Z. Wahrscheinlichkeitstheorie verw. Gebiete
47,213–229 (1979)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Stochastic Differential Equations and Nilpotent Lie Algebras

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1. Introduction

Given C^∞-vector fields $A_i = \sum_{j=1}^{d} A_i^j(x) \frac{\partial}{\partial x^j}$, $0 \leqq i \leqq n$, on $\mathbb{R}^d$ and an n-dimensional Brownian motion $(B_t^1, \ldots, B_t^n)$, we consider the stochastic differential equation

$$dX_t^i = \sum_{j=1}^{n} A_j^i(X_t) \circ dB_t^j + A_0^i(X_t)\, dt, \quad 1 \leqq i \leqq d, \tag{1.1-1}$$

$$X_0 = x \in \mathbb{R}^d, \tag{1.1-2}$$

where the symbol $\circ$ denotes the symmetric stochastic differential of Stratonovich (Itô, K. [4]). We denote by $\mathscr{L}(A_1, \ldots, A_n)$ the Lie subalgebra of $\mathfrak{X}(\mathbb{R}^d)$ generated by $A_1, \ldots, A_n$, where $\mathfrak{X}(\mathbb{R}^d)$ is the Lie algebra of all C^∞-vector fields on $\mathbb{R}^d$ with the bracket product:

$$[X, Y] = XY - YX, \quad X, Y \in \mathfrak{X}(\mathbb{R}^d).$$

We also denote by $C([0, \infty) \to \mathbb{R}^n)$ the set of all continuous functions U_t, $t \in [0, \infty)$, with values in $\mathbb{R}^n$.

Recently, Doss, H. [1, 2] showed that, if the total differential equation

$$\frac{\partial}{\partial \beta^i} h^j(\alpha, \beta) = A_i^j(h(\alpha, \beta)), \quad 1 \leqq i \leqq n, \quad 1 \leqq j \leqq d, \tag{1.2-1}$$

$$h(\alpha, 0) = \alpha \tag{1.2-2}$$

($\alpha \in \mathbb{R}^d$, $\beta \in \mathbb{R}^n$, $h = (h^1, \ldots, h^d)$) is completely integrable, then the solution of (1.1) can be expressed in the form

$$X_t = h(\Phi(x, B.)_t, B_t), \tag{1.3}$$

where the functional

$$\Phi \colon \mathbb{R}^d \times C([0, \infty) \to \mathbb{R}^n) \to C([0, \infty) \to \mathbb{R}^d)$$

0044-3719/79/0047/0213/$03.40

is obtained by solving certain ordinary differential equation. One can easily see that the integrability condition of (1.2) is equivalent to the condition that the Lie algebra $\mathscr{L}(A_1, \ldots, A_n)$ is Abelian. On the other hand, Gaveau, B. [3] treated a special class of stochastic differential equations in the case when $\mathscr{L}(A_1, \ldots, A_n)$ is *not Abelian*. For example, consider the case when

$$A_1=\frac{\partial}{\partial x^1}+2x^2\frac{\partial}{\partial x^3}, \quad A_2=\frac{\partial}{\partial x^2}-2x^1\frac{\partial}{\partial x^3}, \quad A_0=0$$

($d=3$, $n=2$). Then, $\mathscr{L}(A_1, A_2)$ is nilpotent[1] of step 2 and X_t is expressed as a function of multiple Wiener integrals of order $\leqq 2$ (see Example 2.1 of the next section). These works of Doss and Gaveau suggest that there will be a general relation between

(a) the representability of the solutions of stochastic differential equations in a form similar to (1.3) by means of multiple Wiener integrals,
and
(b) the nilpotent property of the associated Lie algebras.

The purpose of this paper is to investigate such a relation in full under a general setting.

Before stating our main results, we must introduce some notations. E denotes either the set $\{0, \ldots, n\}$ or the set $\{1, \ldots, n\}$; it will be decided in each occasion. We put

$$E(p)=\{I=(i_1, \ldots, i_a);\, i_1, \ldots, i_a\in E,\, 1\leqq a\leqq p\}, \quad p=1, 2, \ldots,$$

$$E(\infty)=\bigcup_{p=1}^{\infty} E(p),$$

and define vector fields A_I for $I\in E(\infty)$ inductively by the formula

$$A_{(i_1, \ldots, i_a)}=[A_{(i_1, \ldots, i_{a-1})}, A_{i_a}]. \tag{1.4}$$

For simplicity, we assume that the components of A_I, $I\in E(\infty)$, are Lipschitz continuous on $\mathbb{R}^d$. We also define processes B_t^I, $t\geqq 0$, $I\in E(\infty)$, inductively by the formula

$$B_t^{(i_1, \ldots, i_a)}=\int_0^t B_s^{(i_1, \ldots, i_{a-1})}\circ dB_s^{i_a}, \tag{1.5}$$

where $B_t^0=t$, $t\geqq 0$, by definition, and from now on we write $A_{i_1, \ldots, i_a}$ and $B_t^{i_1, \ldots, i_a}$ instead of $A_{(i_1, \ldots, i_a)}$ and $B_t^{(i_1, \ldots, i_a)}$, respectively.

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[1] A Lie algebra $\mathscr{L}$ is said to be nilpotent of step p if the p-th term of the series:

$$[\mathscr{L}, \mathscr{L}]\supset[\mathscr{L},[\mathscr{L}, \mathscr{L}]]\supset[\mathscr{L}, [\mathscr{L}, [\mathscr{L}, \mathscr{L}]]]\supset\ldots$$

vanishes, where

$$[\mathscr{A}, \mathscr{B}]=\left\{\sum_{i=1}^{k} [a_i, b_i];\, a_i\in A, b_i\in B, i=1, \ldots, k, k=1, 2, \ldots\right\}$$

for each $\mathscr{A}, \mathscr{B}\subset\mathscr{L}$

The main result (Theorem 2.1) of Section 2 is now stated as follows: If $\mathscr{L}(A_0, \ldots, A_n)$ is nilpotent of step p, then there exist a subset F of $E(p)$ ($E = \{0, \ldots, n\}$) and a function $h \in C^\infty(\mathbb{R}^d \times \mathbb{R}^{\#F} \to \mathbb{R}^d)$ [2] with the property that

$$X_t = h(x, (B_t^I)_{I \in F})$$

is the solution of stochastic differential equation (1.1) for each $x \in \mathbb{R}^d$. The proof of this theorem will be given in Section 3. The converse of this theorem is also true as will be proved in Section 4. An extension of a result of Doss, H. [1, 2] will then be presented in Section 5. Namely, we will prove the following theorem: If $\mathscr{L}(A_1, \ldots, A_n)$ is nilpotent of step p, then there exist a subset F of $E(p)$ ($E = \{1, \ldots, n\}$), a function $h \in C^\infty(\mathbb{R}^d \times \mathbb{R}^{\#F} \to \mathbb{R}^d)$ and a functional

$$\Phi \colon \mathbb{R}^d \times C([0, \infty) \to \mathbb{R}^{\#F}) \to C([0, \infty) \to \mathbb{R}^d)$$

having the property that

$$X_t = h(\Phi(x, (B_\cdot^I)_{I \in F})_t, (B_t^I)_{I \in F}), \quad t \geqq 0,$$

is the solution of (1.1) for each $x \in \mathbb{R}^d$.

2. Construction of a Functional when $\mathscr{L}(A_0, \ldots, A_n)$ is Nilpotent

In Sections 2 and 3, we put $E = \{0, \ldots, n\}$. We fix a positive integer p. The set

$$\{y = (y^I)_{I \in E(p)};\ y^I \in \mathbb{R},\ I \in E(p)\}$$

will be identified with $\mathbb{R}^m$, where $m = \# E(p)$. The coordinate system on $\mathbb{R}^m$ is also denoted by y^I, $I \in E(p)$. We define vector fields Q_i, $i \in E$, on $\mathbb{R}^m$ by

$$Q_i = \frac{\partial}{\partial y^i} + \sum_{\substack{a+1 \leq p \\ j_1, \ldots, j_a \in E}} y^{j_1, \ldots, j_a} \frac{\partial}{\partial y^{j_1, \ldots, j_a, i}}. \tag{2.1}$$

We will denote by $\mathscr{L}(Q_i + A_i, i \in E)$ the Lie algebra generated by the vector fields $Q_i + A_i$, $i \in E$, on $\mathbb{R}^{m+d}$.

Let $\mathbb{R}(E)$ be the linear space with basis E and let $\mathbb{T}(E)$ be the tensor algebra based on $\mathbb{R}(E)$, i.e.,

$$\mathbb{T}(E) = \mathbb{R} \oplus \mathbb{R}(E) \oplus (\mathbb{R}(E) \otimes (\mathbb{R} E)) \oplus \ldots$$

Define the bracket product in $\mathbb{T}(E)$:

$$[a, b] = a \otimes b - b \otimes a, \quad a, b \in \mathbb{T}(E).$$

Let $\mathbb{L}(E)$ be the Lie subalgebra of $\mathbb{T}(E)$ generated by E. We denote by τ and λ the injections: $E \to \mathbb{T}(E)$ and $E \to \mathbb{L}(E)$ respectively. Recall that $(\mathbb{T}(E), \tau)$ is a free algebra generated by E, i.e., for each algebra $\mathscr{A}$ and a mapping $\theta \colon E \to \mathscr{A}$, there

[2] $C^\infty(\mathbb{R}^d \times \mathbb{R}^{\#F} \to \mathbb{R}^d)$ is the set of all C^∞-functions: $\mathbb{R}^d \times \mathbb{R}^{\#F} \to \mathbb{R}^d$, where $\#F$ is the number of elements of F

exists a unique homomorphism $\theta': \mathbb{T}(E) \to \mathscr{A}$ such that $\theta' \circ \tau = \theta$. Recall also that $(\mathbb{L}(E), \lambda)$ is a free Lie algebra generated by E, i.e., for each Lie algebra $\mathscr{L}$ and a mapping $\theta: E \to \mathscr{L}$, there exists a unique homomorphism $\theta': \mathbb{L}(E) \to \mathscr{L}$ such that $\theta' \circ \lambda = \theta$. We define $[i_1, \ldots, i_a] \in \mathbb{L}(E)$ for $(i_1, \ldots, i_a) \in E(\infty)$ by

$$[i_1, \ldots, i_a] = [[i_1, \ldots, i_{a-1}], i_a]$$

inductively. Each $[i_1, \ldots, i_a]$ is expressed as

$$[i_1, \ldots, i_a] = \sum_{(j_1, \ldots, j_b) \in E(\infty)} c_{i_1, \ldots, i_a}^{j_1, \ldots, j_b} j_1 \otimes \ldots \otimes j_b \tag{2.2}$$

and coefficients $c_{i_1, \ldots, i_a}^{j_1, \ldots, j_b}$ are uniquely determined by (2.2). We denote by $C(E, p)$ the matrix $(c_I^J)_{I, J \in E(p)}$.

Since $c_i^j = \delta_i^j$, $i, j \in E$, we can always take a subset $F \subset E(p)$ that satisfies

Property 2.1. *F is a maximal subset of $E(p)$ such that the column vectors of $C(E, p)$: $(c_I^J)_{I \in E(p)}$ for $J \in F$ are linearly independent.*

Let r be the rank of the matrix $C(E, p)$ and fix a bijection:

$$v: F + E(p) \setminus F + \{1, \ldots, d\} \to \{1, \ldots, m+d\} \tag{2.3}$$

with $v(F) = \{1, \ldots, r\}$, $v(E(p) \setminus F) = \{r+1, \ldots, m\}$, $v(\{1, \ldots, d\}) = \{m+1, \ldots, m+d\}$, where $F + E(p) \setminus F + \{1, \ldots, d\}$ is the direct sum of these sets.

Proposition 2.1. *Suppose that $\mathscr{L}(A_i, i \in E)$ is nilpotent of step p. Then we have*

i) *$\mathscr{D} \equiv \{\mathscr{L}(Q_i + A_i, i \in E)_q;\ q \in \mathbb{R}^{m+d}\}$ is an r-dimensional differential system that satisfies the integrability condition.*

ii) *For each $F \subset E(p)$ with Property 2.1, there exists a unique function $f \in C^\infty(\mathbb{R}^{m+d} \times \mathbb{R}^r \to \mathbb{R}^{m+d})$ satisfying the followings:*

a) *$f^i(q; u) = u^i$ for each $1 \leqq i \leqq r$, $q \in \mathbb{R}^{m+d}$, $u \in \mathbb{R}^r$.*

b) *$M_q \equiv \{f(q; u); u \in \mathbb{R}^r\}$ is a leaf[3] of $\mathscr{D}$, for each $q \in \mathbb{R}^{m+d}$.*

c) *$f(q; q^1, \ldots, q^r) = q$ for each $q = (q^1, \ldots, q^{m+d}) \in \mathbb{R}^{m+d}$.*

Now we can state

Theorem 2.1. *Suppose that $\mathscr{L}(A_0, \ldots, A_n)$ is nilpotent of step p. Let F be a subset of $E(p)$ $(E = \{0, \ldots, n\})$ with Property 2.1. Then*

$$X_t^i = f^{v(i)}(\underbrace{0, \ldots, 0}_{m}, x; B_t^F), \qquad 1 \leqq i \leqq d, \quad t \geqq 0,$$

is the solution of (1.1) for each $x \in \mathbb{R}^d$, where $B_t^F = (B_t^I)_{I \in F}$.

We present

Example 2.1. Let $d = 3$, $n = 2$, $A_1 = \frac{\partial}{\partial x^1} + 2x^2 \frac{\partial}{\partial x^3}$, $A_2 = \frac{\partial}{\partial x^2} - 2x^1 \frac{\partial}{\partial x^3}$, $A_0 = 0$.

[3] A maximal integral manifold of $\mathscr{D}$ is called a leaf of $\mathscr{D}$

Then $A_{1,2} = -4\frac{\partial}{\partial x^3}$, $A_{1,2,1} = A_{1,2,2} = 0$. *Hence* $\mathscr{L}(A_0, A_1, A_2)$ *is nilpotent of step* 2. *We may take* $F = \{0, 1, 2, (0, 1), (0, 2), (1, 2)\}$. *Noting that* (1.1-1) *takes the form*:

$$\begin{bmatrix} dX_t^1 \\ dX_t^2 \\ dX_t^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2X_t^2 & -2X_t^2 \end{bmatrix} \circ \begin{bmatrix} dB_t^1 \\ dB_t^2 \end{bmatrix},$$

we have

$$X_t = \begin{bmatrix} x^1 + B_t^1 \\ x^2 + B_t^2 \\ x^3 + 2(x^2 B_t^1 - x^1 B_t^2) + 2(B_t^1 B_t^2 - 2B_t^{1,2}) \end{bmatrix},$$

where $x = (x^1, x^2, x^3)$.

3. Proof of Results in Section 2

First we prepare several lemmas. Recall that Q_i, $i \in E$, was defined by (2.1). We define

$$Q_{i_1, \ldots, i_a} = [Q_{i_1, \ldots, i_{a-1}}, Q_{i_a}]$$

inductively for $(i_1, \ldots, i_a) \in E(\infty)$.

Lemma 3.1. i) *For each* $(i_1, \ldots, i_a) \in E(p)$, *we have*

$$Q_{i_1, \ldots, i_a} = \sum_{j_1, \ldots, j_a \in E} c_{i_1, \ldots, i_a}^{j_1, \ldots, j_a} \left(\frac{\partial}{\partial y^{j_1, \ldots, j_a}} + \sum_{\substack{b+a \leq p \\ k_1, \ldots, k_b \in E}} y^{k_1, \ldots, k_b} \frac{\partial}{\partial y^{k_1, \ldots, k_b, j_1, \ldots, j_a}} \right).$$

ii) *For each* $(i_1, \ldots, i_a) \in E(\infty) \setminus E(p)$, *we have*

$$Q_{i_1, \ldots, i_a} = 0.$$

Proof. i) We can easily verify

$$Q_{i_1, \ldots, i_a} = \sum_{j_1, \ldots, j_a \in E} c_{i_1, \ldots, i_a}^{j_1, \ldots, j_a} Q_{j_1} \cdots Q_{j_a}.$$

Hence, it is enough to prove

$$Q_{j_1} \cdots Q_{j_a} \sim \frac{\partial}{\partial y^{j_1, \ldots, j_a}} + \sum_{\substack{b+a \leq p \\ k_1, \ldots, k_b \in E}} y^{k_1, \ldots, k_b} \frac{\partial}{\partial y^{k_1, \ldots, k_b, j_1, \ldots, j_a}},$$

where $\sim$ denotes coincidence except differential operators of degree $\geqq 2$. We have, for each $b\leqq p-1$ and $j_1, \dots, j_b, j\in E$,

$$\left(\frac{\partial}{\partial y^{j_1,\dots,j_b}}+\sum_{\substack{c+b\leqq p\\ k_1,\dots,k_c\in E}} y^{k_1,\dots,k_c}\frac{\partial}{\partial y^{k_1,\dots,k_c,j_1,\dots,j_b}}\right)Q_j$$

$$\sim\left(\frac{\partial}{\partial y^{j_1,\dots,j_b}}+\sum_{\substack{c+b\leqq p\\ k_1,\dots,k_c\in E}} y^{k_1,\dots,k_c}\frac{\partial}{\partial y^{k_1,\dots,k_c,j_1,\dots,j_b}}\right)$$

$$\times\left(\sum_{\substack{a+1\leqq p\\ i_1,\dots,i_a\in E}} y^{i_1,\dots,i_a}\frac{\partial}{\partial y^{i_1,\dots,i_a,j}}\right)$$

$$\sim\frac{\partial}{\partial y^{j_1,\dots,j_b,j}}+\sum_{\substack{c+b+1\leqq p\\ k_1,\dots,k_c\in E}} y^{k_1,\dots,k_c}\frac{\partial}{\partial y^{k_1,\dots,k_c,j_1,\dots,j_b,j}}.$$

Thus, we have proved i).

ii) Noting

$$Q_{i_1,\dots,i_p}=\sum_{j_1,\dots,j_p\in E} c^{j_1,\dots,j_p}_{i_1,\dots,i_p}\frac{\partial}{\partial y^{j_1,\dots,j_p}}$$

and (2.1), we easily obtain ii). Q.E.D.

Let F be a subset of $E(p)$ with Property 2.1. We choose a subset G of $E(p)$ with r elements such that the matrix $C(G, F)=(c^J_I)_{I\in G, J\in F}$ is invertible.

Lemma 3.2. *If $\mathscr{L}(A_i, i\in E)$ is nilpotent of step p, then, $(Q_I+A_I)_q$, $I\in G$ form a basis of $\mathscr{D}_q\equiv\mathscr{L}(Q_i+A_i, i\in E)_q$ for each $q\in\mathbb{R}^{m+d}$.*

Proof. Set

$$E^a=\{(i_1, \dots, i_a); i_1, \dots, i_a\in E\}, \qquad a=1, 2, \dots, \tag{3.1}$$

and

$$C^b_a=(c^J_I)_{I\in E^a, J\in E^b}, \qquad a, b=1, 2, \dots. \tag{3.2}$$

By Lemma 3.1, vector fields Q_I+A_I, $I\in E(\infty)$, are represented as the row vectors of the matrix:

$$\begin{array}{c} \\ E^1\\ E^2\\ \vdots\\ E^p\\ \bigcup_{a=p-1}^{\infty} E^a\end{array}\begin{array}{c}\begin{array}{ccccc} E^1 & E^2 & \dots & E^p & 1\dots d\end{array}\\ \left[\begin{array}{c|c|c|c|c} C^1_1 & * & * & * & *\\ \hline 0 & C^2_2 & * & * & *\\ \hline 0 & 0 & \ddots & * & *\\ \hline 0 & 0 & 0 & C^p_p & *\\ \hline 0 & 0 & 0 & 0 & 0\end{array}\right]\end{array}. \tag{3.3}$$

Hence, $(Q_I+A_I)_q$, $I\in G$, are linearly independent. Since $\mathscr{L}(Q_i+A_i, i\in E)$ is spanned by Q_I+A_I, $I\in E(p)$, it is enough to prove that each Q_I+A_I, $I\in E(p)$, is a linear combination of Q_J+A_J, $J\in G$. By the definition of G, $\{[j_1, \dots, j_b]; (j_1, \dots, j_b)\in G\}$ form a basis of the linear subspace of $\mathscr{L}(E)$ spanned by

$$\{[i_1, \dots, i_a]; (i_1, \dots, i_a)\in E(p)\}.$$

So that the homomorphism: $\mathbb{L}(E)\to\mathscr{L}(Q_i+A_i, i\in E)$ gives us the desired result. Q.E.D.

For each $I\in E(p)$, let $Q_I^J(y)$, $J\in E(p)$, be components of Q_I. Set $Q(G, F)=(Q_I^J)_{I\in G, J\in F}$. We see by (3.3) that there exists the inverse matrix of $Q(G, F)$, which will be denoted by $R=(R_I^J)_{I\in F, J\in G}$. Recalling (3.1), we put

$$F^a=E^a\cap F, \quad G^a=E^a\cap G, \quad a=1, \dots, p. \tag{3.4}$$

Lemma 3.3. *Let $1\leqq a, b\leqq p$ and let $I\in F^a$, $J\in G^b$. Then we have*

i) *R_I^J is a polynomial of y^K, $K\in E(b-a)$, if $b>a$.*

ii) *R_I^J is a constant, if $b=a$.*

iii) *$R_I^J=0$, if $b<a$.*

Proof. Let $R^{(0)}$ be the inverse matrix of $C(G, F)$. Set $S=Q(G, F)\cdot R^{(0)}$ and define $M^{(a)}=(M_I^{(a)J})_{I, J\in G}$, $a=1, \dots, p-1$, by

$$M_I^{(a)J}=\begin{cases}-S_I^J, & \text{if } I\in G^a \text{ and } J\in G^{a+1}\cup\dots\cup G^p,\\ 1, & \text{if } I=J,\\ 0, & \text{otherwise.}\end{cases}$$

Since multiplication: $\times M^{(a)}$ means the operations:

addition of $(-S_I^J)\times(I\text{-th column})$ to $(J\text{-th column})$,
for $I\in G^a$ and $J\in G^{a+1}\cup\dots\cup G^p$,

we have

$$SM^{(1)}\dots M^{(p-1)}=\begin{bmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{bmatrix}.$$

So that, if we put

$$R^{(a)}=R^{(0)}M^{(1)}\dots M^{(a)}, \quad a=1, \dots, p-1,$$

we obtain $Q(G, F)^{-1}=R^{(p-1)}$. Since

$$S_I^J=\sum_{K\in F^b} Q_I^K R_K^{(0)J}, \quad J\in G^b,$$

Lemma 3.1 implies that S_I^J is a polynomial of y^K, $K\in E(b-a)$ for each $I\in G^a$ and $J\in G^b$ with $a<b$. Then, it is easy to verify that $R_I^{(c)J}$ is a polynomial of y^K, $K\in E(b-a)$, for $I\in F^a$, $J\in G^b$ with $a<b$ $(c=1, \dots, p-1)$. Q.E.D.

Let μ be the inverse mapping of ν in (2.3). We define functions $T_\rho^i(z)$, $z \in R^{m+d}$ for $r+1 \leqq i \leqq m+d$, $1 \leqq \rho \leqq r$ by

$$T_\rho^i(z)=\begin{cases} \sum_{I\in G} R_{\mu(\rho)}^I(z^1,\dots,z^m)\,Q_I^{\mu(i)}(z^1,\dots,z^m), & \text{if } r+1\leqq i\leqq m \\ \sum_{I\in G} R_{\mu(\rho)}^I(z^1,\dots,z^m)\,A_I^{\mu(i)}(z^{m+1},\dots,z^{m+d}), & \text{if } m+1\leqq i\leqq m+d. \end{cases} \tag{3.5}$$

Lemma 3.4. *Let $i \leqq m$ and $\mu(i) \in E^b$. Then values of T_ρ^i, $1 \leqq \rho \leqq r$, depend only on $z^{\nu(I)}$, $I \in E(b-1)$.*

Proof. Let $\mu(i) \in E^b$ and $\mu(\rho) \in F^a$. Then we have

$$T_\rho^i=\begin{cases} \sum_{I\in G^a\cup\dots\cup G^b} R_{\mu(\rho)}^I\,Q_I^{\mu(i)}, & \text{if } a\leqq b, \\ 0, & \text{if } a>b, \end{cases}$$

by Lemma 3.1 and 3.3. Using these lemmas again, we see that $R_{\mu(\rho)}^I(z^1, \dots, z^m)$ and $Q_I^{\mu(i)}(z^1, \dots, z^m)$ are functions of $z^{\nu(J)}$, $J \in E(b-1)$, when $a \leqq b$ and $I \in G^a \cup \dots \cup G^b$. Q.E.D.

Denoting by $z=(z^1, \dots, z^{m+d})$ the system of coordinates in $\mathbb{R}^{m+d}$, we have

Lemma 3.5. *Suppose that $\mathscr{L}(A_i, i \in E)$ is nilpotent of step p. Then we have*

i) $\left(\frac{\partial}{\partial z^\rho}+\sum_{i=r+1}^{m+d} T_\rho^i \frac{\partial}{\partial z^i}\right)_q$, $1 \leqq \rho \leqq r$, *form a basis of $\mathscr{D}_q$, for each $q \in \mathbb{R}^{m+d}$.*

ii) $\left(dz^i-\sum_{\rho=1}^{r} T_\rho^i dz^\rho\right)_q$, $r+1 \leqq i \leqq m+d$, *form a dual basis of $\mathscr{D}_q$, for each $q \in \mathbb{R}^{m+d}$.*

iii) *The system of total differential equations*:

$$dv^i=\sum_{\rho=1}^{r} T_\rho^i(u^1, \dots, u^r, v^{r+1}, \dots, v^{m+d})\,du^\rho, \qquad r+1 \leqq i \leqq m+d, \tag{3.6}$$

is completely integrable.

iv) *For each solution v of* (3.6) *defined on an open set $\mathcal{O} \subset \mathbb{R}^r$, the set* $\{(u, v(u));$ *$u \in \mathcal{O}\}$ is an r-dimensional integral manifold of $\mathscr{D}$.*

Proof. By the definition of T_ρ^i, we have

$$\sum_{J\in G} R_I^J(Q_J+A_J)=\frac{\partial}{\partial z^{\nu(I)}}+\sum_{i=r+1}^{m+d} T_{\nu(I)}^i \frac{\partial}{\partial z^i}$$

for $I \in F$. Since R is invertible, Lemma 3.2 gives us i) and ii). Noting that $[Q_I + A_I, Q_J + A_J]$ is a linear combination of $Q_K + A_K$, $K \in G$, for each $I, J \in E(p)$, we have iii) by the theory of complete system. It is easy to show iv) by i). Q.E.D.

Lemma 3.6. *Suppose that $\mathscr{L}(A_i, i\in E)$ is nilpotent of step p. Then the Eq.* (3.6) *with initial condition*

$$v(q^1, \ldots, q^r) = (q^{r+1}, \ldots, q^{m+d}) \tag{3.7}$$

has a unique solution defined on $\mathbb{R}^r$, for each $q=(q^1, \ldots, q^{m+d})\in\mathbb{R}^{m+d}$.

Proof. First we will show a procedure to solve the equation

$$dv^i = \sum_{\rho=1}^{r} T^i_\rho(u^1, \ldots, u^r, v^{r+1}, \ldots, v^{m+d})\, du^\rho, \tag{3.8-1}$$

$$v^i(q^1, \ldots, q^r) = q^i, \tag{3.8-2}$$

for $r+1\leqq i\leqq m$. When $i\in\nu(E^2\setminus F^2)$, Lemma 3.4 implies that each T^i_ρ, $1\leqq\rho\leqq r$, is a function of $z^{\nu(I)}$, $I\in E^1$. Then, noting that $F^1=E^1$, the Eq. (3.8-1) for $i\in\nu(E^2\setminus F^2)$ takes the form:

$$dv^i = \sum_{1\leqq\rho\leqq r} T^i_\rho(u^1, \ldots, u^r)\, du^\rho,$$

which gives us solutions v^i, $i\in\nu(E^2\setminus F^2)$, defined on $\mathbb{R}^r$. Now suppose that we have obtained the unique solution v^i for $i\in\nu(E^2\setminus F^2)\cup\ldots\cup\nu(E^a\setminus F^a)$. Then it is easy to obtain v^i, $i\in\nu(E^{a+1}\setminus F^{a+1})$, by Lemma 3.4. To prove uniqueness and existence of global solution of (3.8) for $i\geqq m+1$, it is enough to note that

$$T^i_\rho(u^1, \ldots, u^r, v^{r+1}, \ldots, v^{m+d}) = \sum_{I\in G} R^I_{\mu(\rho)}(u^1, \ldots, u^r, v^{r+1}, \ldots, v^m)\, A^{\mu(i)}_{I'}(v^{m+1}, \ldots, v^{m+d})$$

for $m+1\leqq i\leqq m+d$ and that the components of A_I, $I\in G$, are Lipschitz continuous. Q.E.D.

Proof of Proposition 2.1. By Lemma 3.2, $\mathscr{D}$ is an r-dimensional differential system. Since $\mathscr{L}(Q_i+A_i, i\in E)$ is a Lie algebra, $\mathscr{D}$ satisfies the integrability condition. Now, let G be a subset of $E(p)$ such that $C(G, F)$ is invertible. Then the solution of (3.6) and (3.7) gives us a function $f(q, u)=(f^i(q, u))_{1\leqq i\leqq m+d}$, $q\in\mathbb{R}^{m+d}$, $u\in\mathbb{R}^r$, defined by

$$f^i(q, u) = u^i, \qquad 1\leqq i\leqq r,$$

and

$$f^i(q, u) = v^i(u), \qquad r+1\leqq i\leqq m+d.$$

Then, Lemma 3.5 iv) implies that

$$M_q = \{f(q, u);\, u\in\mathbb{R}^r\} \tag{3.9}$$

is an r-dimensional manifold of $\mathscr{D}$, for each $q\in\mathbb{R}^{m+d}$. Now, fix any $q\in\mathbb{R}^{m+d}$ and let M be an r-dimensional integral manifold of $\mathscr{D}$ that contains q. Let w be the restriction of the mapping $(z^1, \ldots, z^r)$: $\mathbb{R}^{m+d}\to\mathbb{R}^r$ to M. Let $(\zeta^1, \ldots, \zeta^r)$ be a system of local coordinates of M around q. Since

$$\frac{\partial}{\partial \zeta^\rho}=\sum_{1\leqq i\leqq m+d}\frac{\partial z^i}{\partial \zeta^\rho}\frac{\partial}{\partial z^i}$$

and

$$\left(\frac{\partial}{\partial z^\sigma}+\sum_{r+1\leqq i\leqq m+d}T^i_\sigma\frac{\partial}{\partial z^i}\right)_q,\qquad 1\leqq\sigma\leqq r,$$

form a basis of $T_q M$, we have

$$\left(\frac{\partial}{\partial \zeta^\rho}\right)_q=\sum_{1\leqq\sigma\leqq r}\left(\frac{\partial w^\sigma}{\partial \zeta^\rho}\right)_q\left(\frac{\partial}{\partial z^\sigma}+\sum_{r+1\leqq i\leqq m+d}T^i_\sigma\frac{\partial}{\partial z^i}\right)_q.$$

Consequently, there exists w^{-1} around $(z^1(q),\dots,z^r(q))$ and further it follows from Lemma 3.5 ii) that w^{-1} is a solution of (3.6) and (3.7). Thus, M coincides with M_q in a neighborhood of q. So that, we obtain the maximality of M_q, $q\in\mathbb{R}^{m+d}$. The above argument also shows that the function f is independent of the choice of G. Q.E.D.

We put

$$Y_t=(B^I_t)_{I\in E(p)}. \tag{3.10}$$

To prove Theorem 2.1, we prepare

Lemma 3.7. *Y_t is the solution of the stochastic differential equation*:

$$dY^I_t=\sum_{j\in E}Q^I_j(Y_t)\circ dB^j_t,\qquad I\in E(p), \tag{3.11-1}$$

$$Y_0=0. \tag{3.11-2}$$

Proof. By the definition of $B^{i_1,\dots,i_a}_t$, we have

$$dB^{i_1,\dots,i_a}_t=B^{i_1,\dots,i_{a-1}}_t\circ dB^{i_a}_t$$

when $a>1$. On the other hand, we have

$$Q^{j_1,\dots,j_a}_i=\begin{cases}y^{j_1,\dots,j_{a-1}}\,\delta^{j_a}_i, & \text{if } a>1,\\ \delta^{j_a}_i, & \text{if } a=1,\end{cases} \tag{3.12}$$

by the definition of Q_i. Hence, $(B^I_t)_{I\in E(p)}$ satisfies (3.11). Q.E.D.

Lemma 3.8. *Let f be the function in Proposition* 2.1. *Then we have*

$$f^i(\underbrace{0,\dots,0}_{m},x;B^F_t)=B^{\mu(i)}_t,\qquad r+1\leqq i\leqq m,\quad t\geqq 0, \tag{3.13}$$

for each $x\in\mathbb{R}^d$.

Proof. Set $V_t=(f_i(0,\ldots,0,x;B_t^F))_{r+1\leqq i\leqq m}$. Since $V_0=0$, it is enough to prove

$$dV_t^i=\sum_{j\in E} Q_j^i(Y_t)\circ dB_t^j, \tag{3.14}$$

for $r+1\leqq i\leqq m$. When $\mu(i)\in E^b$, we have

$$dV_t^i=\sum_{1\leqq\rho\leqq r} T_\rho^i(B_t^F,(V_t^j)_{j\in\nu(E(b-1)\setminus F)})\circ dB_t^{\mu(\rho)} \tag{3.15}$$

by Eq. (3.6) and Lemma 3.4. When $b=2$, we have

$$dV_t^i=\sum_{1\leqq\rho\leqq r} T_\rho^i(B_t^F)\circ dB_t^{\mu(\rho)},$$

since $E(1)\setminus F=\phi$. Then, Eq. (3.11-1) gives us

$$\begin{aligned}dV_t^i&=\sum_{1\leqq\rho\leqq r,\,I\in G,\,j\in E}\{(R_{\mu(\rho)}^I Q_I^{\mu(i)})(Y_t)\,Q_j^{\mu(\rho)}(Y_t)\}\circ dB_t^j\\&=\sum_{j\in E,\,I\in G}\{\delta_j^I Q_I^{\mu(i)}(Y_t)\}\circ dB_t^j.\end{aligned}$$

Hence we have

$$V_t^i=B_t^{\mu(i)},\qquad i\in\nu(E(2)\setminus F).$$

Now suppose that we have proved

$$V_t^i=B_t^{\mu(i)}\quad\text{ for } i\in\nu(E(2)\setminus F)\cup\ldots\cup\nu(E(a)\setminus F).$$

Then it is easy to prove (3.14) for $i\in\nu(E(a+1)\setminus F)$ by Eq. (3.15). Q.E.D.

Proof of Theorem 2.1. Put

$$Z_t^i=f^i(0,\ldots,0,x;B_t^F),\quad 1\leqq i\leqq m+d,\qquad X_t^i=Z_t^{\nu(i)},\quad 1\leqq i\leqq d,$$

$$Z_t=(Z_t^1,\ldots,Z_t^{m+d}),\qquad X_t=(X_t^1,\ldots,X_t^d).$$

Then we have

$$Z_t=(Y_t^{\mu(1)},\ldots,Y_t^{\mu(m)},X_t^{\mu(m+1)},\ldots,X_t^{\mu(m+d)}) \tag{3.16}$$

by Lemma 3.8 and condition a) for f. Equation (3.6) gives us

$$\begin{aligned}dX_t^i&=\sum_{1\leqq\rho\leqq r} T_\rho^{\nu(i)}(Z_t)\circ dB_t^{\mu(\rho)}\\&=\sum_{1\leqq\rho\leqq r}\{T_\rho^{\nu(i)}(Z_t)\,Q_j^{\mu(\rho)}(Y_t)\}\circ dB_t^j.\end{aligned}$$

Then we have

$$\begin{aligned}dX_t^i&=\sum_{1\leqq\rho\leqq r,\,I\in G,\,j\in E}\{R_{\mu(\rho)}^I(Y_t)\,A_I^i(X_t)\,Q_j^{\mu(\rho)}(Y_t)\}\circ dB_t^j\\&=\sum_{j\in E,\,I\in G}\{\delta_j^I A_I^i(X_t)\}\circ dB_t^j,\end{aligned}$$

as $R=Q(G,F)^{-1}$. Hence we obtain (1.1-1). It is easy to see $X_0=x$ by (3.7). Q.E.D.

4. Converse of Theorem 2.1

We put $E=\{0, \ldots, n\}$ and $m(p)=\#E(p)$, $p=1, 2, \ldots$. We prove in this section

Theorem 4.1. *Suppose that there exists a function* $h \in C^{\infty}(\mathbb{R}^d \times \mathbb{R}^{m(p)} \to \mathbb{R}^d)$ *such that* $X_{t,x}=h(x, (B_t^I)_{I \in E(p)})$ *is the solution of* (1.1) *for each* $x \in \mathbb{R}^d$. *Then,* $\mathscr{L}(A_0, \ldots, A_n)$ *is nilpotent of step* p.

First we prove

Lemma 4.1. *For each* $h \in C^{\infty}(\mathbb{R}^{m(p)} \to \mathbb{R})$, *we have*

$$dh(Y_t)=\sum_{i \in E} (Q_i h)(Y_t) \circ dB_t^i,$$

where $Y_t=(B_t^I)_{I \in E(p)}$ *and* Q_i, $i \in E$, *are vector fields defined by* (2.1).

Proof. Applying Itô's formula, we have

$$dh(Y_t)=\sum_{I \in E(p)} \frac{\partial h}{\partial y^I}(Y_t) \circ dB_t^I.$$

Then, Eq. (3.11-1) gives us

$$\begin{aligned} dh(Y_t) &= \sum_{I \in E(p),\, i \in E} \left\{ Q_i^I(Y_t) \frac{\partial h}{\partial y^I}(Y_t) \right\} \circ dB_t^i \\ &= \sum_{i \in E} (Q_i h)(Y_t) \circ dB_t^i. \quad \text{Q.E.D.} \end{aligned}$$

Lemma 4.2. *Let* X_t *be the solution of* (1.1). *Suppose that there exist* $g \in C^{\infty}(\mathbb{R}^d \to \mathbb{R})$ *and* $h \in C^{\infty}(\mathbb{R}^{m(p)} \to \mathbb{R})$ *such that*

$$g(X_t)=h(Y_t), \quad t \geqq 0. \tag{4.1}$$

Then we have

$$(A_i g)(X_t)=(Q_i h)(Y_t), \quad i \in E, \quad t \geqq 0. \tag{4.2}$$

Proof. Taking stochastic differential of (4.1), we have

$$\sum_{i \in E} (A_i g)(X_t) \circ dB_t^i = \sum_{i \in E} (Q_i h)(Y_t) \circ dB_t^i \tag{4.3}$$

by Lemma 4.1. Hence, we have

$$\sum_{1 \leqq i \leqq n} \{(A_i g)(X_t)-(Q_i h)(Y_t)\}^2=0,$$

so that

$$(A_i g)(X_t)=(Q_i h)(Y_t), \quad 1 \leqq i \leqq n. \tag{4.4}$$

Then, (4.3) and (4.4) give us

$$(A_0 g)(X_t)=(Q_0 h)(Y_t). \quad \text{Q.E.D.}$$

Proof of Theorem 4.1. Suppose that $X_t = h(x, Y_t)$ is the solution of (1.1) for each $x \in \mathbb{R}^d$. Lemma 4.2 gives us

$$A_i^j(X_t) = (Q_i h^j)(Y_t), \quad i \in E, \quad 1 \leqq j \leqq d.$$

Using Lemma 4.2 again, we obtain

$$(A_{i_1} \dots A_{i_p} A_{i_{p+1}}^j)(X_t) = (Q_{i_1} \dots Q_{i_{p+1}} h^j)(Y_t), \qquad i_1, \dots, i_{p+1} \in E, \quad 1 \leqq j \leqq d. \tag{4.5}$$

By (2.2), we obtain

$$A_{j_1, \dots, j_{p+1}} = \sum_{i_1, \dots, i_{p+1} \in E} c_{j_1, \dots, j_{p+1}}^{i_1, \dots, i_{p+1}} A_{i_1} \dots A_{i_{p+1}},$$

$$Q_{j_1, \dots, j_{p+1}} = \sum_{i_1, \dots, i_{p+1} \in E} c_{j_1, \dots, j_{p+1}}^{i_1, \dots, i_{p+1}} Q_{i_1} \dots Q_{i_{p+1}}.$$

Hence

$$A_{j_1, \dots, j_{p+1}}^j(X_t) = (Q_{j_1, \dots, j_{p+1}} h^j)(Y_t), \qquad j_1, \dots, j_{p+1} \in E, \quad 1 \leqq j \leqq d.$$

Then Lemma 3.1 ii) gives us

$$A_{j_1, \dots, j_{p+1}}^j = 0, \quad j_1, \dots, j_{p+1} \in E, \quad 1 \leqq j \leqq d, \quad x \in \mathbb{R}^d.$$

Hence $\mathscr{L}(A_i, i \in E)$ is nilpotent of step p. Q.E.D.

5. The Functional when $\mathscr{L}(A_1, \dots, A_n)$ is Nilpotent

In this section, we suppose that $\mathscr{L}(A_1, \dots, A_n)$ is nilpotent of Step p. So, we put $E = \{1, \dots, n\}$. Meanings of other symbols: m, r etc. are changed according to the change of E. Take a subset F of $E(p)$ that satisfies Property 2.1. Let

$$f(z; u) = (f^i(z; u))_{1 \leqq i \leqq m+d}, \quad z \in \mathbb{R}^{m+d}, \quad u \in \mathbb{R}^r,$$

be the function in Proposition 2.1. $\frac{\partial f}{\partial z^1}, \dots, \frac{\partial f}{\partial z^{m+d}}, \frac{\partial f}{\partial u^1}, \dots, \frac{\partial f}{\partial u^r}$ will be denoted by $\partial_1 f, \dots, \partial_{m+d} f, \partial_{m+d+1} f, \dots, \partial_{m+d+r} f$ respectively. Put $h = (f^{m+1}, \dots, f^{m+d})$.

Proposition 5.1. *For each* $x \in \mathbb{R}^d$ *and* $U = (U_t)_{t \geqq 0} \in C([0, \infty) \to \mathbb{R}^r)$, *there exists* $D = (D_t)_{t \geqq 0} \in C([0, \infty) \to \mathbb{R}^d)$ *which is the unique solution of an ordianry differential equation*:

$$\frac{dD}{dt} = \sum_{1 \leqq i \leqq d} A_0^i(h(\underbrace{0, \dots, 0}_{m}, D_t; U_t)) \times \partial_{m+i} h(f(\underbrace{0, \dots, 0}_{m}, D_t; U_t); \underbrace{0, \dots, 0}_{r}), \tag{5.1-1}$$

$$D_0 = x. \tag{5.1-2}$$

To prove Proposition 5.1, we prepare

Lemma 5.1. *For each compact* $K \subset \mathbb{R}^r$, *we have*

i) $\sup_{\substack{x \in \mathbb{R}^d \\ u \in K}} \|\partial_{m+i} h(f(\underbrace{0, \ldots, 0}_{m}, x; u); \underbrace{0, \ldots, 0}_{r})\|$ [4] $< \infty, \qquad 1 \leqq i \leqq d,$

ii) $\sup_{\substack{X \in \mathbb{R}^d \\ u \in K}} \|h(\underbrace{0, \ldots, 0}_{m}, x; u)\| / (1 + \|x\|) < \infty.$

Proof. Noting that the values of $f^i(z; u)$, $r+1 \leqq i \leqq m$, are independent of $z^{m+1}, \ldots, z^{m+d}$, we put

$$g(z^1, \ldots, z^m; u) = (f^i(z; u))_{r+1 \leqq i \leqq m}.$$

Then we have

$$f(0, x; u) = (u, g(0; u), h(0, x; u)).$$

To prove i), note that

$$\begin{aligned} &\partial_{m+i} h(f(0, x; u); (1-t) u) \\ &\quad = \partial_{m+i} h(u, g(0; u), h(0, x; u); u) \\ &\quad\quad - \sum_{1 \leqq \rho \leqq r} u^\rho \int_0^t (\partial_{m+d+\rho} \partial_{m+i} h)(f(0, x; u); (1-s) u)\, ds. \end{aligned}$$

Since

$$h^j(u, g(0; u), z^{m+1}, \ldots, z^{m+d}; u) = z^{m+j}, \qquad 1 \leqq j \leqq d,$$

we have

$$\partial_{m+i} h^j(u, g(0; u), h(0, x; u); u) = \delta_i^j, \qquad 1 \leqq i, j \leqq d.$$

Next, Eq. (3.6) gives us

$$\begin{aligned} &(\partial_{m+d+\rho} \partial_{m+i} h^j)(z; u) \\ &\quad = \frac{\partial}{\partial z^{m+i}} \{\sum_{I \in G} R^I_{\mu(\rho)}(u, g(z^1, \ldots, z^m; u)) A^j_I(h(z; u))\} \\ &\quad = \sum_{I \in G,\, 1 \leqq k \leqq d} R^I_{\mu(\rho)}(u, g(z^1, \ldots, z^m; u)) \frac{\partial}{\partial x^k} A^j_I(h(z; u)) \\ &\quad\quad \times \partial_{m+i} h^k(z; u). \end{aligned}$$

Hence, we have

$$\begin{aligned} &\left\| u^\rho \int_0^t (\partial_{m+d+\rho} \partial_{m+i} h)(f(0, x; u); (1-s) u)\, ds \right\| \\ &\qquad \leqq \text{const} \int_0^t \|\partial_{m+i} h(f(0, x; u); (1-s) u)\|\, ds \end{aligned}$$

4 $\|\xi\| = \{(\xi^1)^2 + \ldots + (\xi^d)^2\}^{1/2}$ for $\xi = (\xi^1, \ldots, \xi^d) \in \mathbb{R}^d$

for $x \in \mathbb{R}^d$, $u \in K$, $0 \leqq t \leqq 1$. Then, Gronwall's inequality gives us i). To prove ii), observe that

$$h(0, x; tu) = x + \sum_{1 \leqq \rho \leqq r} \int_0^t u^\rho \partial_{m+d+\rho} h(0, x; su)\, ds.$$

Then Eq. (3.6) gives us

$$\partial_{m+d+\rho} h^i(0, x; su) = \sum_{I \in G} R^I_{\mu(\rho)}(su, g(0; su))\, A^i_I(h(0, x; su)).$$

Hence we have

$$\left\| \int_0^t u^\rho \partial_{m+d+\rho} h(0, x; su)\, ds \right\| \leqq \text{const} \left(\int_0^t \|h(0, x; su)\|\, ds + 1 \right)$$

for $x \in \mathbb{R}^d$, $u \in K$, $0 \leqq t \leqq 1$. So that we obtain ii) by virtue of Gronwall's inequality. Q.E.D.

Proof of Proposition 5.1. Fix T, $a > 0$ and $U \in C([0, \infty) \to \mathbb{R}^r)$. Put

$$F(t, x) = \sum_{1 \leqq i \leqq d} A^i_0(h(0, x; U_t))\, \partial_{m+i} h(f(0, x; U_t); 0), \qquad t \in [0, T], \quad x \in \mathbb{R}^d. \tag{5.2}$$

By Lemma 5.1, there exists a constant $b > 0$ such that

$$\|F(t, x)\| \leqq b(\|x\| + 1), \qquad x \in \mathbb{R}^d.$$

Hence, if we put $\omega(t) = (a+1) \exp bt - 1$, $t \in [0, T]$, ω satisfies

$$\frac{d\omega}{dt} \geqq \|F(t, x)\| \quad (t \in [0, T], \|x\| = \omega(t)).$$

Then, applying Perron's theorem, we obtain a solution of (5.1) defined on the interval $[0, T]$ for each x: $\|x\| \leqq a$. Uniqueness follows from the local Lipschitz continuity of $F(t, x)$. Q.E.D.

Now, Proposition 5.1 gives us a functional

$$\Phi: \mathbb{R}^d \times C([0, \infty) \to \mathbb{R}^r) \to C([0, \infty) \to \mathbb{R}^d)$$

defined by

$$\Phi(x, U)_t = D_t, \quad x \in \mathbb{R}^d, \qquad U \in C([0, \infty) \to \mathbb{R}^r), \quad t \geqq 0.$$

Theorem 5.1. *Suppose that $\mathscr{L}(A_1, \ldots, A_n)$ is nilpotent of step p. Then*

$$X_t = h(0, \ldots, 0, \Phi(x, B^F_\cdot)_t; B^F_t), \qquad t \geqq 0,$$

is the solution of (1.1) *for each* $x \in \mathbb{R}^d$.

To prove Theorem 5.1, we prepare

Lemma 5.2. *We have*

$$\sum_{1\leqq j\leqq d} \partial_{m+i} h^j(f(0, x; u); 0)\, \partial_{m+j} h^k(0, x; u) = \delta_i^k \tag{5.3}$$

for each $x\in\mathbb{R}^d$, $u\in\mathbb{R}^r$, $1\leqq i, k\leqq d$.

Proof. By the condition c) for f, we have

$$h(f(0, x; u); 0) = x.$$

Differentiating by $\dfrac{\partial}{\partial x^i}$, we obtain (5.3). Q.E.D.

Now we present

Proof of Theorem 5.1. Set

$$D_t = \Phi(X, B_\cdot^F)_t$$

and

$$X_t = h(0, D_t; B_t^F).$$

We have

$$\begin{aligned} dX_t = &\sum_{1\leqq i\leqq d} \frac{dD^i}{dt} \partial_{m+i} h(0, D_t; B_t^F)\, dt \\ &+ \sum_{1\leqq \rho\leqq r} \partial_{m+d+\rho} h(0, D_t; B_t^F) \circ dB_t^{\mu(\rho)}. \end{aligned}$$

Equations (5.1) and (5.3) give us

$$\begin{aligned} \sum_{1\leqq i\leqq d} \frac{dD^i}{dt} \partial_{m+i} h^j(0, D_t; B_t^F) &= \sum_{1\leqq k\leqq d} \delta_k^j A_0^k(h(0, D_t; B_t^F)) \\ &= A_0^j(X_t). \end{aligned}$$

Then, as the proof of Theorem 2.1, we obtain

$$\begin{aligned} &\sum_{1\leqq \rho\leqq r} \partial_{m+d+\rho} h^j(0, D_t; B_t^F) \circ dB_t^{\mu(\rho)} \\ &= \sum_{1\leqq \rho\leqq r,\, i\in E} \{T_\rho^{m+j}(f(0, D_t; B_t^F))\, Q_i^{\mu(\rho)}(Y_t)\} \circ dB_t^i \\ &= \sum_{1\leqq \rho\leqq r,\, i\in E,\, I\in G} \{R_{\mu(\rho)}^I(Y_t)\, A_I^j(h(0, D_t; B_t^F))\, Q_i^{\mu(\rho)}(Y_t)\} \circ dB_t^i \\ &= \sum_{1\leqq i\leqq n} A_i^j(X_t) \circ dB_t^i. \end{aligned}$$

It is easy to see that $X_0 = x$. Q.E.D.

Acknowledgment. The author wishes to express thanks to Professor N. Ikeda for his encouragement.

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Received July 7, 1978