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Exit Kernels and Quasi-left-continuity of Standard Markov Processes

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Let X be a standard Markov process (see [2]) with the property that almost all paths have left-hand-limits everywhere on $(0,\infty)$. It is well known (see [2] p. 50, [5] p. 208) that smoothness conditions (such as 'Feller', 'quasi Feller') on the semigroup (P_t) of X imply that X is quasi-left-continuous on $[0,\infty)$, hence a Hunt process (in this context see also [3]).

Starting with a $\mathfrak{P}$-harmonic space $(E,\mathscr{H}^*)$ with $1\in\mathscr{H}^*$ one can always choose a process X_0 whose semigroup is quasi Feller (see [1]). Every standard process Y associated with $(E,\mathscr{H}^*)$ (i.e. $_+\mathscr{H}^*$ coincides with the set of all excessive functions of Y) can be obtained from X_0 by means of a time change. So the question arises if every such process is already a Hunt process.

In general smoothness properties of the semigroup as well as quasi-left-continuity on $[0,\infty)$ are not invariant under (strict continuous) time changes.

In order to give 'invariant' sufficient conditions for X to be a Hunt process the above smoothness conditions on (P_t) are replaced by smoothness conditions on the exit kernels of X.

This yields that every standard Markov process associated with a 'nondegenerated' standard balayage space (see [1]) is a Hunt process. In particular this is true for every $\mathfrak{P}$-harmonic space.

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In the following let $(\mathscr{B}_+, \mathscr{B}_b)$ $\mathscr{B}$ denote the class of all (positive, bounded) Borel measurable functions on E, and $\mathfrak{U}$ the class of all nonempty relatively compact open subsets of E.

For standard Markov processes X with state space E the notations follow essentially those of Blumenthal-Getoor [2] (e.g.: ζ, (θ_t) and $\mathscr{F}$ denote life time and shift operators of X and the σ-field generated by all X_t, $t\geqq 0$, respectively). But Π_A will denote the exit kernel corresponding to the first exit time $D_{\complement A}$ from a set A (i.e. $D_{\complement A}=\inf\{t\geqq 0; X_t\notin A\}$ and $\Pi_A=E^x[f\circ X_{D_{\complement A}}]$, if defined).

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Lemma. *Let X be a standard Markov process having left-hand-limits on $(0,\infty)$ a.s., let (T_n) be an increasing sequence of stopping times with bounded supremum T and define $L:=\lim X_{T_n}$. Then the following holds:*

a) *For $x\in E$, for all $f\in\mathscr{B}_+$ which are $L(P^x)$-a.s. continuous, and for all $0\leqq Y\in\mathscr{F}$, we have*

$$E^x[\liminf E^{X_{T_n}}[Y]\,f(L)]\leqq E^x[\limsup Y\circ\theta_{T_n} f(L)].$$

b) *If G is an open subset of E, then for all but finitely many $n\in\mathbb{N}$*

$$D_{\complement G}\circ\theta_{T_n}=T-T_n+D_{\complement G}\circ\theta_T \qquad \text{a.s. on } [L\in G].$$

Proof. a) If f is continuous, $L(P^x)$-a.s., then P^x-a.s.

$$\liminf E^{X_{T_n}}[Y]\,f(L)=\liminf(E^{X_{T_n}}[Y]\,f(X_{T_n}))$$

and the same with lim sup. Hence the result follows from the strong Markov property and Fatou's lemma.

b) It suffices to consider $\omega\in[T_n<T$ for all $n]$. But then $L(\omega)=X_{T-}(\omega)$, and $X_{T-}(\omega)\in G$ implies the existence of an integer k such that

$$X([T_n(\omega),T(\omega)[,\omega)=X([0,T(\omega)-T_n(\omega)[,\theta_{T_n}(\omega))\subset G$$

for all $n\geqq k$. Consequently

$$\theta_{T_n}(\omega)\in[T(\omega)-T_n(\omega)\leqq D_{\complement G}]$$

hence

$$\begin{aligned}D_{\complement G}(\theta_{T_n}(\omega))&=T(\omega)-T_n(\omega)+D_{\complement G}(\theta_{T(\omega)-T_n(\omega)}(\theta_{T_n}(\omega)))\\&=T(\omega)-T_n(\omega)+D_{\complement G}(\theta_T(\omega)).\end{aligned}$$

(Observe that the quasi-left-continuity on $[0,\zeta)$ is not needed in the above proof.) □

Proposition. *Let X be a standard Markov process. Then X is a Hunt process if the following holds:*

i) *Almost all paths have left-hand-limits on $(0,\infty)$.*

ii) *For every $x\in E$ there is a neighbourhood U of x such that $\Pi_U 1(x)>0$.*

iii) *$\Pi_U 1$ is l.s.c. on U for all $U\in\mathfrak{U}$.*

Proof. Suppose X to be not quasi-left-continuous on $[0,\infty)$. According to ([2] p. 50) we may (and do) assume that the stopping times involved are bounded[1].

Then there exists a point $y\in E$ and an increasing sequence (T_n) of stopping times with bounded limit T such that

$$P^y[L\neq X_T]>0 \tag{1}$$

where $L=\lim X_{T_n}$.

Since the paths have left-hand-limits on $(0,\infty)$, L does exist. Since X is quasi-left-continuous on $[0,\zeta)$ (1) yields

$$P^y[L\in E, T\geqq\zeta]>0. \tag{2}$$

[1] The author is obliged to a referee for this remark

Furthermore, from ii) and iii) we know that, for all $x \in E$, there is an open neighbourhood V of x such that

$$\inf \Pi_V 1 = \inf_V \Pi_V 1 > 0. \tag{3}$$

Moreover, V may be chosen to be a $L(P^y)$-continuity set (i.e. $P^y[L \in \partial V] = 0$). Hence part a) of the lemma applies to $f = 1_V$.

Define $D := D_{\complement V}$, then

$$\begin{aligned} E^y[\Pi_V 1 \circ L; L \in V] &\leqq E^y[\liminf \Pi_V 1 \circ X_{T_n}; L \in V], \quad \text{by iii)} \\ &= E^y[\liminf P^{X_{T_n}}[D < \zeta]; L \in V] \\ &\leqq P^y[\limsup [D \circ \theta_{T_n} < \zeta \circ \theta_{T_n}], L \in V], \quad \text{by part a)} \end{aligned}$$

of the lemma. By part b), applied to the open sets V and E simultaneously, this equals

$$\begin{aligned} &P^y[D \circ \theta_T < \zeta \circ \theta_T, L \in V] \\ &\quad = P^y[D \circ \theta_T < \zeta \circ \theta_T, L \in V, T < \zeta] \\ &\quad = P^y[D \circ \theta_T < \zeta \circ \theta_T, X_T = L \in V, T < \zeta] \\ &\quad = E^y[\Pi_V 1 \circ L, X_T = L \in V, T < \zeta] \\ &\quad \leqq E^y[\Pi_V 1 \circ L, L \in V] - (\inf \Pi_V 1) P^y[L \in V, T \geqq \zeta]. \end{aligned}$$

Consequently $P^y[L \in V, T \geqq \zeta] = 0$ for all $L(P^y)$-continuity sets V which satisfy (3). But E can be covered by a sequence of such sets, which contradicts (2). □

The following remarks discuss the conditions ii), iii) and the quasi-left-continuity on $[0, \zeta)$ assumed in the proposition, and give an application to harmonic spaces.

Remarks. 1) Since $\sup\{\Pi_U 1(x); U \downarrow \{x\}\} = P^x[D_{\complement\{x\}} < \zeta]$, ii) holds if and only if X has no absorbing points.

2) The conditions ii) and iii) are obviously not necessary for X to be a Hunt process. But they cannot simply be dropped:

2a) Let X be uniform motion to the right on $\mathbb{R}_+$ but terminating with probability 1/2 just before reaching the point 1. Then X satisfies all but condition iii). Moreover $\Pi_U 1$ is bounded away from zero for all $U \in \mathfrak{U}$.

2b) Let $E = [0, 1] \subset \mathbb{R}$ be the state space of the process X which is uniform motion on $[0, 1)$ terminating just before reaching the point 1 and which terminates with some intensity $\lambda > 0$ if starting at 1. Then X satisfies all but condition ii).

3) The conditions i)–iii) alone do not imply the quasi-left-continuity on $[0, \zeta)$-even if left-limits are in E:

3a) Let $E = [0, 1] \cup [2, \infty) \subset \mathbb{R}$ be the state space of the following process X: X begins uniform motion to the right on $E \setminus \{1\}$ jumping to the point 2 just before reaching 1, and starting at 1 X jumps to 2 with some strictly positive intensity. Then $\Pi_U 1 = 1$ for all $U \in \mathfrak{U}$ and the paths have left-hand-limits (in E) on $(0, \infty)$. But X is not a standard process.

3b) The 'same' process with state space $[0,1)\cup[2,\infty)$ instead of the above E gives an analoguous example without holding points. (In this case the left-hand-limits exist as well but are not necessarily in E).

4) As we have seen, it is not possible to drop the quasi-left-continuity on $[0,\zeta)$ from the assumptions. In the case that it is replaced by

(*) $[T<\zeta]\subset[X_{T-}\in E]$ a.s. for all stopping times T

one gets the same result, if the conditions ii) and iii) are replaced by

ii)′ All points are instantaneous for X.

iii)′ $\Pi_V 1_U$ is l.s.c. on V for all $\bar{V}\subset U$ (V, $U\in\mathfrak{U}$).

This can be proved with the same methods as before but I shall not do so since I don't know any reasonable condition for (*) different from quasi-left-continuity on $[0,\zeta)$.

5) The proposition applies easily to nondegenerated standard balayage spaces $(E,\mathcal{S})$ in the sense of [1]:

Every standard Markov process associated with $(E,\mathcal{S})$ in the usual sense (i.e. the set of excessive functions coincides with $\mathcal{S}$) is a Hunt process.

Proof. It suffices to ensure condition i) of the proposition. But it is known from [1] that the following approximation theorem holds:

$$\{s_1-s_2;\ s_i\in\mathcal{S}\cap\mathscr{C}_b,\ s_1-s_2\in\mathscr{C}_c\}\quad\text{is dense in } \mathscr{C}_c$$

w.r.t. the sup-norm ($\mathscr{C}_b$ and $\mathscr{C}_c$ denote the class of continuous functions which are bounded and which have compact support, respectively).

This, in turn, yields a sufficient condition of i) in the proposition:

There exists a sequence $(d_n)\subset\mathscr{C}_c$ such that (d_n) separates the points of E and, for all $n\in\mathbb{N}$, $t\to d_n\circ X_t$ has left-hand-limits a.s. on $(0,\infty)$. □

5a) In particular every standard Markov process associated with a $\mathfrak{P}$-harmonic space (see [4]) is a Hunt process.

5b) It should be possible[2] to show the above conclusion for general standard balayage spaces, i.e. to eliminate the restrictive nondegeneracy, but it seems that this requires a different method.

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[2] It is: Look for the forthcoming "Markov Processes on Standard Balayage Spaces" by W. Hansen