Z. Wahrscheinlichkeitstheorie verw. Gebiete
54, 15–20 (1980)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

The Average Measure of the Intersection of Two Sets

H. Groemer*

Department of Mathematics, The University of Arizona, Tucson, Arizona 85721, USA

1. Introduction

Many problems of geometric probability theory and integral geometry concern integrals of the form

$$\int_G \phi(gA \cap B)\, d\mu(g),$$

where A and B are subsets of a space that has a transformation group G with invariant measure μ, and where ϕ is a real valued function defined for all (or almost all) sets $gA \cap B$ ($g \in G$). For example, if A and B are compact convex subsets of euclidean n-dimensional space R^n, and if G is the group of rigid motions of R^n equipped with Haar measure μ then the mean projection measures w_k satisfy the relations

$$\int_G w_k(gA \cap B)\, d\mu(g) = \sum_{n-k}^{n} \alpha_{ik}\, w_{k+i-n}(A)\, w_{n-i}(B), \tag{1}$$

where $k = 0, 1, \ldots n$ and the coefficients α_{ik} are known constants (depending on the normalization of μ). If v denotes the volume and s the surface area for convex sets in R^n then the case $k=0$ of (1) can be restated as

$$\int_G v(gA \cap B)\, d\mu(g) = v(A)\, v(B), \tag{2}$$

and the case $k=1$ as

$$\int_G s(gA \cap B)\, d\mu(g) = v(a)\, s(b) + s(a)\, v(B). \tag{3}$$

Detailed presentations of these matters can be found in the books of Hadwiger [5] and Santaló [9]. It has been pointed out repeatedly that (2) holds also if G is taken to be the group T of translations of R^n. In this case the sets A and B can

* Supported by National Science Foundation Research Grant MCS 76-06111

0044-3719/80/0054/0015/$01.20

be identified with sets in T and (2) can be viewed as a special case of the "average theorem for measurable groups" (see Halmos [6, p. 261]). This formula can be stated as

$$\int_H \gamma(hM \cap N)\, d\gamma(h) = \gamma(M^{-1})\,\gamma N \tag{4}$$

where γ is an invariant measure and M, N are suitable subsets of a group H. Hence (2) is essentially a measure theoretic relation that does not depend on special geometric properties of the sets A, B (see also Balanzat [1]). It can be shown that (3) holds also if the group of rigid motions is replaced by the group of translations of R^n, but all previously given proofs of this and similar results depend on the convexity or other geometric regularity assumptions on the sets A, B (see Schneider [10], Streit [12], Groemer [4]).

It is the aim of the present paper to prove in a completely measure theoretic setting an integral relation that contains as special cases (2), (3), (4) and various other formulas of this kind. These special formulas are immediate consequences of an integral relation that can be formulated in terms of the Hausdorff measure of the pertinent sets. Some of these results generalize integral geometric relations that have been proved by Federer [2], [3, p. 248] under additional regularity assumptions (rectifiability). However, it should be pointed out that Federer's formulas are more general as far as the dimensions of the admissible sets are concerned. There appears to be yet another possibility to obtain our main results, namely by the use of a suitably generalized version of a theorem of Stein [11] concerning "reciprocal functions".

The following section contains our main result and some immediate consequences. Applications to integral geometric relations for Hausdorff measures in R^n and spherical spaces are presented in Section 3.

2. Main Theorems

It will always be assumed that we are given a group G with a left invariant measure μ, and that G acts transitively on some set S (the underlying "space"). The image of any $s \in S$ under the application of an $g \in G$ will be denoted by gs. If $X \subset S$ we say that X *has induced measure* $\hat{\mu}X$ if for some $q \in S$ the set $\{g\colon g^{-1}q \in X,\ g \in G\}$ is μ-measurable and $\hat{\mu}X = \mu\{g\colon g^{-1}q \in X,\ g \in G\}$. Due to the transitivity of G and the invariance of μ this definition does not depend on the special choice of the element q. Instead of "left invariant" we shall say simply "invariant". When we are given two measure spaces and we refer to the *product measure* we mean the not necessarily complete product measure that is defined on the smallest σ-algebra generated by the measurable rectangles (cf. Halmos [5, Sect. 35]).

We can now formulate our principal result.

Theorem 1. *Let S be a set, A and B subsets of S, and G a group that acts transitively on S. Let μ be a σ-finite invariant measure on G, and v a σ-finite measure on B. Moreover, let us assume that the set $\{(g, b)\colon g^{-1} b \in A,\ g \in G,\ b \in B\}$ is measurable with respect to the product measure $\mu \times v$. Then,*

(i) *A has induced measure* $\hat{\mu}A$,
(ii) $gA \cap B$ *is* ν*-measurable for every* $g \in G$,
(iii) $\nu(gA \cap B)$ *is a* μ*-measurable function of* g,
(iv)
$$\int_G \nu(gA \cap B)\, d\mu(g) = \hat{\mu}A\, \nu B.$$

Proof. For any $T \subset S$ let χ_T denote the characteristic function (indicator function) of T. Because of $\{(g, b)\colon g^{-1} b \in A,\ g \in G,\ b \in B\} = \{(g, b)\colon \chi_{gA \cap B}(b) = 1\}$ the function that maps (g, b) onto $\chi_{gA \cap B}(b)$ is measurable with respect to $\mu \times \nu$. Since both μ and ν are assumed to be σ-finite it follows from one version of the Theorem of Fubini (cf. Halmos [6, p. 147]) that

$$\int_B \int_G \chi_{gA \cap B}(b)\, d\mu(g)\, d\nu(b) = \int_G \int_B \chi_{gA \cap B}(b)\, d\nu(b)\, d\mu(g) \tag{5}$$

where the inner integrals exist for every b and every g, respectively, and are measurable functions of these variables. For the inner integral on the right hand side of (5) we obtain for every g

$$\int_B \chi_{gA \cap B}(b)\, d\nu(b) = \nu(gA \cap B).$$

As a consequence of this equality and the previous statements about existence and measurability we obtain (ii) and (iii). Moreover, it follows that (5) can be written in the form

$$\int_B \int_G \chi_{gA \cap B}(b)\, d\mu(g)\, d\nu(b) = \int_G \nu(gA \cap B)\, d\mu(g). \tag{6}$$

If $b \in B$ we find

$$\begin{aligned}\mu\{g\colon \chi_{gA \cap B}(b) = 1,\ g \in G\} &= \mu\{g\colon b \in gA,\ b \in B,\ g \in G\}\\ &= \mu\{g\colon g^{-1} b \in A,\ g \in G\} = \hat{\mu}A\end{aligned}$$

and therefore

$$\int_G \chi_{gA \cap B}(b)\, d\mu(g) = \hat{\mu}A. \tag{7}$$

This shows in particular that (i) is correct. If (6) is combined with (7) we obtain immediately (iv). Thus all parts of the theorem have been proved.

Before we turn to another version of Theorem 1 we consider first a rather special case. Let us take $S = G$, $\mu = \nu$. Then, if the assumptions of Theorem 1 are satisfied, A has induced measure $\hat{\mu}A = \mu\{g\colon g^{-1} x \in A\} = \mu\{g\colon g \in A^{-1}\} = \mu(A^{-1})$. The following corollary is now an immediate consequence of Theorem 1. It is one version of the average theorem for measurable groups (cf. Halmos [6, p. 261]).

Corollary. *Let* $(G, \mathscr{G}, \mu)$ *be a measure space where* G *is a group and* μ *an invariant* σ*-finite measure. Assume that* $A \subset G$, $B \in \mathscr{G}$ *and that* $\{(g, b)\colon g^{-1} b \in A,\ b \in B\}$ *is* $\mu \times \mu$*-measurable. Then the sets* A^{-1} *and* $gA \cap B$ *are measurable,* $\mu(gA \cap B)$ *is a measurable function of* g, *and*

$$\int_G \mu(gA \cap B)\, d\mu(g) = \mu(A^{-1})\, \mu B.$$

For most applications of Theorem 1 it is rather inconvenient to check whether the set $\{(g, b): g^{-1} b \in A, g \in G, b \in B\}$ is $\mu \times \nu$-measurable. This condition can be removed if suitable topological assumptions are made. We formulate one such possibility as a theorem. The class of all Borel sets of a topological space X (i.e., the smallest σ-algebra containing the open subsets of X) will be denoted by $\mathscr{B}(X)$. More generally, if $Y \subset X$ we denote by $\mathscr{B}(Y)$ the Borel sets of Y corresponding to the relative topology of Y induced by X.

Theorem 2. *Let S be a topological space and G a topological group that acts transitively and continuously on S, and suppose that both S and G satisfy the second countability axiom. Moreover, let A be a Borel subset and B a subset of S, and let μ be an invariant σ-finite measure on $\mathscr{B}(G)$ and ν a σ-finite measure on $\mathscr{B}(B)$. Then*

(i) *A has induced μ-measure $\hat{\mu}A$,*
(ii) *$gA \cap B \in \mathscr{B}(B)$ for every $g \in G$,*
(iii) *$\nu(Ag \cap B)$ is a μ-measurable function of g,*
(iv) $$\int_G \nu(gA \cap B)\, d\mu(g) = \hat{\mu}A\, \nu B.$$

Proof. Theorem 2 follows obviously from Theorem 1 if we can show that the set $C = \{(g, b): g^{-1} b \in A, g \in G, b \in B\}$ is $\mu \times \nu$-measurable. To prove this we consider first the function $\phi(g, b) = g^{-1} b$ that maps the space $G \times B$ into S. (Here B is assumed to carry the relative topology induced by S; the space $G \times B$ is supposed to have the product topology induced by G and B.) From the assumption that G act continuously on S and that G itself be a topological group it follows that ϕ is a continuous function. This implies that ϕ^{-1} maps Borel sets onto Borel sets. Because of $A \in \mathscr{B}(S)$ it follows therefore that $\phi^{-1}(A) \in \mathscr{B}(G \times B)$. However, $\phi^{-1}(A) = C$ and consequently

$$C \in \mathscr{B}(G \times B).$$

Hence, to complete the proof we have only to show that the Borel subsets of $G \times B$ are $\mu \times \nu$-measurable. For this purpose we note that it is a simple consequence of the second countability axiom for G and S that every open subset of $G \times B$ can be written as a countable union of sets of the form $X \times Y$ where X is an open subset of G, and Y a (relatively) open subset of B. Since the sets $X \times Y$ are $\mu \times \nu$-measurable we find that the open subsets of $G \times B$ are $\mu \times \nu$-measurable. This implies obviously that the Borel sets are also measurable.

3. Euclidean and Spherical Spaces

For $0 \leqq p \leqq n$ we denote by λ_p the p-dimensional Hausdorff measure on R^n or on the n-dimensional unit sphere S^n. It is a well-known fact that λ_p serves as a useful generalization of volume or area of a p-dimensional manifold. In order to avoid the introduction of undesirable constants into our formulas we use suitable normalizations of the pertinent Haar measures. If G is the group of translations or the group of rigid motions of R^n we normalize the Haar measure of G so that for a unit cube Q and a point $q \in R^n$ the measure of the set $\{g: gq \in Q, g \in G\}$ is 1. If G is the group of all rotations of S^n (i.e., proper isometries of S^n) we normalize the Haar measure on G so that for a spherical cap C on S^n with $\lambda_n C = 1$ and a point $q \in S^n$ the set $\{g: gq \in C, g \in G\}$ has measure 1. In all three cases we refer to this measure on G as the *normalized Haar measure.*

The following theorem enables one to evaluate the average value of $\lambda_p(gA \cap B)$ for $0 \leqq p \leqq n$. This theorem is, on the one hand, more general than previously known relations of this kind since A and B are only assumed to be Borel sets. On the other hand, it is more special since we impose the restriction that B have σ-finite λ_p-measure (cf. Santaló [8], [9, p. 258] and Federer [2], [3, p. 248]). For the case of sets in R^n and $p=0$ or $p=n$ see also Balanzat [1].

Theorem 3. *Let A and B be two Borel subsets of R^n or of S^n, and assume that B has σ-finite p-dimensional Hausdorff measure $(0 \leqq p \leqq n)$. If A and B are in R^n let G be the group of translations or the group of rigid motions of R^n, and if A and B are in S^n let G be the group of rotations of S^n. Let μ denote the normalized Haar measure on G. Then, $\lambda_p(gA \cap B)$ is a measurable function of $g \in G$ and*

$$\int_G \lambda_p(gA \cap B)\, d\mu(g) = \lambda_n A\, \lambda_p B. \tag{8}$$

Proof. If $B \subset R^n$ the class $\{X \cap B: X \in \mathscr{B}(R^n)\}$ is a σ-algebra of subsets of B that contains the relatively open subsets of B. This shows that every set from $\mathscr{B}(B)$ is of the form $X \cap B$ with $X \in \mathscr{B}(R^n)$. Hence, $\mathscr{B}(B) \subset \mathscr{B}(R^n)$ and similarly $\mathscr{B}(B) \subset \mathscr{B}(S^n)$ if $B \subset S^n$. Consequently λ_p can be viewed as a measure on $\mathscr{B}(B)$ and it is now clear that all the assumptions of Theorem 2 are satisfied. It remains only to find $\hat{\mu}A$. This can be done by considering the cases of R^n and S^n separately and explicit evaluation of $\hat{\mu}A$. However, one can also treat all cases at the same time by noting that $\hat{\mu}$ is an invariant measure on the homogeneous spaces R^n or S^n with respect to G. Since λ_n is also such a measure the desired result $\hat{\mu}A = \lambda_n A$ follows immediately from known uniqueness theorems for such spaces and the normalization assumption (see Nachbin [6, p. 138]).

We note that it is often more convenient to work with integrals of the form $2\int_G \lambda_p(A \cap gB)\, d\mu(g)$ rather than $\int_G \lambda_p(gA \cap B)\, d\mu(g)$. These two integrals are equal since $\lambda_p(gA \cap B) = \lambda_p(A \cap g^{-1}B)$ and for every measurable subset H of any of the three groups G under consideration we have $\mu H = \mu(H^{-1})$.

As a final application of Theorem 2 we formulate and prove a rather general version of the surface area formula (3). The closure of a set X in a topological space will be denoted by $\bar{X}$, its boundary by ∂X.

Theorem 4. *Let A and B be two subsets of R^n or of S^n, and assume that for some p with $0 \leqq p < n$ the boundaries ∂A and ∂B have σ-finite p-dimensional Hausdorff measure. If A and B are in R^n let G be the group of translations or the group of rigid motions of R^n; if A and B are in S^n let G be the group of rotations of S^n. Let μ denote the normalized Haar measure on G. Then, $\lambda_p \partial(gA \cap B)$ is a measurable function of $g \in G$ and*

$$\int_G \lambda_p(\partial(gA \cap B))\, d\mu(g) = \lambda_n \bar{A}\, \lambda_p \partial B + \lambda_n \bar{B}\, \lambda_p \partial A.$$

Proof. Since ∂A and $\bar{B}$ are closed sets it follows from Theorem 3 that

$$\int_G \lambda_p(g\bar{A}\cap\partial B)\,d\mu(g)=\lambda_n\bar{A}\,\lambda_p\partial B. \tag{9}$$

Similarly we obtain

$$\int_G \lambda_p(g\partial A\cap\bar{B})\,d\mu(g)=\lambda_n\bar{B}\,\lambda_p\partial A \tag{10}$$

and, using the fact that $\lambda_n\partial A=0$ (since $p<n$), we have also

$$\int_G \lambda_p(g\partial A\cap\partial B)\,d\mu(g)=\lambda_n\partial A\,\lambda_p\partial B=0. \tag{11}$$

Now, from the definitions of closure and boundary it follows easily that

$$\partial(gA\cap B)=(g\bar{A}\cap\partial B)\cup((\partial gA\cap\bar{B})\setminus(\partial gA\cap\partial B)).$$

Since the right hand side of this identity consists of the union of two disjoint sets it follows that

$$\lambda_p\partial(gA\cap B)=\lambda_p(g\bar{A}\cap\partial B)+\lambda_p((\partial gA\cap\bar{B})\setminus\partial gA\cap\partial B))$$

and therefore

$$\lambda_p\partial(gA\cap B)+\lambda_p(\partial gA\cap\partial B)=\lambda_p(g\bar{A}\cap\partial B)+\lambda_p(\partial gA\cap\bar{B}).$$

From this relation, together with (9), (10), (11), we obtain immediately the desired result

$$\int \lambda_p\partial(gA\cap B)\,d\mu(g)=\lambda_m\bar{A}\,\lambda_p\partial B+\lambda_n\bar{B}\,\lambda_p\partial A.$$

References

1. Balanzat, M.: Sur quelques formules de la géométrie intégral des ensembles dans un espace à n dimensions. Portugal. Math. **3**, 87–94 (1942)
2. Federer, H.: Some integral geometric theorems. Trans. Amer. Math. Soc. **77**, 238–261 (1954)
3. Federer, H.: Geometric Measure Theory. New York: Springer 1969
4. Groemer, H.: On translative integral geometry. Arch. Math. **39**, 324–330 (1977)
5. Hadwiger, H.: Vorlesungen über Inhalt, Oberfläche und Isoperimetrie. Berlin-Göttingen-Heidelberg: Springer 1957
6. Halmos, P.R.: Measure Theory. Princeton-New York-Toronto-London: D. Van Nostrand 1950
7. Nachbin, L.: The Haar Integral. Princeton-New York-Toronto-London: D. Van Nostrand 1965
8. Santaló, L.A.: Geometría integral en espacios de curvatura constante. Publ. Com. Nac. Energía Atómica, Ser. Mat. **1**, fasc. 1 (1952)
9. Santaló, L.A.: Integral Geometry and Geometric Probability. Reading, Ma: Addison-Wesley 1976
10. Schneider, R.: Eine Verallgemeinerung des Differenzenkörpers. Monatsh. Math. **74**, 258–272 (1970)
11. Stein, S.: A measure-theoretic relation between a function and its reciprocal. Amer. Math. Monthly **58**, 691–693 (1951)
12. Streit, F.: Results on the intersection of randomly located sets. J. Appl. Probab. **12**, 817–823 (1975)

Received October 18, 1979; in revised form April 21, 1980