldsymbol{\gamma}_2$), it has also asymptotically the best constant power on such surfaces and, finally, it is an asymptotically most stringent test.

5. ADF Tests for Parallelism of Several Regression Surfaces

As a multivariate generalization of the univariate problem treated in Sen (1969), we consider here the following. Let $\mathbf{X}_i^{(k)}$, $k=1,\dots,n_k$ be n_k independent rv's with continuous c.d.f.'s

$$F_i^{(k)}(\mathbf{x})=P\{\mathbf{X}_i^{(k)}\leqq \mathbf{x}\}=F(\mathbf{x}-\boldsymbol{\alpha}_k-\boldsymbol{\beta}_k\mathbf{c}_i^{(k)}), \tag{5.1}$$

for $i=1,\dots,n_k$, $k=1,\dots,s(\geqq 2)$. We desire to test the null hypothesis

$$H_0\colon\ \boldsymbol{\beta}_1=\cdots=\boldsymbol{\beta}_s=\boldsymbol{\beta}(\text{unknown}) \tag{5.2}$$

where $\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_s$ and $\boldsymbol{\beta}$ are treated as nuisance parameters. If we let $\boldsymbol{\beta}_k=\boldsymbol{\beta}_1+\boldsymbol{\beta}_k^*$, $k=1,\dots,s$ (so that $\boldsymbol{\beta}_1^*=\mathbf{0}$), $q=st$ (where each of the $\mathbf{c}_i^{(k)}$ in (5.1) is a t-vector), then, we are in a position to use the theory developed in Sections 3 and 4. Therefore, without giving the details of derivations, we present the main results in this case as follows.

Based on the n_k observations in the k-th sample, we define $\mathbf{S}_{n_k}^{(k)}$ as in (2.2) and $\mathbf{S}_{n_k}^{(k)}(\mathbf{B})$ as in (2.14)–(2.16). Let then $n=n_1+\cdots+n_s$ and

$$\bar{\mathbf{S}}_n(\mathbf{B})=\sum_{k=1}^{s}\mathbf{S}_{n_k}^{(k)}(\mathbf{B}) \qquad \text{for } \mathbf{B}\in R^{pt}. \tag{5.3}$$

Under H_0, we estimate the common $\boldsymbol{\beta}$ as follows: as in (3.7)–(3.8), we let

$$\mathbf{D}_n=\left\{\mathbf{B}\colon \sum_{j=1}^{p}\sum_{m=1}^{t}|\bar{S}_{n,jm}(\mathbf{b}_j)|=\text{minimum}\right\}; \tag{5.4}$$

$$\hat{\boldsymbol{\beta}}_n=\text{centre of gravity of } \mathbf{D}_n. \tag{5.5}$$

Let then

$$\hat{\mathbf{S}}_{n_k}^{(k)}=\mathbf{S}_{n_k}^{(k)}(\hat{\boldsymbol{\beta}}_n), \qquad k=1,\dots,s; \tag{5.6}$$

$$\mathbf{H}_{n_k}^{(k)}=((\hat{S}_{n_k,jr}^{(k)}\hat{S}_{n_k,j'r'}^{(k)}))_{j,j'=1,\dots,p;\,r,r'=1,\dots,t}; \tag{5.7}$$

$$\mathbf{C}_{n_k}^{(k)}=\sum_{i=1}^{n}(\mathbf{c}_i^{(k)}-\bar{\mathbf{c}}_{n_k})(\mathbf{c}_i^{(k)}-\bar{\mathbf{c}}_{n_k})' \tag{5.8}$$

$$\hat{\mathbf{M}}_n=(n-s)^{-1}\left(\left(\sum_{k=1}^{s}\sum_{i=1}^{n_k}\{a_{n_k}^{(j)}(\hat{R}_{ji}^{(k)})-\bar{a}_{n_k}^{(j)}\}\{a_{n_k}^{(j')}(\hat{R}_{j'i}^{(k)})-\bar{a}_{n_k}^{(j')}\}\right)\right); \tag{5.9}$$

$$\mathbf{G}_{n_k}=\hat{\mathbf{M}}_n\otimes\mathbf{C}_{n_k}^{(k)} \qquad \text{for } k=1,\dots,s, \tag{5.10}$$

where the $\bar{\mathbf{c}}_{n_k}$ and $\bar{a}_{n_k}^{(j)}$ are the averages of the regression vectors and the scores, and are defined as in Section 2; $\hat{R}_{ji}^{(k)}$ is the rank of $X_{ji}^{(k)}-\sum_{r=1}^{t}\hat{\beta}_{n,jr}c_{ir}^{(k)}$ among the n_k aligned observations on the j-th variate in the k-th sample, for $i=1,\dots,n_k$, $j=1,\dots,p$; $k=1,\dots,s$. Then, the aligned rank order test-statistic is

$$\mathscr{L}_n=\sum_{k=1}^{s}\operatorname{Trace}(\mathbf{H}_{n_k}^{(k)}\mathbf{G}_{n_k}^{-1}). \tag{5.11}$$

Under H_0 in (5.2), $\mathscr{L}_n$ has asymptotically a chi-square distribution with $p(s-1)t$ degrees of freedom, while under the sequence of alternatives $\{K_n\}$ where

$$K_n\colon\ \boldsymbol{\beta}_k=\boldsymbol{\beta}+n^{-\frac{1}{2}}\gamma_k,\qquad k=1,\ldots,s;\ \sum_{k=1}^{s}\mathbf{C}_{n_k}^{(k)}\gamma_k=\mathbf{0},\tag{5.12}$$

it has the corresponding non-central distribution with the non-centrality parameter

$$\Delta_{\mathscr{L}}=\sum_{k=1}^{s}\operatorname{Trace}(\bar{\Gamma}_k(\mathbf{T}(F)\otimes\mathbf{C}_k)^{-1}),\tag{5.13}$$

where

$$\bar{\Gamma}_k=((\gamma_{jr}^{(k)}\gamma_{j'r'}^{(k)}))\quad\text{and}\quad\mathbf{C}_k=\lim_{n\to\infty}n^{-1}\mathbf{C}_{n_k}^{(k)},\qquad k=1,\ldots,s,\tag{5.14}$$

which we assume to exist. Asymptotic optimality results hold under the same setup as in the later part of Section 4.

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